CASTELNUOVO–MUMFORD REGULARITY
OF SIMPLICIAL SEMIGROUP RINGS
WITH ISOLATED SINGULARITY

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Abstract. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n \geq 2$ variables over a field $K$ and $m$ its graded maximal ideal. Let $f_1, \ldots, f_m \in S$ be homogeneous polynomials of degree $d - 1 \geq 2$ generating an $m$-primary ideal, and let $g_1, \ldots, g_r \in S$ be arbitrary homogeneous polynomials of degree $d$. In the present paper it will be proved that the Castelnuovo–Mumford regularity of the standard graded $K$-algebra $A = K[[f_{ij}]]_{i=1, \ldots, m, j=1, \ldots, n}$ is at most $(d - 2)(n - 1)$. By virtue of this result, it follows that the regularity of a simplicial semigroup ring $K[C]$ with isolated singularity is at most $e(K[C]) - \text{codim}(K[C])$, where $e(K[C])$ is the multiplicity of $K[C]$ and $\text{codim}(K[C])$ is the codimension of $K[C]$.

Introduction

Castelnuovo–Mumford regularity of graded rings and ideals is one of the most active research topics in computational commutative algebra and computational algebraic geometry.

Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n \geq 2$ variables over a field $K$, and let $M$ be a finitely generated graded $S$-module. If

$$\cdots \rightarrow F_j \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

is the graded minimal free $S$-resolution of $M$, then the Castelnuovo–Mumford regularity $\text{reg}(M)$ of $M$ is the nonnegative integer $\text{reg}(M) = \max\{b_j - j: j = 0, 1, \ldots\}$, where $b_j$ is the maximal degree of the generators of the graded free $S$-module $F_j$.

We are especially interested in the Castelnuovo–Mumford regularity of the standard graded $K$-algebra $A = S/I$, where $I$ is a homogeneous ideal of $S$. Eisenbud and Goto conjectured in their paper [3] that if $A$ is an integral domain, then $\text{reg}(A)$ satisfies the inequality

$$\text{reg}(A) \leq e(A) - \text{codim}(A),$$

where $e(A)$ is the multiplicity of $A$ and $\text{codim}(A)$ is the codimension of $A$. The Eisenbud–Goto conjecture turns out to be true in several special cases considered
in algebraic geometry; see [3] and [4]. However, the conjecture is widely open in general; even in the case that \(A\) is an affine semigroup ring.

Let \(\mathfrak{m}\) denote the graded maximal ideal of \(S\). In the present paper, we pay attention to the Castelnuovo–Mumford regularity of the standard graded \(K\)-algebra \(A = K[f_1, \ldots, f_m]\), where \(I = (f_1, \ldots, f_m) \subset S\) is a homogeneous ideal generated in degree \(d\) such that \(I^k = \mathfrak{m}^d\) for some \(k > 0\). For such a \(K\)-algebra \(A\) one has \(e(A) = d^{n-1}\) and \(\text{codim}(A) \leq \left(\frac{d+n-1}{n-1}\right) - n\).

For a particular class of such \(K\)-algebras we can bound the regularity. This is shown in Theorem I.1. As a consequence we obtain in Corollary [1.3] that for such a \(K\)-algebra \(A\), one has \(\text{reg}(A) \leq e(A) - \text{codim}(A)\), if \(n \geq 3\).

Recently, in Hoa and St"uckrad [4] the regularity of simplicial semigroup rings was studied. Their work strongly stimulates the research to find reasonable classes of simplicial semigroup rings satisfying inequality [1]. As a conclusion of Corollary 1.3 and a simple counting argument, we show in our final Corollary 2.2 that the Eisenbud–Goto conjecture holds for simplicial semigroup rings with isolated singularity.

1. Regularity of Certain Graded Rings Generated by \(d\)-Forms

Let \(K\) be a field and \(S = K[x_1, \ldots, x_n]\) the polynomial ring in \(n \geq 2\) variables over \(K\) with the graded maximal ideal \(\mathfrak{m} = (x_1, \ldots, x_n)\).

