**DISCRETE SPECTRA OF C*-ALGEBRAS AND COMPLEMENTED SUBMODULES IN HILBERT C*-MODULES**

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Abstract. Let $A$ be a $C^*$-algebra and let $X$ be a full (right) Hilbert $A$-module. It is shown that if the spectrum $\hat{A}$ of $A$ is discrete, then every closed $\mathcal{K}(X)$-$A$-submodule of $X$ is complemented in $X$, and conversely that if $\hat{A}$ is a $T_1$-space and if every closed $\mathcal{K}(X)$-$A$-submodule of $X$ is complemented in $X$, then $\hat{A}$ is discrete.

1. Introduction

Hilbert $C^*$-modules first appeared in the work of Kaplansky [3]. His idea was to generalize Hilbert spaces by allowing the inner product to take values in a $C^*$-algebra rather than in the field of complex numbers. In the sequel, about twenty years later, an outstanding work in which Hilbert $C^*$-modules played a crucial role in the use of them was Rieffel’s work [10] on induced representations of $C^*$-algebras. Now Hilbert $C^*$-modules are getting to become a more important, standard tool in various research areas in $C^*$-algebras, for example, KK-theory and Morita equivalence (see [2] for KK-theory, [7] for the $C^*$-algebraic approach to quantum groups, and [9] for Morita equivalence).

Although a Hilbert $C^*$-module is a generalization of a Hilbert space, we cannot, a priori, expect that Hilbert $C^*$-modules behave like Hilbert spaces, and in some ways they do. In fact, as is mentioned below, there is an essential way in which Hilbert $C^*$-modules differ from Hilbert spaces. Let $X$ be a Hilbert $C^*$-module equipped with inner product $\langle \cdot , \cdot \rangle$ taking values in the $C^*$-algebra and let $Y$ be a closed subspace $Y$ of $X$. We denote by $Y^\perp$ the orthogonally complemented subspace of $Y$ in $X$, i.e.,

$$ Y^\perp = \{ x \in X \mid \langle x , y \rangle = 0 \text{ for all } y \in Y \}. $$

Then $Y^\perp$ is also a closed subspace of $X$, and $(Y^\perp)^\perp$ is usually larger than $Y$. In general, $X$ is not equal to the direct sum $Y \oplus Y^\perp$ of $Y$ and $Y^\perp$. We say that a closed subspace $Y$ of a Hilbert $C^*$-module $X$ is complemented in $X$ if $X$ coincides with $Y \oplus Y^\perp$. Throughout this paper, by a complemented subspace of $X$ we mean one that is orthogonally complemented. Since the whole theory of Hilbert spaces and their linear operators is based on the use of the orthogonally complemented...
subspaces, it is clear that there will be obstacles in developing a theory of Hilbert $C^*$-modules analogous to that of Hilbert spaces. Hence, when it is necessary to obtain an analogous theory which works for Hilbert $C^*$-modules, what we must first know is when every closed submodule of a Hilbert $C^*$-module is complemented. At present, as for related results, there are only a few results on what kind of closed submodule is complemented in a Hilbert $C^*$-module (see, for example, [7, Chapter 3]). As far as we know, however, there are no answers to the problem of when every closed submodule of a Hilbert $C^*$-module is complemented. The aim of this paper is to provide a few sufficiently satisfactory answers to such a problem.

On the other hand, when some property in $C^*$-algebras $A$ and $B$ is preserved under Morita equivalence, it is natural to expect that an $A$-$B$-imprimitivity bimodule $X$ must contain sufficient information on that property of both $A$ and $B$ (for example, [5, Theorem 2.3], [6, Theorem 3.3]). Hence it would be very interesting to attempt to characterize $X$ in terms of some property in both $A$ and $B$. In fact, in connection with the problem mentioned in the preceding paragraph, for a full (right) Hilbert $C^*$-module $X$ over a $C^*$-algebra $A$, we show that if the spectrum $\hat{A}$ of $A$ is discrete in the Jacobson topology, then every closed $\mathcal{K}(X)$-$A$-submodule of $X$ is complemented, and conversely that if $\hat{A}$ is a $T_1$-space and if every closed $\mathcal{K}(X)$-$A$-submodule of $X$ is complemented in $X$, then $\hat{A}$ is discrete in the Jacobson topology.

2. Results

Recall the definition of a Hilbert $C^*$-module. Let $A$ be a $C^*$-algebra. By a left Hilbert $A$-module, we mean a left $A$-module $X$ equipped with an $A$-valued pairing $\langle \cdot , \cdot \rangle$, called an $A$-valued inner product, satisfying the following conditions:

(a) $\langle \cdot , \cdot \rangle$ is sesquilinear. (We make the convention that $\langle \cdot , \cdot \rangle$ is linear in the first variable and is conjugate-linear in the second variable.)

