DIVERGENT CESÀRO AND RIESZ MEANS OF JACOBI AND LAGUERRE EXPANSIONS

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Abstract. We show that for $\delta$ below certain critical indices there are functions whose Jacobi or Laguerre expansions have almost everywhere divergent Cesàro and Riesz means of order $\delta$.

1. Introduction

1.1. Orthogonal expansions. Suppose that $(X, \mu)$ is a positive measure space, $(\varphi_n)_{n=0}^{\infty}$ is an orthogonal subset of $L^2(X, \mu)$, and $h_n = \|\varphi_n\|_2^2$ for all $n \geq 0$. If $f$ is a function on $X$ for which all the products $f\varphi_n$ are $\mu$-integrable, then $f$ has an orthogonal expansion

$$
\sum_{n=0}^{\infty} c_n(f)h_n^{-1}\varphi_n(x)
$$

with coefficients

$$
c_n(f) = \int_X f(x)\varphi_n(x)\,d\mu(x), \quad \forall n \geq 0.
$$

1.2. Cesàro means. As described in Zygmund’s book [16, pp. 76–77], the Cesàro means of order $\delta$ of the expansion (1) are defined by

$$
\sigma^\delta_N f(x) = \sum_{n=0}^{N} A^\delta_{N-n} A^\delta_N c_n(f)h_n^{-1}\varphi_n(x),
$$

where $A^\delta_n = \binom{n+\delta}{n}$. Theorem 3.1.22 in [16] says that if the Cesàro means converge, then the terms of the series have controlled growth.

Lemma 1.1. Suppose that $\lim_{N \to \infty} \sigma^\delta_N f(x)$ exists for some $x \in X$ and $\delta > -1$. Then

$$
|c_N(f)h_N^{-1}\varphi_N(x)| \leq C_\delta N^\delta \max_{0 \leq n \leq N} |\sigma^\delta_n f(x)|, \quad \forall N \geq 0.
$$
1.3. **Riesz means.** Hardy and Riesz [6] had proved a similar result for Riesz means. Recall that the Riesz means of order $\delta \geq 0$ are defined for each $r > 0$ by

$$S_r^\delta f(x) = \sum_{0 \leq n < r} \left(1 - \frac{n}{r}\right)^\delta c_n(f) h_n^{-1} \varphi_n(x).$$

Theorem 21 of [6] tells us how the convergence of $S_r^\delta f(x)$ controls the size of the partial sums $S_r^0 f(x)$.

**Lemma 1.2.** Suppose that $f$ has an orthogonal expansion and for some $\delta > 0$ and $x \in X$ its Riesz means $S_r^\delta f(x)$ converges to $c$ as $r \to \infty$. Then

$$|S_r^\delta f(x) - c| \leq A_\delta r^\delta \sup_{0 < t \leq r+1} |S_t^0 f(x)|.$$

Note that we can write

$$c_n(f) h_n^{-1} \varphi_n(x) = (S_n^0 f(x) - c) - (S_n^0 f(x) - c) = O(n^\delta)$$

and obtain the same growth estimates as in Lemma 1.1.

Gergen [5] wrote formulae relating the Riesz and Cesàro means of order $\delta \geq 0$, from which the equivalence of the two methods of summation follows.

1.4. **Uniform boundedness.** Suppose there is a $1 < q \leq \infty$ for which $\varphi_n \in L^q(X, \mu)$ for all $n$. In addition, suppose that there is some positive number $\lambda$ with

$$\|\varphi_n\|_q \geq cn^\lambda, \quad \forall n \geq 1.$$

The formation of the coefficient $f \mapsto c_n(f)$ is then a bounded linear functional on the dual of $L^q(X, \mu)$ with norm bounded below by a constant multiple of $n^\lambda$. The uniform boundedness principle implies that for $p$ conjugate to $q$ and each $0 \leq \varepsilon < \lambda$ there is an $f \in L^p(X, \mu)$ so that

$$c_n(f)/n^\varepsilon \to \infty \text{ as } n \to \infty.$$

1.5. **Cantor-Lebesgue Theorem.** The following argument is based on [16] Section IX.1. Suppose we have a sequence of functions $F_n$ on an interval in the real line with the asymptotic property

$$F_n(\theta) = c_n(\cos(M_n \theta + \gamma_n) + o(1)), \quad \forall n \geq 0,$$

uniformly on a set $E$ of finite positive measure, and with $M_n \to \infty$ as $n \to \infty$. Integrating $|F_n|^2$ over $E$ gives

$$\int_E |F_n(\theta)|^2 \, d\theta = |c_n|^2 \left( \int_E \cos^2(M_n \theta + \gamma_n) \, d\theta + o(1) \right) = |c_n|^2 \left( \frac{|E|}{2} + \frac{e^{2i\gamma_n}}{4} \chi_E(2M_n) + \frac{e^{-2i\gamma_n}}{4} \chi_E(-2M_n) + o(1) \right).$$