**Theorem 1.1.** Let \(f_1, \ldots, f_m \in S\) be homogeneous polynomials of degree \(d - 1 \geq 2\) generating an \(\mathfrak{m}\)-primary ideal, and let \(g_1, \ldots, g_r \in S\) be arbitrary homogeneous polynomials of degree \(d\). Then the regularity of the standard graded \(K\)-algebra \(A = K[\{f_i x_j\}_{i=1, \ldots, m}, g_1, \ldots, g_r]\) is at most \((d-2)(n-1)\).

**Proof.** We may assume that \(K\) is an infinite field. Let \(J = (f_1, \ldots, f_m)\). Then there exists an ideal \(L \subset J\) generated by a regular sequence of length \(n\) consisting of elements of degree \(d - 1\). For a finite length graded \(S\)-module \(N\) we set \(s(N) = \max\{i : N_i \neq 0\}\). It is known that \(\text{reg}(N) = s(N)\). Therefore for \(k\) it follows that

\[
\text{reg}(J^k) = \text{reg}(S/J^k) + 1 \leq \text{reg}(S/L^k) + 1 = \text{reg}(L^k).
\]

Let \(L = (\ell_1, \ldots, \ell_n)\). Since \(L\) is generated by a regular sequence of length \(n\) of elements of degree \(d - 1\), the resolution of \(L^k\) is given by the Eagon–Northcott complex (see, e.g., [2]) attached to the \((d-1) \times (n+d-2)\) matrix

\[
\begin{pmatrix}
\ell_1 & \cdots & \ell_n & 0 & \cdots & 0 \\
0 & \ell_1 & \cdots & \ell_n & \cdots & 0 \\
\vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \ell_1 & \cdots & \ell_n
\end{pmatrix}
\]

It follows immediately from this resolution that \(\text{reg}(L^k) = (d-1)k + (d-2)(n-1)\).

For the convenience of the reader we give a direct proof of this fact: The ideal \(\mathfrak{m}^k = (x_1, \ldots, x_n)^k\) has a \(K\)-linear resolution. In particular, the generators in the last step of the resolution are of degree \(k + n - 1\). Consider the flat map \(\varphi : S \to S\) with \(\varphi(x_i) = \ell_i\) for \(i = 1, \ldots, n\). Then \(\varphi(\mathfrak{m}^k) = L^k\), and so the resolution of \(L^k\) is obtained from that of \(\mathfrak{m}^k\) by replacing each \(x_i\) with \(\ell_i\). This implies that all the shifts are multiplied by \(d - 1\). Hence the generators in the last step of the resolution of \(L^k\) are of degree \((k + n - 1)(d - 1)\). From this we conclude that \(\text{reg}(L^k) = (k + n - 1)(d - 1) - (n-1) = (d-1)k + (d-2)(n-1)\).
Let $I = Jm$. We claim that $I^k$ has a linear resolution if $k \geq (d-2)(n-1)$. In fact, $I^k = J^km^k = (J^k)^{(d-1)k+k}$, where for a graded module $M$ we set $M_{\geq j} = \bigoplus_{i \geq j} M_i$. Recall from [5] (or [1] Theorem 4.3.1) that

$$\text{reg } M = \min \{ j : M_{\geq j} \text{ has a linear resolution} \}.$$ 

It follows that $I^k$ has a linear resolution if and only if $(d-1)k + k \geq \text{reg}(J^k)$. In particular, $I^k$ has a linear resolution if $(d-1)k + k \geq (d-1)k + (d-2)(n-1)$, namely if $k \geq (d-2)(n-1)$.

Next we notice that an $m$-primary ideal $H$ generated in one degree, say $h$, has a linear resolution if and only if it is a power of $m$. To see why this is true, we observe that $H$ has a linear resolution if and only if $\text{reg}(H) = h$. But $\text{reg}(H) = s(S/H) + 1 = h$ if and only if $H = m^h$.

Applied to our situation we conclude that $I^k = m^{dk}$ for $k \geq (d-2)(n-1)$. This implies that $A_k = S_{dk}$ for all $k \geq (d-2)(n-1)$. Let $A^*$ be the integral closure of $A$. Then $A^* = S^{(d)}$, the $d$th Veronese subring of $S$, and $A^*/A$ is of finite length with $s(A^*/A) \leq (d-2)(n-1) - 1$.

