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ON THE DENSENESS OF THE INVERTIBLE GROUP IN BANACH ALGEBRAS

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Abstract. We examine the condition that a complex Banach algebra A has dense invertible group. We show that, for commutative algebras, this property is preserved by integral extensions. We also investigate the connections with an old problem in the theory of uniform algebras.

Introduction

In this paper we investigate the condition that a complex Banach algebra, A, has a dense invertible group, $G(A)$. Throughout we shall consider only unital, complex algebras; usually they will also be commutative.

The condition was first explored by Robertson in [23] for C^*-algebras. As remarked there, a complete characterisation of it in terms of the topology of the maximal ideal space has been known for some time in the commutative case. It is so fundamental that we state it now: if X is a compact Hausdorff space, then $C(X)$ has dense invertible group if and only if $\dim X$, the covering dimension of X, is not more than 1. For a proof of this fact see for example [21], Proposition 3.3.2. This book is a useful source of information on dimension.

The condition has since attracted considerable (implicit) attention as the first case of the condition '$\operatorname{tsr} A \leq m$' where m is a natural number, A a topological algebra, and $\operatorname{tsr} A$ its 'topological stable rank'. This notion was introduced in the seminal paper [21] and for a very useful survey and collection of results on this subject we refer the reader to [5].

We also mention that the condition has been used to characterise commutative C^*-algebras in which all elements have square roots ([13]). Furthermore, in Chapter 10 of [21], the dimension of a compact Hausdorff space, X, is described precisely in terms of the set of (uniformly closed) subalgebras of $C(X, \mathbb{R})$ which are integrally closed in $C(X, \mathbb{R})$. (Recall that an element b of an algebra, B, is said to be *integral* over a unital subalgebra, A, if there exists a monic polynomial with coefficients in A which has b as a root.) However we shall only consider complex Banach algebras here.

In general it is not easy to determine the closure of the group of invertible elements of a Banach algebra. Again we quote an example from [23]: when A is

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the disc algebra, $\overline{G(A)}$ is equal to the set of elements of A whose zero set lies in the unit circle. Further introductory examples are given in Section 1 of this paper.

Our study of the denseness of the invertible group was prompted by an observation (Theorem 1.7) which sheds light on the old open problem of whether or not there is a non-trivial uniform algebra on the unit interval whose maximal ideal space is equal to the unit interval. However our main result, proved in Section 2, is that the property of having a dense invertible group is inherited by integral extensions of a commutative Banach algebra.

1. Notation, preliminary examples and results

We begin by setting out some notation and standard terminology.

The Gelfand transform of a Banach algebra, A, will be denoted by $\hat{A}$. The algebra A is said to be *symmetric* if $\bar{f} \in \hat{A}$ whenever $f \in \hat{A}$. We shall use the notation $B_A(a,r)$ for the open ball of centre a and radius r in the normed space A. The unit circle is denoted by S^1; we regard it as a subset of the complex plane.

The following definition, which seems quite standard in the literature, is useful.

Definition 1.1. A subalgebra, B, of an algebra, A, is called *full* if we have $G(B) = B \cap G(A)$.

For example, $R_0(X)$, the algebra of rational functions on a compact plane set, X, with poles off X, and $C^n(I)$ are both full subalgebras of their completions with respect to the uniform norm.

Example 1.2. Let B be the restriction to S^1 of the algebra of complex rational functions with poles off $S^1 \cup \{2\}$. Then B is easily seen to be uniformly dense in $A = C(S^1)$ (by the Stone-Weierstrass theorem for example). However, B is not a full subalgebra of A because, for example, the function given by $z - 2 \in B$ is invertible in A.

For future reference we note the following elementary result.