The Riemann-Lebesgue Theorem [16] Thm. II.4.4] says that the Fourier transforms $\hat{\chi}_E(\pm 2M_n) \to 0$ as $M_n \to \infty$. If we know that there is some function $G$ for which $|F_n(\theta)| \leq G(n)$ uniformly on $E$ for all $n$, then there is an $n_0 > 0$ for which

$$\frac{|E|}{4} |c_n|^2 \leq \int_E |F_n(\theta)|^2 \, d\theta \leq G(n)^2 |E|, \quad \forall n \geq n_0.$$

This shows that $|c_n| \leq 2G(n)$ for all $n \geq n_0$. 

2. Jacobi polynomials

2.1. Notation. Fix real numbers \( \alpha \geq \beta \geq -1/2 \), with \( \alpha > -1/2 \), and let \( \mu \) denote the measure on \([-1, 1]\) defined by
\[
d\mu(x) = (1 - x)^\alpha (1 + x)^\beta \, dx.
\]
Let \( P_n^{(\alpha, \beta)}(x) \) be the Jacobi polynomial of degree \( n \) associated to the pair \((\alpha, \beta)\) as in Szegö’s book [15]. Then \( \left( P_n^{(\alpha, \beta)} \right)_{n=0}^\infty \) is an orthogonal subset of \( L^2([-1, 1], \mu) \).

Equation (43.3) in [15] shows that the normalization terms \( h_n^{(\alpha, \beta)} = \| P_n^{(\alpha, \beta)} \|_2^2 \) satisfy
\[
h_n^{(\alpha, \beta)} \sim c_{\alpha, \beta} n^{-1} \quad \text{as} \quad n \to \infty.
\]
The Jacobi polynomial expansion of \( f \in L^1(\mu) \) is
\[
\sum_{n=0}^\infty c_n(f) \left( h_n^{(\alpha, \beta)} \right)^{-1} P_n^{(\alpha, \beta)}(x),
\]
with coefficients \( c_n(f) = \int_{-1}^1 f(x) P_n^{(\alpha, \beta)}(x) \, d\mu(x) \). We take \( \alpha \) and \( \beta \) as fixed and use \( \sigma_n^\alpha f(x) \) and \( \sigma_n^\beta f(x) \) to denote the Cesàro and Riesz means of this expansion, respectively.


Lemma 2.1. For \( \alpha \geq \beta \geq -1/2 \) and \( \varepsilon > 0 \) the following estimate holds uniformly for all \( \varepsilon \leq \theta \leq \pi - \varepsilon \) and \( n \geq 1 \):
\[
P_n^{(\alpha, \beta)}(\cos \theta) = n^{-1/2} k(\theta) \cos (M_n \theta + \gamma) + O \left(n^{-3/2}\right).
\]
Here \( k(\theta) = \pi^{-1/2} \left( \sin(\theta/2) \right)^{\alpha-1/2} \left( \cos(\theta/2) \right)^{-\beta-1/2} \), \( M_n = n + (\alpha + \beta + 1)/2 \), and \( \gamma = -(\alpha + 1/2) \pi/2 \).

From Egoroff’s theorem and Lemma [14] we can see that if \( \sigma_n^\alpha f(x) \) converges on a set of positive measure in \([-1, 1]\), then there is a set of positive measure \( E \) on which
\[
\left| c_n n^{(1/2)-\delta} \left( \cos (M_n \theta + \gamma) + O(n^{-1}) \right) \right| \leq A
\]
uniformly for \( \cos \theta \in E \). The argument of subsection [1.5] shows that
\[
\left| c_n n^{(1/2)-\delta} \right| \leq A, \quad \forall n \geq 1.
\]

2.3. Norm estimates. Next we recall the calculation of Lebesgue norms of Jacobi polynomials, according to Markett [10] and Dreseler and Soardi [4]. Equation (2.2) in [10] gives the following lower bounds on these norms.