Let $n$ be the graded maximal ideal of $A$. Local cohomology applied to the exact sequence

$$0 \longrightarrow A \longrightarrow A^* \longrightarrow A^*/A \longrightarrow 0$$

yields that $H^n_0(A^*/A) = H^n(A^*)$ and $H^i(A^*) = H^i_n(A^*)$ for $i > 1$. Since $A^*$ is Cohen–Macaulay, one also has $H^i_n(A^*) = 0$ for $i < d = \dim(A) = \dim(A^*)$. Hence

$$\text{reg}(A) = \max\{\text{reg}(A^*), \text{reg}(A^*/A) + 1\}.$$ 

Since $\text{reg}(A^*/A) = s$, it follows that

$$\text{reg}(A) = \max\{\text{reg}(A^*), s(A^*/A) + 1\} \leq (d-2)(n-1),$$

since the regularity of the Cohen–Macaulay algebra $A^*$ is $n$ plus its $a$-invariant, and hence at most $n-1$, and since $n-1 \leq (d-2)(n-1)$ because $d \geq 3$. \hfill \square

Theorem [14] suggests the following question: Let $f_1, \ldots, f_m$ be homogeneous polynomials of degree $d$ and suppose that $A_k = S_{dk}$ for some $k$. Does this imply that $\text{reg}(A) \leq (d-2)(n-1)$?

We shall need the following numerical result.

**Lemma 1.2.** If $n \geq 3$ and $d \geq 3$, then $(d-2)(n-1) \leq d^n - \left( \binom{n+d}{n} - (n+1) \right)$.

**Proof.** Replace $n-1$ with $n$ in the required inequality, and what we must prove is the inequality

$$\text{(2)} \quad (d-2)n \leq d^n - \left( \binom{n+d}{n} - (n+1) \right)$$

for $n \geq 2$ and $d \geq 3$. Inequality (2) is equivalent to the inequality

$$d^n - \left( \binom{n+d}{n} \right) \geq (d-2)n - (n+1).$$

Thus we must prove the inequality

$$\text{(3)} \quad d^n - \prod_{i=1}^{n} (1 + \frac{d}{i}) \geq nd - 3n - 1$$

for $n \geq 2$ and $d \geq 3$.

Fix $d \geq 3$. By using induction on $n \geq 2$ we will prove (3).

If $n = 2$, then the inequality (3) coincides with $(d-3)(d-4) \geq 0$. 

Let $n \geq 2$ and suppose that the inequality (3) is true. Inequality (3) for $n + 1$ then follows from the computation below:

$$\begin{align*}
d^{n+1} - \prod_{i=1}^{n+1} (1 + \frac{d}{i}) &= (d^{n+1} - d^n) - \prod_{i=1}^{n+1} (1 + \frac{d}{i}) + d^n - \prod_{i=1}^{n} (1 + \frac{d}{i}) \\
&\geq d^n (d - 1) - \frac{d}{n+1} \prod_{i=1}^{n} (1 + \frac{d}{i}) + (nd - 3n - 1)
\end{align*}$$

2. Simplicial semigroup rings with isolated singularity

Let $C$ be a positive affine semigroup of rank $n$, i.e., the associated group $\mathbb{Z}C$ is isomorphic to $\mathbb{Z}^n$ and $\{0\}$ is the only subgroup contained in $C$. Let $G$ be the minimal set of generators of $C$. We say that $C$ is standard graded if there exists a hyperplane $H \subset \mathbb{Z}C \otimes \mathbb{Q}$ such that $G \subset H$. Let $P$ be the convex hull of $G$ in $\mathbb{Z}C \otimes \mathbb{Q}$. We say that $C$ is simplicial if $P$ is a simplex. Let $v_1, \ldots, v_n$ be the vertices of $P$. After the choice of a basis of $\mathbb{Z}C$, the vertices $v_i$ can be identified with integral vectors. Let

$$A = (v_1^t, \ldots, v_n^t)$$

be the $n \times n$ matrix whose columns are the transpose of the vertices $v_i$. We denote by $A^*$ the adjoint matrix of $A$. Let $\delta = \det(A)$. Then $\delta \neq 0$ and $A^* A = \delta E_n$, where $E_n$ is the unit matrix of size $n$.

