A NOTE ON THE IMBEDDING THEOREM OF BROWDER AND TON

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(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. The imbedding theorem of Browder and Ton states that for any real separable Banach space $X$ there exist a real separable Hilbert space $H$ and a compact linear injection $\psi : H \rightarrow X$ such that $\psi(H)$ is dense in $X$. We shall give a short and elementary new proof to this result. We also briefly discuss the corresponding result without the completeness assumption.

1. Introduction

The imbedding theorem of Browder and Ton [7] can be viewed as an abstract version of classical imbedding theorems familiar in the context of function spaces. Indeed, let

$$W^{m,p}(\Omega) = \{ u | D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq m \},$$

where $\Omega$ is a bounded open set in $\mathbb{R}^n$ satisfying the uniform cone condition. If $1 \leq p < \infty$, $k - m \geq 1$ and

$$\frac{1}{p} > \frac{1}{2} - \frac{k - m}{n},$$

then by the Sobolev imbedding theorem (see [1], for instance) the natural injection $i : W^{k,2}(\Omega) \rightarrow W^{m,p}(\Omega)$ is a compact linear map having a dense range in $W^{m,p}(\Omega)$.

In 1968 F. Browder and B.A. Ton proved the abstract version of the imbedding theorem. It states, on a purely abstract level, that for any real separable Banach space $X$ there exist a real separable Hilbert space $H$ and a compact linear injection $\psi : H \rightarrow X$ such that $\psi(H)$ is dense in $X$.

In their original paper, Browder and Ton used the imbedding theorem to obtain the so-called ‘elliptic super-regularization’ for operators from $X$ into the dual space $X^*$. Their approach is a generalization of the method of elliptic regularization used by Lions, Nirenberg and others (see the references given in [7]). The idea is to replace a given nonlinear elliptic equation by a mildly nonlinear elliptic equation of higher order, in which the nonlinear term is considered as a perturbation. A similar idea is later used for instance in [3], [2], [4], [5], [6], [8], [10] and [11].

The original proof of the imbedding theorem in [7] is quite lengthy. A shorter version based on the same reasoning can be found in [9]. We give a short and elementary new proof. Let $X$ be a real separable Banach space and $S = \{ v_1, v_2, \ldots \}$ an infinite set of linearly independent vectors such that $\|v_k\|_X = 1$ for all $k \in \mathbb{Z}_+$.

Received by the editors May 30, 2002.

2000 Mathematics Subject Classification. Primary 47H05, 78M99.

Key words and phrases. Compact imbedding.
and \( \text{sp} \, S \) is dense in \( X \). Taking into account a suitably restricted set of infinite linear combinations of vectors of \( S \) we find a linear space \( V \) such that \( \text{sp} \, S \subset V \subset X \) and \( V \) can be naturally identified with a compact injective image of a closed subspace of \( l^2 \).

Actually, we shall give a variant of the imbedding theorem without the completeness assumption. The imbedding theorem of Browder and Ton is then obtained as a corollary.

2. THE RESULT

Let \( X \) be a real normed space and \( \tilde{X} \) the essentially unique completion of \( X \). The norm in \( \tilde{X} \) is denoted by \( \| \cdot \|_{\tilde{X}} \) and \( \| x \|_{\tilde{X}} = \| x \|_X \) whenever \( x \in X \).

**Theorem 2.1.** Let \( X \) be a normed space and \( S \subset X \) a countable subset. Then there exist a separable Hilbert space \( H \) and a compact linear injection \( \psi : H \to \tilde{X} \) such that \( \text{sp} \, S \subset \psi(H) \cap X \).

**Proof.** Without loss of generality we can assume that \( S = \{v_1, v_2, \ldots\} \) is an infinite set of linearly independent vectors such that \( \| v_k \|_X = 1 \) for all \( k \in \mathbb{Z}_+ \). Let \( (a_k)_{k=1}^{\infty} \) be a real sequence such that \( (a_k)_{k=1}^{\infty} \in l^2 \). Then the series

\[
\sum_{k=1}^{\infty} \frac{a_k}{k} v_k
\]

converges in \( \tilde{X} \). Indeed, denoting \( s_n = \sum_{k=1}^{n} \frac{a_k}{k} v_k \) we have

\[
\| s_{n+p} - s_n \|_X \leq \sum_{k=n+1}^{n+p} \frac{|a_k|}{k} \leq \sqrt{\sum_{k=n+1}^{n+p} \frac{1}{k^2}} \sqrt{\sum_{k=n+1}^{n+p} |a_k|^2}
\]

for all \( n \in \mathbb{Z}_+ \) and \( p = 1, 2, 3, \ldots \). Hence \( (s_n) \) is a Cauchy sequence in \( X \) and it converges in \( \tilde{X} \). Note that the representation \( u = \sum_{k=1}^{\infty} \frac{a_k}{k} v_k \), \( (a_k)_{k=1}^{\infty} \in l^2 \), is not necessarily unique. Define the map \( i : l^2 \to \tilde{X} \) by setting

