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A TAUBERIAN THEOREM FOR VILENKIN SERIES

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Abstract. There are a number of papers in the literature which contain Cesàro analogues of results already known for martingale sums of Vilenkin-Fourier series. We show that for Vilenkin systems of bounded type, these are not merely analogues but actually generalizations. Indeed, we prove that convergence of the Cesàro means of a Vilenkin series S implies convergence of martingale partial sums of S itself.

§1. Introduction

Let $\mathbf{N} := \{0, 1, 2, \cdots\}$, and $\mathcal{P} := \{p_0, p_1, \cdots\}$ be any sequence of integers which satisfies $p_n \geq 2$. For each $n \in \mathbf{N}$ set $P_n := p_0 p_1 \cdots p_{n-1}$, where the empty product is by definition 1. The *multiplicative Vilenkin group* associated with $\mathcal{P}$ is the set $G := \{(x_0, x_1, \cdots) : x_k \in \mathbf{N} \text{ and } 0 \leq x_k < p_k\}$ together with the operation

$$x \overset{\bullet}{+} y := (x_0 \oplus y_0, x_1 \oplus y_1, \cdots),$$

where $x = (x_0, x_1, \cdots)$, $y = (y_0, y_1, \cdots)$ and, for each k, $x_k \oplus y_k$ represents the sum of x_k and y_k modulo p_k. The dual group of G is the system $(w_n, n \in \mathbf{N})$ defined for $x = (x_0, x_1, \cdots)$ by

$$w_n(x) := \prod_{k=0}^{\infty} \exp\left(\frac{2\pi i n_k x_k}{p_k}\right),$$

where the coefficients n_k are integers which satisfy $0 \leq n_k < p_k$ and $n = \sum_{k=0}^{\infty} n_k P_k$ (see Vilenkin [4] for details). When $p_k := 2$ for all k, the group G is called the *dyadic group* and the characters w_n are called the *Walsh system*. When $p_k = O(1)$, the system $\{w_n\}$ is called a (multiplicative) Vilenkin system of *bounded type*.

It is well known that G is a compact group for each collection of radices $\mathcal{P}$, and that the corresponding *Vilenkin system* $\{w_n\}$ is a complete orthonormal system on G. Moreover, the group G can be identified with the interval $[0, 1)$ by taking an $x = (x_0, x_1, \cdots) \in G$ to the number

$$\overline{x} := \sum_{k=0}^{\infty} x_k P_{k+1}^{-1}.$$

Under this identification, Haar measure on G is taken to Lebesgue measure on $[0, 1)$.

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A *Vilenkin series* is a series of the form $S := \sum_{k=0}^{\infty} a_k w_k$, where a_k is some sequence of complex numbers. For each $x \in G$ and $n \in \mathbf{N}$, the *partial sums* of a Vilenkin series S are defined by

$$S_n(x) := \sum_{k=0}^{n-1} a_k w_k(x).$$

The partial sums S_{P_n} form a martingale in $L^2(G)$ which allows one to use martingale convergence theorems on Vilenkin series.

§2. Preliminaries

For each nonnegative integer n, define intervals on G by $I_0(0) := G$, and

$$I_n(j) := \left\{ x = (x_0, x_1, \cdots) \in G : \sum_{k=0}^{n-1} x_k P_{k+1}^{-1} = \frac{j}{P_n} \right\} \tag{1}$$

for $j = 0, 1, \cdots, P_n - 1$, $n = 1, 2, \cdots$. Recall that $\{I_n(0)\}_{n=0}^{\infty}$ is a nested sequence of subgroups of G which forms a neighborhood base at the origin, and for each n, $\{I_n(j)\}_{j=0}^{P_n-1}$ is a collection of pairwise disjoint compact sets in G whose union is G. In particular, given $x \in G$ and $n \in \mathbf{N}$, there is a unique $0 \le j < P_n$ such that $x \in I_n(j)$. We shall denote this interval by $I_n(x)$.

Denote the Haar measure of a subset E of G by $m(E)$ and the Lebesgue measure of a subset E of $[0, 1)$ by $|E|$. Notice that under the identification of G with $[0, 1)$, the interval $I_n(j)$ corresponds to the interval $[jP_n^{-1}, (j+1)P_n^{-1})$. In particular,

$$m(I_n(j)) = P_n^{-1} \tag{2}$$

for $0 \le j < P_n$ and $n \in \mathbf{N}$.

Let $x \in G$. A sequence of measurable sets E_j in G is said to *shrink nicely* to x if there exist integers $r_j \to 0$, as $j \to \infty$, and an absolute constant $\alpha > 0$ such that $E_j \subset I_{r_j}(x)$ and

$$m(E_j) \ge \alpha \cdot m(I_{r_j}(x)),$$

for $j = 1, 2, \ldots$. By using the identification of G with the unit interval, and of Haar measure with Lebesgue measure, it is easy to check that if E is a measurable subset of G and h represents the characteristic function of E, i.e., $h(x) = 1$ for $x \in E$ and $h(x) = 0$ for $x \notin E$, then

$$\frac{1}{m(E_j)} \int_{E_j} h \, dm \to 1$$

almost everywhere $[m]$ on E for any sequence of sets E_j which shrinks nicely to x (see Rudin [2], p. 140). In particular, if E_j shrinks nicely to x, then

$$\lim_{j \to \infty} \frac{m(E \cap E_j)}{m(E_j)} = 1 \tag{3}$$

for almost every $x \in E$.