Proposition 1.3. *Let A be a Banach algebra with maximal ideal space Ω such that $\hat{A}$ is dense in $C(\Omega)$ (for example if A is symmetric). Give $\hat{A}$ the uniform norm. Then $\hat{A}$ has dense invertible group if and only if $C(\Omega)$ has.*

Proof. It is elementary that $\hat{A}$ is a full subalgebra of $C(\Omega)$. Moreover, if $\hat{A}$ is dense in $C(\Omega)$, then, since $G(C(\Omega))$ is open in $C(\Omega)$, we have from elementary topology that
$$\overline{G(\hat{A})} = \overline{\hat{A} \cap G(C(\Omega))} = \overline{G(C(\Omega))}$$
where the closures are taken in $C(\Omega)$. This shows that if $G(C(\Omega))$ is dense in $C(\Omega)$, then the invertible elements of $\hat{A}$ are dense in $\hat{A}$. The converse is trivial. □

It follows from facts mentioned in Section 1.2 of [5] that if A is a commutative Banach algebra with maximal ideal space Ω and A is regular (that is, the hull-kernel topology on Ω is the Gelfand topology), then the denseness of $G(A)$ in A implies that $C(\Omega)$ also has dense invertible group. We do not know if the converse is true for regular Banach algebras. Neither do we know any examples of non-regular Banach algebras for which $G(A)$ is dense in A but $C(\Omega)$ does not have dense invertible group.

Example 1.4. In [22] it is shown that if A is a unital C^*-algebra, then A has dense invertible group if and only if $M_n(A)$ has for every $n \in \mathbb{N}$. We remark that the proof given in [22] applies to all Banach algebras.

In view of the fact that $M_n(A) = M_n(\mathbb{C})\hat{\otimes}A$ (see for example [20], p. 43), where $\hat{\otimes}$ is the projective tensor product, one might be tempted to conjecture that if A_1 and A_2 were commutative unital Banach algebras with dense invertible groups, then $A_1\hat{\otimes}A_2$ would have dense invertible group. However if we let $A_1 = A_2 = C(I)$, then by a standard result (see for example [6], Proposition 2.3.7) their tensor product, B, has a maximal ideal space homeomorphic to $I \times I$ (dimension 2) and B is easily seen to be symmetric. Therefore B could not have dense invertible group by Proposition 1.3 above.

Proposition 1.5. *Suppose that A is a commutative unital Banach algebra which is rationally generated by a single element, a. (See* [6]*, Definition* 2.2.7*: we mean that the algebra of elements obtained by applying the rational functions with poles off $\sigma(a)$ to a is dense in A.) If $\sigma(a)$ has empty interior, then A has dense invertible group.*

Proof. This follows from the fact that if K is a subset of $\mathbb{C}$ with empty interior and ϕ is a function holomorphic on a neighbourhood of K, then $\phi(K)$ also has no interior. Thus A has a dense set of elements whose spectra have no interior, which, as is noted in [7], is equivalent to the condition $\overline{G(A)} = A$. □

For the definition of a weight sequence and Beurling algebras, which appear in the next example, we refer the reader to [6], Definition 4.6.6.

Corollary 1.6. *Let ω be a weight sequence on $\mathbb{Z}$ and let A be the associated Beurling algebra, $\ell^1(\mathbb{Z},\omega)$. Then A has dense invertible group if and only if $\rho_- := \lim_{n\to+\infty} \omega_n^{1/n} = \lim_{n\to+\infty} \omega_{-n}^{1/-n} =: \rho_+$.*

Proof. It is standard (see for example [6], Theorem 4.6.7) that A is unitally polynomially generated by $\left\{\delta_1, {\delta_1}^{-1}\right\}$ and so rationally generated by $\{\delta_1\}$ where δ_1 is the characteristic function of $\{1\}$. It is also standard that the maximal ideal space of A can be identified with the annulus $\{w \in \mathbb{C} : \ \rho_- \le |w| \le \rho_+\}$. By standard results in dimension theory $\Omega(A)$ has dimension 1 if $\rho_- = \rho_+$ and dimension 2 otherwise. If A has dense invertible group, then so has $\hat{A}$ and we must have $\rho_- = \rho_+$ by Proposition 1.3, as A is plainly symmetric.