Let $\varphi : \mathbb{Z}C \to \mathbb{Z}^n$ be the linear map associated with $A^*$. It then follows that the simplex $P' = \varphi(P) \subset \mathbb{Z}^n$ has the vertices $\delta \varepsilon_i$, where $\varepsilon_i$ denotes the $i$th standard unit vector of $\mathbb{Q}^n$. Let $C' = \varphi(C)$ and $G' = \varphi(G)$. Note that $C'$ is isomorphic to $C$ and that $G'$ is the minimal set of generators of $C'$ with $\{\delta \varepsilon_1, \ldots, \delta \varepsilon_n\} \subset G' \subset P'$. Let $t$ be the greatest common divisor of all the components of all the vectors belonging to $G'$, and set $d = \delta / t$. Denote by $C'' \subset \mathbb{Z}$ the semigroup generated by $G'' = \frac{1}{t}G'$. 

Corollary 1.3. Let $A$ be the $K$-algebra as defined in Theorem 1.1 and assume that $n \geq 3$. Then

$$\text{reg}(A) \leq e(A) - \text{codim}(A).$$

Proof. If $n \geq 3$, then the assertion follows from Theorem 1.1 together with Lemma 1.2 because $e(A) = e(S(d)) = d^{n-1}$ and $\text{codim}(A) \leq \binom{n+d-1}{n-1} - n$. 

\[\square\]
Then it is clear that the convex hull of $G''$ is the simplex with vertices $d\varepsilon_1, \ldots, d\varepsilon_n$, that $C'' \cong C$ and that $[\mathbb{Z}^n : \mathbb{Z}C''] = d$. We call $d$ the index of $C$. It is in fact an invariant of $C$, i.e., does not depend on the particular basis of $\mathbb{Z}C$ which was chosen to define the matrix $A$. We say that the simplicial semigroup $C''$ is standard embedded.

**Theorem 2.1.** Let $C$ be a simplicial semigroup of rank $n > 1$ with index $d > 2$. Let $K$ be a field and $K[C]$ the semigroup ring associated with $C$. Suppose that $K[C]$ is a $K$-algebra with isolated singularity. Then

$$\text{reg}(K[C]) \leq (d-2)(n-1).$$

**Proof.** Since $K[C] \cong K[C']$ we may assume that the embedding of $C$ itself is standard. Let $[n] = \{1, \ldots, n\}$. Write $x^w = x_1^{w_1} \cdots x_n^{w_n}$ if $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$.

For each $1 \leq i \neq j \leq n$, we write $q_i^j \geq 1$ for the biggest integer satisfying

$$d\varepsilon_i + \frac{1}{q_i^j}(d\varepsilon_j - d\varepsilon_i) \in G.$$ 

Since the localization

$$K[C]_{x_1^i} = K[x_1^d, \frac{1}{x_1^d}][\{x^w\}_{w \in G}] = K[x_1^d, \frac{1}{x_1^d}][\{x^{w - d\varepsilon_i}\}_{w \in G}]$$

is regular if and only if

$$K[C]_{x_1^i} = K[x_1^d, \frac{1}{x_1^d}][\{x^{\frac{j}{q_i^j}(d\varepsilon_j - d\varepsilon_i)}\}_{j \in [n] \setminus \{i\}}],$$

and since $K[C]$ is a $K$-algebra with isolated singularity, it follows that, for any $w \in G$ and for any $1 \leq i < n$, there exists $0 \leq p_j \in \mathbb{Z}$, $j \in [n] \setminus \{i\}$, such that

$$w - d\varepsilon_i = \sum_{j \in [n] \setminus \{i\}} \frac{p_j}{q_i^j}(d\varepsilon_j - d\varepsilon_i).$$

This simple observation yields the crucial result that $q_i^j = q_k^f$ for all $i, j, k, f$ with $i \neq j$ and $k \neq f$. In fact, in case of $1 \leq i \neq j \leq n$, since

$$(d\varepsilon_j + \frac{1}{q_i^j}(d\varepsilon_i - d\varepsilon_j)) - d\varepsilon_i = \frac{p_j}{q_i^j}(d\varepsilon_j - d\varepsilon_i), \quad 0 \leq p \in \mathbb{Z},$$

one has $q_i^j = q_j^i(q_i^j - p)$. Thus $q_i^j$ divides $q_j^i$. Similarly, $q_k^j$ divides $q_i^j$. Hence $q_i^j = q_j^i$.