§3. The main result

For each $x \in G$ and each integer $n > 0$, the Cesàro means of S are defined by

$$\sigma_n(x) := \frac{1}{n}\sum_{k=1}^{n} S_k(x).$$

It is well known (and easy to see) that if S_k converges to some limit f, as $k \to \infty$, then $\sigma_n \to f$, as $n \to \infty$. We will obtain the following partial converse to this result. (The Walsh version of this result was obtained by Shaginyan [3] in 1974.)

Theorem 1. *Suppose E is a measurable subset of G and S is a Vilenkin series whose Cesàro means satisfy*

$$\lim_{n\to\infty} \sigma_{P_n}(x) = f(x), \qquad x \in E, \tag{4}$$

for some finite-valued, measurable function f. If the Vilenkin system is of bounded type, then

$$\lim_{n\to\infty} S_{P_n}(x) = f(x)$$

for almost every $x \in E$.

Proof. We may suppose that E is of positive Haar measure, i.e., $m(E) > 0$. Fix $0 < \epsilon < m(E)/2$. By hypothesis, $f_n(x) := \sigma_{P_{n+1}}(x)$ converges to $f(x)$ for $x \in E$, as $n \to \infty$. Thus by Egoroff's Theorem, we can choose a measurable subset E_0 of E such that $f_n \to f$ uniformly on E_0 and the measure of E_0 satisfies $m(E_0) > m(E) - \epsilon$.

Let $x \in E_0$ be a point which satisfies (3) for every sequence E_j which shrinks nicely to x. For each fixed positive integer j, let $\mathcal{W}_j$ represent the collection of indices ℓ which satisfy $I_{j+1}(\ell) \subset I_j(x)$. If $\ell \in \mathcal{W}_j$ and

$$B_j := I_{j+1}(\ell) \neq A_j := I_{j+1}(x),$$

let E_j represent the union of A_j and B_j. By construction, $E_j \subset I_j(x)$ and by (2),

$$m(E_j) = 2m(I_{j+1}(x)) = \frac{2}{p_j P_j} \geq \frac{2}{MP_j} = \frac{2}{M} \cdot m(I_j(x)),$$

where $M := \sup\{p_i : i \in \mathbf{N}\}$. Thus A_j and E_j both shrink nicely to x, as $j \to \infty$. It follows from (3) that $m(E_0 \cap A_j)/m(A_j) \to 1$ and $m(E_0 \cap E_j)/m(E_j) \to 1$ as $j \to \infty$.

Let $n \in \mathbf{N}$ be so large that $m(E_0 \cap A_n)/m(A_n) < 5/4$ and $m(E_0 \cap E_n)/m(E_n) > 3/4$. Since $B_n = E_n \setminus A_n$ and $m(E_n) = 2m(A_n) = 2m(B_n)$, such a choice for n implies

$$\begin{aligned}\frac{m(E_0 \cap B_n)}{m(B_n)} &= \frac{m(E_0 \cap E_n) - m(E_0 \cap A_n)}{m(B_n)} \\ &= \frac{2m(E_0 \cap E_n)}{m(E_n)} - \frac{m(E_0 \cap A_n)}{m(A_n)} > \frac{6}{4} - \frac{5}{4} > 0.\end{aligned}$$

In particular, $E_0 \cap B_n$ is uncountable.

We have proved that if n is sufficiently large, for each $\ell \in \mathcal{W}_n$ which satisfies $I_{n+1}(\ell) \neq I_{n+1}(x)$, there exists an $x_n(\ell) \in E_0 \cap I_{n+1}(\ell)$. The restriction $I_{n+1}(\ell) \neq I_{n+1}(x)$ is not needed. Indeed, if ℓ satisfies $I_{n+1}(\ell) = I_{n+1}(x)$, then $x_n(\ell) := x$ also belongs to $E_0 \cap I_{n+1}(\ell)$. Thus, for n sufficiently large, it is possible to choose points $x_n(\ell) \in E_0 \cap I_{n+1}(\ell)$ for all $\ell \in \mathcal{W}_n$.