Lemma 2.2. For real numbers \( \alpha \geq \beta \geq -1/2 \), with \( \alpha > -1/2 \), \( 1 \leq q < \infty \), and \( r > -1/q \),
\[
\left( \int_0^1 \left| P_n^{(\alpha, \beta)}(x) (1-x)^r \right|^q \, dx \right)^{1/q} \sim \begin{cases} n^{-1/2} & \text{if } r > \alpha/2 + 1/4 - 1/q, \\ n^{-1/2} (\log n)^{1/q} & \text{if } r = \alpha/2 + 1/4 - 1/q, \\ n^{\alpha - 2r - 2/q} & \text{if } r < \alpha/2 + 1/4 - 1/q. \end{cases}
\]
2.4. Main result. There are critical indices, as used in [11],
\[ p_c = \frac{4(\alpha + 1)}{2\alpha + 3} \quad \text{and its conjugate} \quad p'_c = \frac{4(\alpha + 1)}{2\alpha + 1}. \]
Taking \( r = \alpha/q \) in Lemma 2.2 we have that
\[ n^{\alpha - 2\alpha/q - 2} \]
for \( \alpha/q < \alpha/2 + 1/4 - 1/q \). This last inequality can be rewritten as
\[ q > \frac{4(\alpha + 1)}{\alpha + 1} = p'_c. \]
We can now prove that below the critical index there are functions with almost everywhere divergent Cesaro and Riesz means.

**Theorem 2.3.** For real numbers \( \alpha \geq \beta \geq -1/2 \), with \( \alpha > -1/2 \),
\[ 1 \leq p < p_c = \frac{4(\alpha + 1)}{2\alpha + 3}, \quad \text{and} \quad 0 \leq \delta < \frac{(2\alpha + 2)}{p} - \frac{(2\alpha + 3)}{2}, \]
there is an \( f \in L^p(\mu) \), supported in \([0, 1]\), whose Cesaro means \( \sigma_{N}^\delta f(x) \) and Riesz means \( S_{\delta}^\mu f(x) \) are divergent almost everywhere on \([-1, 1]\).

**Proof.** Let \( q \) be conjugate to \( p \), so that \( 1/p = (q - 1)/q \). Suppose that
\[ \delta < \frac{(2\alpha + 2)}{p} \quad \text{and} \quad 0 \leq \frac{(2\alpha + 3)}{2} \]
Then
\[ \delta + \frac{1}{2} < \frac{2\alpha + 2}{p} - \frac{2\alpha - 2}{q} \]
which is the exponent of \( n \) in the inequality (10). Now apply the argument given in subsection 1.4. The norms of the Jacobi polynomials in Lemma 2.2 are calculated over \([0, 1]\) and so we can find \( f \) in \( L^p([-1, 1], \mu) \), supported on \([0, 1]\), for which the coefficients satisfy
\[ c_n(f)/n^{\delta - 1/2} \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty. \]
Combine this with Lemmas 1.1 and 1.2 and the argument around inequality (9) to see that for this \( f \) both \( \sigma_{N}^\delta f(x) \) and \( S_{\delta}^\mu f(x) \) are divergent almost everywhere. This argument follows the methods used in [11, 8, 7, 12].

2.5. Remarks. Convergence results above the critical index are contained in the work of Bonami and Clerc [1], Colzani, Taibleson and Weiss [3], and Chanillo and Muckenhoupt [2]. In particular, in [2, Thm. 1.4] it is shown that for
\[ 1 \leq p < p_c = \frac{4(\alpha + 1)}{2\alpha + 3} \quad \text{and} \quad \delta = \frac{(2\alpha + 2)}{p} - \frac{(2\alpha + 3)}{2}, \]
the maximal operator \( f \mapsto \sup_{N \geq 0} |\sigma_{N}^\delta f(x)| \) is of weak type \((p, p)\).

For \( \delta = 0 \) and \( p = p_c \), divergence was proved in [11].

