$p$–HYPERSONALITY IS NOT TRANSLATION–INVARIANT

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Abstract. In this note we provide an example of a semi-hyponormal Hilbert space operator $T$ for which $T - \lambda$ is not $p$–hyponormal for some $\lambda \in \mathbb{C}$ and all $0 < p \leq \frac{1}{2}$.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be $p$–hyponormal if

\[(T^*T)^p - (TT^*)^p \geq 0.\]

If $p = 1$, $T$ is hyponormal and if $p = \frac{1}{2}$, $T$ is called semi-hyponormal. It is well known that $q$–hyponormal operators are $p$–hyponormal for $p \leq q$, by Löwner’s Theorem. Throughout this note we assume $0 < p < 1$. The notion of semi-hyponormality was introduced by D. Xia [Xi] and the notion of $p$–hyponormal operators was introduced by A. Aluthge [Al]. The $p$–hyponormal operators have been studied by many authors (cf. [Al], [AlW], [Ch], [ChH], [ChI], [ChJ]). An operator $T \in B(\mathcal{H})$ is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in \mathcal{H}$. It is well known (cf. [An], [ChJ], [FHM]) that

\[(0.1) \quad p$–hyponormal $\implies$ paranormal.

The $p$–hyponormal operators share many properties with hyponormal operators: for example, if $T$ is $p$–hyponormal, then (cf. [Al], [Ch], [ChH])

(i) $T$ is normaloid (i.e., norm equals spectral radius);
(ii) $T$ is reduced by its eigenspaces;
(iii) if $T$ is invertible, then $T^{-1}$ is also $p$–hyponormal.

However, it has remained open whether $p$–hyponormality is translation–invariant. For a long time researchers have guessed that this is not true. But no concrete example has been found. The purpose of this note is to provide an example which shows that $p$–hyponormality is not translation–invariant.

Concrete examples of non–hyponormal $p$–hyponormal are scarce in the literature before [AlW]. But the following result in [AlW] helps provide an abundancy of examples.

**Theorem 1** ([AlW, Corollary 1]). *If $T \in B(\mathcal{H})$ is $p$–hyponormal and $n$ is a positive integer, then $T^n$ is $\frac{p}{n}$–hyponormal for every $0 < p \leq 1$.**

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To see how Theorem 1 gives rise to many examples of operators which are semi-hyponormal but not hyponormal, recall the definition of Toeplitz operators on the Hardy space $H^2(T)$ of the unit circle $T$. Recall that the Hilbert space $L^2(T)$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(T)$ is the closed linear span of $\{e_n : n = 0, 1, \cdots \}$. An element $f \in L^2(T)$ is said to be analytic if $f \in H^2(T)$, and co-analytic if $f \in L^2(T) \ominus H^2(T)$. If $P$ denotes the orthogonal projection from $L^2(T)$ to $H^2(T)$, then for every $\varphi \in L^\infty(T)$ the operator $T_\varphi$ defined by

$$T_\varphi g := P(\varphi g) \quad (g \in H^2(T))$$

is called the Toeplitz operator with symbol $\varphi$. If $\varphi$ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^N a_n z^n$, where $a_{-m}$ and $a_N$ are nonzero, then $T_\varphi$ is called a trigonometric Toeplitz operator.

The following theorem gives a useful necessary condition for hyponormality and normality of trigonometric Toeplitz operators.

**Theorem 2** (Fal). Suppose that $\varphi$ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^N a_n z^n$, where $a_{-m}$ and $a_N$ are nonzero.

(i) If $T_\varphi$ is hyponormal, then $m < N$ and $|a_{-m}| \leq |a_N|$.  
(ii) If $T_\varphi$ is normal, then $m = N$ and $|a_{-m}| = |a_N|$.

The following theorem together with Theorem 1 gives abundant examples which are semi-hyponormal operators but not hyponormal.

**Theorem 3** ([CuL, Theorem 3.2]). Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic.

For example, if $\varphi$ is a trigonometric polynomial of the form

$$\varphi(z) = \sum_{n=-m}^N a_n z^n \quad (a_{-m} \neq 0 \text{ and } a_N \neq 0)$$

and satisfies

(i) $m < N$,
(ii) $|a_{-m}| = |a_N|$,
(iii) $\begin{pmatrix} a_{-1} & \cdots & a_{-m} \\ \vdots & \ddots & \vdots \\ a_{N-m} & \cdots & a_{N} \end{pmatrix} \begin{pmatrix} N \\ N-m+1 \\ \vdots \\ N-m+2 \end{pmatrix} = a_{-m} \begin{pmatrix} N-m+1 \\ N-m+2 \vdots \vdots \end{pmatrix}$,

then $T_\varphi$ is hyponormal by an argument of [Fal, Theorem 1.4], but by Theorem 3, $T_\varphi^2$ is not hyponormal since $T_\varphi$ is neither analytic nor normal. However by Theorem 1, $T_\varphi^2$ is semi-hyponormal.

We now have

**Theorem 4.** $p$–hyponormality is not translation–invariant; more precisely, there exists an operator $T$ satisfying that $T$ is semi–hyponormal, but $T - \lambda$ is not $p$–hyponormal for some $\lambda \in \mathbb{C}$ and any $p > 0$.

**Proof.** If $U$ is the unilateral shift on $\ell_2$, define

$$S = 4 U^2 + U^{*2} + 2 U U^* + 2.$$  

We now claim that (i) $S$ is semi-hyponormal, and (ii) $S - 4$ is not $p$–hyponormal for any $p > 0$.  

(i) Define $\varphi(z) = 2z + z^{-1}$. Then it is easy to see that $T_\varphi$ is hyponormal, but $T_\varphi^2$ is not hyponormal by Theorem 2. A straightforward calculation shows that $T_\varphi^2 = S$. Thus, by Theorem 1, $S$ is semi–hyponormal.

(ii) To show that $S - 4$ is not $p$–hyponormal for any $p > 0$, in view of (0.1) it suffices to show that $S - 4$ is not paranormal. Indeed,

$$\| (S - 4) e_0 \|^2 = \| (4U^2 + U^{*2} + 2UU^* - 2) e_0 \|^2$$

$$= \| 4e_2 - 2e_0 \|^2 = 20$$

and

$$\| (S - 4)^2 e_0 \| = \| (4U^2 + U^{*2} + 2UU^* - 2)(4e_2 - 2e_0) \|$$

$$= \| 8e_0 - 8e_2 + 16e_4 \| = \sqrt{384},$$

which shows that $\| (S - 4) e_0 \|^2 > \| (S - 4)^2 e_0 \|$. This implies that $S - 4$ is not paranormal.

\[ \square \]

References


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