Conversely suppose that $\rho_- = \rho_+$. Then the spectrum of δ_1 has empty interior and Proposition 1.5 applies. □

Recall that a *uniform algebra*, A, is a Banach subalgebra of $C(X)$ with the uniform norm such that A separates the points of X (a compact Hausdorff space) and contains the constants. A (unital) *Banach function algebra* satisfies the same axioms as a uniform algebra except that the complete norm on the algebra need not be the uniform norm. The Banach function algebra A is called *natural* if all of its characters are given by evaluation at points of X and *trivial* if $A = C(X)$.

In the following result we make use of the Arens-Royden theorem (see for example [20], p. 411) which implies that whenever A is a natural Banach function algebra on a compact Hausdorff space, X, there is an isomorphism

$$G(A) \;/\; e^A \to G(C(X)) \;/\; e^{C(X)}$$

and that these groups are isomorphic to the homotopy classes of maps $X \to S^1$. These groups are also isomorphic to $H^1(X, \mathbb{Z})$, the first Čech cohomology group of X with coefficients in $\mathbb{Z}$. (Further information about Čech cohomology groups can be found in Section 10 of [24].) For example when X is a compact plane set, then $H^1(X, \mathbb{Z}) = 0$ if and only if $\mathbb{C} \setminus X$ is connected.

Theorem 1.7. *Let A be a natural Banach function algebra on a compact Hausdorff space, X, such that $H^1(X, \mathbb{Z}) = 0$. Then:*

(i) *If X is locally connected and A has dense invertible group, then A is uniformly dense in $C(X)$ (and so trivial if A is a uniform algebra).*
(ii) *If X contains a locally connected closed subspace whose topological dimension is at least two, then A does not have a dense invertible group.*

Proof. We prove (ii); one can prove (i) by repeating parts of the argument.

Let E be a non-empty, closed, locally connected subspace of X of topological dimension at least two and suppose for a contradiction that A has a dense invertible group. By the comments above, $e^A = G(A)$. Let B be the algebra of functions on E which are restrictions of elements of A. Then B is isomorphic to the quotient of A by the (closed) ideal of functions which vanish on E. We regard B as a Banach function algebra on E with respect to this quotient norm.

By hypothesis the set $\mathcal{F} = \left\{ f \in C(E) : \text{ there exists } \tilde{f} \in e^A \text{ with } \tilde{f}|_E = f \right\}$ is dense in B. Now let C be the closure of B in $C(E)$. It is easy to check that C is a uniform algebra on E such that $\mathcal{F}$ is dense in C. Clearly every element in $\mathcal{F}$ has a square root in C and E is locally connected so it follows from Čirka's theorem (see [24], pp. 131-134) that $C = C(E)$. Since $C(E)$ has a dense set of invertible elements we must have $\dim E \leq 1$ by the fundamental result stated in the introduction, a contradiction. □

It is a famous open question (see Section 3) whether or not there exists a non-trivial natural uniform algebra on the unit interval, I. In connection with this we have the following corollary:

Corollary 1.8. *Let A be a natural uniform algebra on the unit interval. If A has dense invertible group, then $A = C(I)$.*

Now it follows quite generally and elementarily that elements of $A \setminus \overline{G(A)}$ have spectra whose interiors contain 0. Thus the set of space-filling curves of a non-trivial natural uniform algebra on the unit interval must have interior in A. It has certainly been known for a long time ([14]) that every such uniform algebra must contain a function whose image has positive Lebesgue measure. One can also compare the result of Čirka ([15], p. 670) which shows (using standard theory) that a non-trivial doubly generated natural uniform algebra on I must contain a dense set of functions whose images have non-empty interior.