Also, in case of $i, k, \ell \in [n]$ with $i \neq k$, $k \neq \ell$ and $i \neq \ell$, since

$$d\varepsilon_k - d\varepsilon_i + \frac{1}{q_k^i}(d\varepsilon_i - d\varepsilon_k) = \sum_{j \in [n] \setminus \{i, k\}} \frac{p_j}{q_i^j}(d\varepsilon_j - d\varepsilon_i), \quad 0 \leq p_j \in \mathbb{Z},$$

one has $\frac{1}{q_k^i} = \frac{p_k}{q_i^j}$ and $q_i^j$ divides $q_k^i$. Similarly, $q_k^j$ divides $q_i^j$. Hence $q_i^j = q_k^i$.

Let $q = q_i^j$ for all $1 \leq i \neq j \leq n$. Then $0 < \frac{d}{q} \in \mathbb{Z}$ divides each component of any vector belonging to $G$. Since the embedding of $C$ is standard, it follows that $q = d$. We now conclude that

$$(4) \quad K[C] = K[\{x_j^{d-1}x_i\}_{i=1,\ldots,n, j=1,\ldots,n}, g_1, \ldots, g_r],$$

where $g_1, \ldots, g_r$ are monomials of degree $d$. Hence we are in the situation of Theorem 1.1 with $f_i = x_i^{d-1}$ for $i = 1, \ldots, n$. \qed
The following final result follows partly from Corollary 1.3.

**Corollary 2.2.** Let $K[C]$ be a simplicial semigroup ring with isolated singularity. Then

$$\text{reg}(K[C]) \leq e(K[C]) - \text{codim}(K[C]).$$

**Proof.** Fix a standard embedding of $C$. Let rank $C = n$. For $n \geq 3$, the assertion follows from (4) and Corollary 1.3.

Now let $n = 2$. Then

$$K[C] = K\{x_1^{d-a_i}x_2^{a_i} \}_{i=0,\ldots, r+1},$$

with $0 = a_0 < 1 = a_1 < a_2 < \cdots < d - 1 = a_r < d = a_{r+1}$.

Therefore, $e(K[C]) = d$, and $\text{codim}(K[C]) = r$. Thus we need to show that $\text{reg}(K[C]) \leq d - r$, or equivalently, that

$$K[C]_{d-r} = K\{x_1^{(d-r)-j}x_2^{r} \}_{j=0,\ldots,(d-r)d}.$$

Set $k = d - r$, and let $X = \{j: x_1^{kd-j}x_2^{r} \in K[C]_{d-r}\}$. Since $a_0 = 0$, it follows that

$$X = \{\sum_{i=1}^{r+1} k_ia_i: k_i \geq 0, \sum_{i=1}^{r+1} k_i \leq k\},$$

and we have to show that $X = \{0, \ldots, kd\}$.

For any two integers $a \leq b$ we set $[a, b] = \{c \in \mathbb{Z}: a \leq c \leq b\}$. Fix a number $j \in \{0, \ldots, k\}$. Then $a_i + jd \in X \cap [jd, (j+1)d]$ for $i = 0, \ldots, r+1$.

Next we notice that $a_i + ja_{r+1} + la_1 = a_i + jd + l \in X$ for $l = 0, \ldots, k - 1 - j$, and that $a_{i+1} + la + (j-l)a_{r+1} = a_{i+1} + jd - l \in X$ for $l = 0, \ldots, j$. Thus we see that

$$(a_i + jd, a_i + jd + (k - 1 - j)) \cup [a_{i+1} + jd - j, a_{i+1} + jd] \subset X.$$

Since

$$(a_{i+1} + jd - j) - (a_i + jd + (k - 1 - j)) = (a_{i+1} - a_i) - (k - 1) \leq (d-r) - (d-r-1) = 1,$$

it follows that

$$[a_i + jd, a_{i+1} + jd] = [a_i + jd, a_i + jd + (k - 1 - j)] \cup [a_{i+1} + jd - j, a_{i+1} + jd],$$

so that by (5), $[a_i + jd, a_{i+1} + jd] \in X$ for all $i = 0, \ldots, r$ and all $j = 0, \ldots, k$. Since $[0, kd] = \bigcup_{j=0,\ldots,k} [a_i + jd, a_{i+1} + jd]$, the assertion follows.\qed

**References**


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