We claim that for these points,

$$\sum_{\ell\in\mathcal{W}_n} \sigma_{P_{n+1}}(x_n(\ell)) = \sigma_{P_n}(x) + (p_n-1)S_{P_n}(x). \tag{5}$$

To verify (5), let $y \in G$ and notice by definition that

$$\begin{aligned}\sigma_{P_{n+1}}(y) &= \frac{1}{P_{n+1}}\sum_{k=1}^{P_{n+1}} S_k(y)\\ &= \frac{1}{p_nP_n}\sum_{k=1}^{P_n} S_k(y) + \frac{1}{p_nP_n}\sum_{k=P_n+1}^{P_{n+1}} S_k(y) \qquad (6)\\ &= \frac{1}{p_n}\sigma_{P_n}(y) + \frac{1}{p_nP_n}\sum_{k=P_n+1}^{P_{n+1}} S_k(y) =: I_1(y) + I_2(y).\end{aligned}$$

Look at a typical term $T(y) := a_k w_k(y)$ in one of these sums where $y = x_n(\ell)$. If T comes from I_1, then k is less than P_n. Since the Vilenkin function w_k is constant on $I_n(x)$ and $x_n(\ell) \in I_n(x)$, it is clear that $T(x_n(\ell)) = T(x)$ for all $\ell \in \mathcal{W}_n$. Since the number of intervals of the form $I_{n+1}(\ell)$ which are subsets of $I_n(x)$ is p_n, it follows that

$$\sum_{\ell\in\mathcal{W}_n} I_1(x_n(\ell)) = \sum_{\ell\in\mathcal{W}_n} \frac{1}{p_n}\sigma_{P_n}(x) = \sigma_{P_n}(x).$$

On the other hand, suppose that T comes from I_2 and $P_n < k \le P_{n+1}$. Notice that $\{I_{n+1}(\ell) : \ell \in \mathcal{W}_n\}$ contains every subinterval of $I_n(x)$, hence $\{w_k(x_n(\ell)), \ell \in \mathcal{W}_n\}$ ranges over every p_nth root of unity. Since the sum of p_nth roots of unity is zero, we have

$$\sum_{\ell\in\mathcal{W}_n} w_k(x_n(\ell)) = 0.$$

Consequently, adding up the values of I_2 at $y = x_n(\ell)$, as ℓ runs over $\mathcal{W}_n$, cancels higher order terms and multiplies lower order terms. Indeed, since the number of intervals of the form $I_{n+1}(\ell)$ which are subsets of $I_n(x)$ is p_n and $P_{n+1} - P_n = (p_n-1)P_n$, we have

$$\sum_{\ell\in\mathcal{W}_n}\sum_{k=P_n+1}^{P_{n+1}} S_k(x_n(\ell)) = \sum_{\ell\in\mathcal{W}_n}\sum_{k=P_n+1}^{P_{n+1}} S_{P_n}(x) = p_n(p_n-1)P_nS_{P_n}(x).$$

It follows from (6) that

$$\sum_{\ell\in\mathcal{W}_n} \sigma_{P_{n+1}}(x_n(\ell)) = \sigma_{P_n}(x) + (p_n-1)S_{P_n}(x),$$

i.e., (5) holds.

Solve (5) for S_{P_n} to write

$$S_{P_n}(x) = \frac{1}{p_n-1}\left(\sum_{\ell\in\mathcal{W}_n} \sigma_{P_{n+1}}(x_n(\ell)) - \sigma_{P_n}(x)\right). \tag{7}$$

By construction, $x_n(\ell)$ belongs to $I_n(x)$, hence $x_n(\ell) \to x$, as $n \to \infty$, for each $\ell \in \mathcal{W}_n$. Since the characters w_k are continuous on the group, $w_k(x_n(\ell)) \to w_k(x)$, as $n \to \infty$, for each $k \in \mathbf{N}$. But f is the uniform limit of f_n on E_0 and each f_n is a Vilenkin polynomial. Since $x_n(\ell) \in E_0$, it follows that $f(x_n(\ell)) \to f(x)$ as $n \to \infty$.

Let g (respectively, g_n) represent the real part of f (respectively, of f_n). By construction and the observation just made, we can choose n so large that

$$\mathrm{Re}\left[\sigma_{P_{n+1}}(x_n(\ell))\right] = g_n(x_n(\ell)) \geq g(x_n(\ell)) - \frac{\epsilon}{2} \geq g(x) - \epsilon. \tag{9}$$

Substituting this estimate back into (7), we obtain

$$\begin{aligned} \mathrm{Re}\,(S_{P_n}(x)) &\geq \frac{1}{p_n - 1}\left(p_n g(x) - p_n\epsilon - \mathrm{Re}\,[\sigma_{P_n}(x)]\right) \\ &\geq \frac{1}{p_n - 1}(p_n g(x) - \mathrm{Re}\,[\sigma_{P_n}(x)]) - M\epsilon. \end{aligned} \tag{8}$$

Take the limit infimum of this inequality as $n \to \infty$, apply hypothesis (4), and then let $\epsilon \to 0$. It follows that

$$\liminf_{n\to\infty} \mathrm{Re}\,[S_{P_n}(x)] \geq g(x) = \mathrm{Re}\,[f(x)]$$

for almost every $x \in E_0$. Since $|E_0| > |E| - \epsilon$, we can replace E_0 by E. Hence, $\liminf_{n\to\infty} \mathrm{Re}\,[S_{P_n}] \geq \mathrm{Re}\,[f]$ almost everywhere on E. By repeating the entire argument with the inequalities in (9) reversed, we see that $\limsup_{n\to\infty} \mathrm{Re}\,[S_{P_n}] \leq \mathrm{Re}\,[f]$ almost everywhere on E. A similar argument works for the imaginary parts. We conclude that $S_{P_n} \to f$ almost everywhere on E as $n \to \infty$. □

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