Compare, too, the above with an unpublished result of Cole ([4]) which states that every non-trivial uniform algebra contains a chain of subalgebras $B_1 \subseteq B_2 \subseteq \cdots$ such that for each $n \in \mathbb{N}$, B_n is isomorphic to the n-dimensional polydisc algebra. We also remind the reader of the well-known examples of (non-natural) uniform algebras A on the unit interval such that every non-constant $f \in A$ has the interior of $f(I)$ non-empty. (See for example [18], p. 200.)

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2. Integral extensions

As hinted at in the Introduction denseness of invertible elements in uniform algebras seems to be connected with the existence of roots of monic polynomials over the algebra. Accordingly we mention the following result stated by Grigoryan in [12]: a uniform algebra A on a compact space, X, is trivial if $C(X)$ is finitely generated and is an integral extension of A.

Our main result is that integral extensions of commutative Banach algebras preserve the property of having dense invertible group.

Recall (or see for example p. 369 of [20]) that if A is a commutative unital normed algebra and $\alpha(x)$ is a monic polynomial over A, then the Arens-Hoffman extension, A_α, of A by $\alpha(x)$ is $A[x]/\big(\alpha(x)\big)$ where $\big(\alpha(x)\big)$ denotes the principal ideal in $A[x]$ generated by $\alpha(x)$. There are infinitely many equivalent norms which make this algebra an isometric extension of A. In fact, for a sufficiently large fixed value of $t > 0$, such an 'Arens-Hoffman' norm is given by

$$\left\|\big(\alpha(x)\big) + \beta(x)\right\| = \sum_{j=0}^{n-1} \|b_j\|\, t^j$$

where $\beta(x) = \sum_{j=0}^{n-1} b_j x^j$ $(b_0, \ldots, b_{n-1} \in A)$. (It can be shown, as on p. 128 of [16] for example, that every element of A_α has a unique representative of degree less than n, so this is well defined.) We shall write $\bar{x}$ for the coset $\big(\alpha(x)\big) + x$ from now on and we shall assume that the norm on A_α is given by an Arens-Hoffman norm.

The proof of our main result relies on resultants; we now recall their definition. Let $\alpha(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n$ be a monic polynomial over a commutative ring A and $\beta(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1} \in A[x]$. By definition (see for example [16], p. 325)

$$\operatorname{res}(\alpha(x), \beta(x)) := \begin{vmatrix} \overbrace{\begin{matrix} 1 & a_{n-1} & \cdots & a_2 \end{matrix}}^{n-1} & \overbrace{\begin{matrix} a_1 & a_0 & 0 & \cdots & 0 \end{matrix}}^{n} \\ \begin{matrix} 0 & 1 & \cdots & \cdots \end{matrix} & \begin{matrix} a_2 & a_1 & a_0 & \cdots & 0 \end{matrix} \\ \begin{matrix} \cdots & \cdots & \cdots & \cdots \end{matrix} & \begin{matrix} \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix} \\ \begin{matrix} 0 & \cdots & \cdots & 1 \end{matrix} & \begin{matrix} a_{n-1} & a_{n-2} & \cdots & \cdots & a_0 \end{matrix} \\ \begin{matrix} b_{n-1} & b_{n-2} & \cdots & b_1 \end{matrix} & \begin{matrix} b_0 & 0 & 0 & \cdots & 0 \end{matrix} \\ \begin{matrix} 0 & b_{n-1} & \cdots & \cdots \end{matrix} & \begin{matrix} b_1 & b_0 & 0 & \cdots & 0 \end{matrix} \\ \begin{matrix} \cdots & \cdots & \cdots & \cdots \end{matrix} & \begin{matrix} \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix} \\ \begin{matrix} 0 & 0 & \cdots & 0 \end{matrix} & \begin{matrix} b_{n-1} & b_{n-2} & \cdots & \cdots & b_0 \end{matrix} \end{vmatrix}.$$

It can be shown (see [2], A.IV §6.6.1) that $\big(\alpha(x)\big) + \beta(x)$ is invertible in A_α if and only if $\operatorname{res}(\alpha(x), \beta(x))$ is invertible in A.