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3. Laguerre Functions

3.1. Notation. For each \( \alpha > -1 \) let \( \mu_\alpha \) be the measure on \([0, \infty)\) defined by
\[
d\mu_\alpha(x) = e^{-x}x^\alpha \, dx.
\]
We denote by \( L_n^{(\alpha)}(x) \) the Laguerre polynomial of degree \( n \), as in [15, Chpt. 5].
The \( L^2(\mu_\alpha) \)-norms of these satisfy the identity
\[
h_n^{(\alpha)} = \| L_n^{(\alpha)} \|_{L^2(\mu_\alpha)}^2 = \Gamma(\alpha + 1) \binom{n + \alpha}{n} \sim n^\alpha.
\]
Fejér's formula [15, Thm. 8.22.1] gives the asymptotic properties of these polynomials. For each \( \alpha > 1 \) and \( 0 < \varepsilon < \omega \),
\[
L_n^{(\alpha)}(x) = \frac{e^{x/2}}{\pi^{1/2}x^{\alpha/2}} n^{\alpha/2-1/4} \cos \left(2(n\varepsilon)^{1/2} - \alpha\pi/2 - \pi/4\right) + O\left(n^{\alpha/2-3/4}\right),
\]
uniformly in \( x \in [\varepsilon, \omega] \). The corresponding normalized functions are
\[
L_n^{(\alpha)}(x) = \sqrt{\frac{(n+1)}{\Gamma(n+\alpha+1)}} e^{-x/2}x^{\alpha/2} L_n^{(\alpha)}(x), \quad \forall x \geq 0, n \geq 0.
\]
These provide an orthonormal subset of \( L^2([0, \infty)) \), where the half line carries Lebesgue measure.

3.2. Norm estimates. Markett [9, Lemma 1] has calculated the Lebesgue norms of the Laguerre functions, for \( \alpha > -1/2 \),
\[
\| L_n^{(\alpha)} \|_q \sim \begin{cases} n^{1/q-1/2}, & \forall \ 1 \leq q < 4, \\ n^{-1/4}(|\log n|)^{1/4}, & \text{if } q = 4, \\ n^{-1/q}, & \forall \ 4 < q \leq \infty. 
\end{cases}
\]

3.3. Divergence result.

Theorem 3.1. If \( \alpha > -1/2 \), \( p > 4 \) and \( 0 < \delta < 1/4 - 1/p \), then there is a function \( f \in L^p(0, \infty) \) whose Laguerre expansion
\[
\sum_{n=0}^{\infty} c_n(f) L_n^{(\alpha)}(x)
\]
has Cesàro and Riesz means of order \( \delta \) which diverge almost everywhere.

Proof. Suppose that the expansion \( \sum_{n=0}^{\infty} c_n(f) L_n^{(\alpha)}(x) \) is either Cesàro or Riesz summable of order \( \delta \) on a set of positive measure in \([0, \infty)\). Then Lemma 1.1 or Lemma 1.2 implies that
\[
c_n(f) L_n^{(\alpha)}(x) = O(n^\delta)
\]
on a set of positive measure. When equations (12) and (13) are combined with the argument of subsection 1.5 we find that
\[
c_n(f) = O(n^{\delta+1/4}).
\]
The case when \( \delta = 0 \) is Lemma 2.3 in Stempak’s paper [14]. Suppose that
\[
\frac{1}{q} - \frac{1}{2} > \delta + \frac{1}{4},
\]
so that \( \delta < \frac{1}{2} - \frac{3}{q^2} = \frac{1-3q}{q^2} \). If \( \frac{1}{q} = 1 - \frac{1}{p} \), then this inequality is \( \delta < \frac{1}{2} - \frac{1}{p} \). The argument of subsection 1.4 shows that if \( p > 4 \) and \( \delta < 1/4 - 1/p \), then there is a function \( f \in L^p(0, \infty) \) for which the inequality (16) fails,

\[
c_n(f)/n^{\delta+1/4} \to \infty \quad \text{as} \quad n \to \infty.
\]

The Laguerre expansion of this function has Cesàro and Riesz means of order \( \delta \) which diverge almost everywhere.

### 3.4. Remarks.

There is an extensive treatment of almost everywhere convergence results for Laguerre expansions in [12]. In particular, [13, Thm. 1.20] implies that if \( p > 4 \) and \( \delta \geq 1/4 - 1/p \), then all \( f \in L^p(0, \infty) \) have almost everywhere convergent Cesàro means of order \( \delta \).

### References


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