\[
i(\tilde{a}) = \sum_{k=1}^{\infty} \frac{a_k}{k} v_k
\]

for all \( \tilde{a} = (a_k)_{k=1}^{\infty} \in l^2 \). Then it is easy to see that \( i \) is linear and by the estimate \( (2.1) \) (with the usual convention that the sum over an empty set is zero)

\[
\| i(\tilde{a}) \|_{\tilde{X}} \leq c_0 \| \tilde{a} \|_{l^2},
\]

where \( c_0 = \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \). Hence the mapping \( i : l^2 \to \tilde{X} \) is continuous. Moreover, the map \( i \) is compact since it is a uniform limit of operators having finite dimensional range. Indeed, denoting \( i_n(\tilde{a}) = \sum_{k=1}^{n} \frac{a_k}{k} v_k \) we get by \( (2.1) \)

\[
\| i(\tilde{a}) - i_n(\tilde{a}) \|_{\tilde{X}} \leq \sqrt{\sum_{k=n+1}^{\infty} \frac{1}{k^2}} \sqrt{\sum_{k=n+1}^{\infty} |a_k|^2}.
\]

Thus

\[
\| i - i_n \| = \sup_{\| \tilde{a} \|_{l^2} = 1} \| i(\tilde{a}) - i_n(\tilde{a}) \|_{\tilde{X}} \leq \sqrt{\sum_{k=n+1}^{\infty} \frac{1}{k^2}},
\]
proving the assertion. Denote $W_0 = \text{Ker}(i)$, which is a closed linear subspace of $l^2$.

Define a real separable Hilbert space $H$ by setting

$$H = W_0^\perp = \{ \tilde{a} \in l^2 | \tilde{a} \perp W_0 \}. $$

Then $H \cong l^2/W_0 = l^2/\text{Ker}(i)$ and consequently the map $\psi = i|_H : H \to \tilde{X}$ is a linear compact injection. Moreover, denoting by $P : l^2 \to H$ the orthonormal projection, we have $i(\tilde{e}_j) = \psi(P\tilde{e}_j) = v_j/j \in X$ for all $j \in \mathbb{Z}_+$, where $\tilde{e}_j = (\delta_{j,k})_{k=1}^\infty$. Clearly the subset $S_H := \psi^{-1}(S)$ of $H$ is countable and $\psi(S_H) = \psi(S)$. Hence $\text{sp} \ S \subset \psi(H) \cap X$, completing the proof.

**Corollary 2.2.** Let $X$ be a real separable space. Then there exist a separable Hilbert space $H$ and a compact linear injection $\psi : H \to \tilde{X}$ such that $\psi(H) \cap X$ is dense in $X$.

*Proof.* Let $S = \{ v_1, v_2, \ldots \}$ be an infinite set of linearly independent vectors such that $\|v_k\| = 1$ for all $k \in \mathbb{Z}_+$ and $\text{sp} \ S$ is dense in $X$. Clearly $\text{sp} \ S$ is also dense in $\tilde{X}$. Thus by Theorem 2.1 there exist a real separable Hilbert space $H$ and a linear compact injection $\psi : H \to \tilde{X}$ such that $\text{sp} \ S \subset \psi(H) \cap X$, completing the proof.

**Corollary 2.3** (Imbedding Theorem of Browder and Ton). Let $X$ be a real separable Banach space. Then there exist a separable Hilbert space $H$ and a compact linear injection $\psi : H \to X$ such that $\psi(H)$ is dense in $X$.

*Proof.* Now $X = \tilde{X}$ and the conclusion follows from Corollary 2.2.

We close this note with a few remarks, which may clarify the reasoning. Let $X$ be a real separable Banach space and let $S = \{ v_1, v_2, \ldots \}$ be an infinite set of linearly independent vectors such that $\|v_k\| = 1$ for all $k \in \mathbb{Z}_+$ and $\text{sp} \ S$ is dense in $X$. In view of the proofs above it is relevant to define a linear subspace of $X$ by setting

$$V = \{ u \in X | u = \sum_{k=1}^{\infty} \frac{a_k}{k} v_k, (a_k)_{k=1}^{\infty} \in l^2 \}. $$

Clearly $\text{sp} \ S \subset V \subset X$ and hence $V$ is dense in $X$. Identifying any pair of sequences $(a_k)_{k=1}^{\infty} \in l^2$ and $(b_k)_{k=1}^{\infty} \in l^2$ such that $\sum_{k=1}^{\infty} \frac{a_k}{k} v_k = \sum_{k=1}^{\infty} \frac{b_k}{k} v_k$ in $X$, gives the quotient space identified with a closed subspace $H$ of $l^2$ needed in Corollary 2.3.

**References**


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