We see from the above that, writing c for b_0, $\operatorname{res}(\alpha(x), \beta(x)) = P(c) = p_0 + p_1 c + \cdots + p_{n-1}c^{n-1} + c^n$ for some $p_0, \ldots, p_{n-1} \in A$ which are polynomials in $b_1, \ldots, b_{n-1}, a_0, \ldots, a_{n-1}$ only with coefficients in $\mathbb{C}$.

We shall usually fix $\alpha(x)$ and allow $\beta(x)$ to vary. We then denote the resultant of $\alpha(x)$ and $\beta(x)$, $\operatorname{res}(\alpha(x), \beta(x))$, by $R_\alpha(\beta(x))$ for a polynomial, $\beta(x)$, over A.

It is a standard fact ([11], p. 398) that $R_\alpha(b_0 + \cdots + b_{n-1}x^{n-1})$ is homogeneous of degree n in $b_0, \ldots, b_{n-1}$ and homogeneous of degree $n-1$ in $a_0, \ldots, a_{n-1}, 1$.

Sometimes in the literature (for example [3] and [8]) the integral extensions considered are obtained by adjoining square roots. In this case we have $\alpha(x) = x^2 - a_0$ and then $R_\alpha(b_0 + b_1 x) = b_0^2 - a_0 b_1^2$.

Theorem 2.1. *Let A be a commutative unital Banach algebra with dense invertible group and $\alpha(x)$ a monic polynomial over A. Then $G(A_\alpha)$ is dense in A_α.*

Proof. We may assume $n \geq 2$. Let $\beta(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1} \in A[x]$ and $\varepsilon > 0$. We have to show that there exists $\tilde{\beta}(x) \in A[x]$ with $\left\|\tilde{\beta}(\bar{x}) - \beta(\bar{x})\right\| < \varepsilon$ and $\tilde{\beta}(\bar{x}) \in G(A_\alpha)$. In fact we shall show that by slightly perturbing b_0 only, we can obtain a polynomial $\tilde{\beta}(x)$ with $R_\alpha(\tilde{\beta}(x)) \in G(A)$.

By the above comments, $P(c) = R_\alpha(c + b_1 x + \cdots + b_{n-1}x^{n-1})$ is a polynomial $p_0 + \cdots + p_{n-1}c^{n-1} + c^n$ where $p_0, \ldots, p_{n-1} \in A$ are independent of c. Consider the n formal derivatives of P as maps $A \to A$:

$$\begin{aligned} P^{(0)}(c) &= P(c), \\ P'(c) &= p_1 + 2p_2 c + \cdots + nc^{n-1}, \\ &\vdots \\ P^{(n-1)}(c) &= n!c. \end{aligned}$$

Set $\tilde{c}_{n-1} = b_0$. Note that, trivially, $P^{(n-1)}$ is a local homeomorphism at $\tilde{c}_{n-1}$. Now let $1 \leq k < n$ and suppose that $\tilde{c}_{n-1}, \ldots, \tilde{c}_{n-k} \in A$ have been chosen so that

(i) $P^{(n-j)}$ is a local homeomorphism at $\tilde{c}_{n-j}$ $(1 \leq j \leq k)$, and
(ii) $\left\|\tilde{c}_{n-j} - \tilde{c}_{n-(j-1)}\right\| < \varepsilon/n$ for $1 < j \leq k$.

We shall now show how to choose $\tilde{c}_{n-(k+1)}$ so that (i) and (ii) become true with 'k' replaced by '$k+1$'.

It is easy to see from the inverse function theorem for Banach spaces (see for example [1], Chapter 7, Theorem 8) that $P^{(n-(k+1))}$ is a local homeomorphism at $a \in A$ if $P^{(n-k)}(a)$ is invertible. (This fact is also stated in [7].)

By hypothesis, $P^{(n-k)}$ is a local homeomorphism at $\tilde{c}_{n-k}$ so some open neighbourhood of $\tilde{c}_{n-k}$ is mapped onto an open set in A. Since $G(A)$ is dense in A there is some $\tilde{c}_{n-(k+1)} \in B_A(\tilde{c}_{n-k}, \varepsilon/n)$ with $P^{(n-k)}(\tilde{c}_{n-(k+1)}) \in G(A)$.

Thus $\tilde{c}_{n-(k+1)}$ has the required properties and by induction we can choose $\tilde{c}_{n-1}, \ldots, \tilde{c}_0 \in A$ with $\|\tilde{c}_k - \tilde{c}_{k-1}\| < \varepsilon/n$ $(k = 1, \ldots, n-1)$, $\tilde{c}_{n-1} = b_0$, and P a local homeomorphism at $\tilde{c}_0$.

Again, since P is a local homeomorphism at $\tilde{c}_0$, we can find $\tilde{b}_0 \in B_A(\tilde{c}_0, \varepsilon/n)$ with $P(\tilde{b}_0) \in G(A)$. Since

$$\left\|\tilde{b}_0 - b_0\right\| \leq \left\|\tilde{b}_0 - \tilde{c}_0\right\| + \|\tilde{c}_0 - \tilde{c}_1\| + \cdots + \|\tilde{c}_{n-2} - \tilde{c}_{n-1}\| < \varepsilon,$$

the result is proved. □

We also remark that the method of proving Theorem 2.1 gives another way to see that if A is a commutative unital Banach algebra with $\overline{G(A)} = A$, then for every $n \in \mathbb{N}$, $M_n(A)$ also has dense invertible group. In fact we can approximate any $B = [b_{ij}] \in M_n(A)$ by an invertible matrix by perturbing only n entries $b_{i_1,j_1}, \ldots, b_{i_n,j_n}$ provided that $k \mapsto i_k, j_k$ are both permutations. We leave the details to the reader.

Corollary 2.2. *Let A and B be commutative normed algebras and suppose that B is an integral extension of A. Suppose that A is a full subalgebra of its completion, $\widetilde{A}$, and that A has dense invertible group. Then B has dense invertible group.*

Proof. It is sufficient to prove the case when B is an Arens-Hoffman extension of A, for if C is a normed integral extension of A and $c \in C$ is a root of the monic polynomial $\alpha(x) \in A[x]$, then there is a continuous unital homomorphism $\theta : A_\alpha \to C$ with $\theta(\bar{x}) = c$; the result quickly follows from this.

Let $b \in A_\alpha$ and let $\varepsilon > 0$. It is not hard to show that the universal property of Arens-Hoffman extensions which has just been mentioned allows us to identify $(\tilde{A})_\alpha$ with $\widetilde{A_\alpha}$. By Theorem 2.1 there exists $b' \in G((\widetilde{A})_\alpha)$ with $\|b' - b\| < \varepsilon/2$. Suppose $b' = \sum_{k=0}^{n-1} b'_k \bar{x}^k$ where n is the degree of $\alpha(x)$ and $b'_0, \ldots, b'_{n-1} \in \widetilde{A}$. Since $G((\widetilde{A})_\alpha)$ is open (the Arens-Hoffman extension of a Banach algebra is complete) we can perturb $b'_0, \ldots, b'_{n-1}$ to obtain $b''_0, \ldots, b''_{n-1} \in A$ so that $b'' = \sum_{k=0}^{n-1} b''_k \bar{x}^k$ is invertible in $(\widetilde{A})_\alpha$ and $\|b'' - b'\| < \varepsilon/2$. But now $R_\alpha(b'') \in G(\widetilde{A}) \cap A = G(A)$ so b'' is invertible in A_α and $\|b'' - b\| < \varepsilon$. □

The following theorem follows directly from this corollary:

Theorem 2.3. *Let B be a commutative Banach algebra which is an integral extension of the commutative Banach algebra A. If A has dense invertible group, then so has B.*

The converse of Theorem 2.1 seems harder to investigate. However the method of the proof gives at least partial information in this direction:

Proposition 2.4. *Suppose that A is a commutative unital Banach algebra and $\alpha(x)$ is a monic polynomial of degree n over A. If A_α has dense invertible group, then $\{b \in A :$ there exists $a \in A$ such that $b = a^n\}$, the set of nth powers, is contained in the closure of $G(A)$.*

Proof. Fix $a \in A$ and let $\varepsilon > 0$. Let the norm parameter for the Arens-Hoffman extension be $t > 0$. Then there exists $\beta_\varepsilon(x) = b_{\varepsilon,0} + \cdots + b_{\varepsilon,n-1}x^{n-1} \in A[x]$ such that $R_\alpha(\beta_\varepsilon(x)) \in G(A)$ and

$$\|\beta_\varepsilon(x) - a\| = \|b_{\varepsilon,0} - a\| + \sum_{j=1}^{n-1} \|b_{\varepsilon,j}\| \, t^j < \varepsilon.$$

Now as we mentioned before, $R_\alpha(b_0 + \cdots + b_{n-1}x^{n-1})$ is homogeneous of degree n in $b_0, \ldots, b_{n-1}$. Therefore, writing $R_\alpha(\beta(x)) = P(b_0) = \sum_{j=0}^{n-1} p_j b_0^j + b_0^n$ as in Theorem 2.1, we have that each coefficient p_j $(j = 0, \ldots, n-1)$ is a sum of elements of A each of which has b_k as a factor for some $k \in \{1, \ldots, n-1\}$.

Thus, letting $\varepsilon \to 0$, we obtain invertible elements $R_\alpha(\beta_\varepsilon(x))$ in A which tend to a^n. □

It is clear that the property of having a dense invertible group passes to quotients and completions of normed algebras. The following is also clear.

Proposition 2.5. *Let A be a direct limit of normed algebras with dense invertible groups and where the connecting homomorphisms are unital isometric monomorphisms. Then A has dense invertible group.*

Proof. This is clear. □

We mention the consequences of these results for some of the examples of uniform algebras constructed by Lindberg, Karahanjan, Cole, and Feinstein ([19], [17], [3], [8]). Specifically we are referring to those obtained by taking Banach algebra direct limits of systems of uniform algebras in which the intermediate algebras are completions (or quotients followed by completions) of integral extensions of their predecessors. In many of these the initial algebra has dense invertible group and so the final algebra also has this property. In particular our results show that there is no need to invoke the theory of 'dense thin systems' as developed in [17].

3. Open questions

1. The question of whether or not there is more than one natural uniform algebra on the unit interval is now about fifty years old. It seems to have been first formally asked by Gelfand in 1957 ([10]). The answer is not known even if we assume that the algebra is regular. See [9] and the references cited there for material on this problem.

2. Let A be a commutative Banach algebra with maximal ideal space Ω and Šilov boundary S. A lack of examples describing $\overline{G(A)}$ leads to the following conjectures:

(a) If $\overline{G(A)} = A$, then $S = \Omega$.
(b) If $\overline{G(A)} = A$, then $C(\Omega)$ has dense invertible group.
(c) If $S = \Omega$ and the invertible elements of $C(\Omega)$ are dense in $C(\Omega)$, then $\overline{G(A)} = A$.

It follows from standard properties about the maximal ideal space and Šilov boundary of the uniform closure of a Banach function algebra (see for example [6], p. 447) that we may assume A is a uniform algebra in questions 2(a) and 2(b). The answer to 2(c) may depend on the category.

3. The converse of Theorem 2.1 remains open.

4. We do not know if the word 'full' is redundant in Corollary 2.2.

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