(b) $\langle x , y \rangle = \langle y , x \rangle^*$ for all $x, y \in X$.

(c) $\langle ax , y \rangle = a \langle x , y \rangle$ for all $x, y \in X$ and $a \in A$.

(d) $\langle x , x \rangle \geq 0$ for all $x \in X$, and $\langle x , x \rangle = 0$ implies that $x = 0$.

(e) $X$ is complete with respect to the norm $\|x\| = \|\langle x , x \rangle\|^{\frac{1}{2}}$.

We remark that the Hilbert $A$-module is always assumed to be a vector space over the field of complex numbers. Furthermore, $X$ is said to be full if $X$ satisfies an additional condition:

(f) the closed linear span of \{ $\langle x , y \rangle$ $|$ $x, y \in X$ \} coincides with $A$.

Let $B$ be a $C^*$-algebra. Right Hilbert $B$-modules are defined similarly except that we require that $B$ should act on the right of $X$, that $\langle \cdot , \cdot \rangle$ should be conjugate-linear in the first variable, and that $\langle x , yb \rangle = \langle x , y \rangle b$ for all $x, y \in X$ and $b \in B$.

Let $A$ and $B$ be $C^*$-algebras. We denote by $\lambda \langle \cdot , \cdot \rangle$ the $A$-valued inner product on the left Hilbert $A$-module and by $\langle \cdot , \cdot \rangle_B$ the $B$-valued inner product on the right Hilbert $B$-module, respectively. By an $A$-$B$-imprimitivity bimodule $X$, we mean a full left Hilbert $A$-module and full right Hilbert $B$-module $X$ satisfying

(g) $\lambda \langle xb , y \rangle = \lambda \langle x , yb^* \rangle$ and $\langle ax , y \rangle_B = \langle x , a^*y \rangle_B$ for all $x, y \in X, a \in A$ and $b \in B$;

(h) $\lambda \langle x , y \rangle \cdot z = x \cdot \langle y , z \rangle_B$ for all $x, y, z \in X$.

Throughout this paper, by an $A$-$B$-Hilbert module we mean a left Hilbert $A$-module and right Hilbert $B$-module.
Two $C^*$-algebras $A$ and $B$ are said to be Morita equivalent if there exists an $A$-$B$-imprimitivity bimodule. We remark that in this paper, Morita equivalence means strong Morita equivalence in the sense of Rieffel (cf. [9, Remark 3.15]). The reader is referred to [3] for Hilbert $C^*$-modules and Morita equivalence.

We denote by $\text{Prim}(A)$ the primitive spectrum of $A$, that is, the set of primitive ideals of $A$ equipped with the Jacobson topology, which is defined as the topology for which the class $\{\text{hull}(I) | I \text{ is a closed ideal of } A\}$ form the closed sets, where

$$\text{hull}(I) = \{t \in \text{Prim}(A) | t \supset I\}$$

(see [8, 4.1] for details). The primitive spectrum is always locally compact and is a $T_0$-space. In general, however, it is not necessarily a $T_1$-space. We denote by $\hat{A}$ the spectrum of $A$, that is, the set of (unitary) equivalence classes of nonzero irreducible representations of $A$ equipped with the Jacobson topology, where the Jacobson topology on $\hat{A}$ is defined as the topology for which the canonical map from $\hat{A}$ onto $\text{Prim}(A)$ is open and continuous. We note that $\hat{A}$ is a locally compact space, not necessarily a $T_0$-space. We remark that $\hat{A}$ is a $T_0$-space if and only if the canonical map from $\hat{A}$ onto $\text{Prim}(A)$ is injective. The reader is referred to [1], [8] for the spectrum of a $C^*$-algebra.

**Lemma 2.1.** Let $A$ and $B$ be $C^*$-algebras and let $X$ be an $A$-$B$-imprimitivity bimodule. If a closed $A$-$B$-submodule $Y$ of $X$ is complemented in $X$ with respect to $\langle \cdot , \cdot \rangle^*_A$ and with respect to $\langle \cdot , \cdot \rangle^*_B$ respectively, then we have

$$Y^\perp = \{ x \in X | \langle x , y \rangle = 0 \text{ for all } y \in Y \} = \{ x \in X | \langle x , y \rangle^*_A = 0 \text{ for all } y \in Y \}.$$

**Proof.** By symmetry, it suffices to show that if $\langle x , y \rangle = 0$ for all $y \in Y$, then $\langle x , y \rangle^*_B = 0$. Let $x = y_0 \oplus y'$ for $y_0 \in Y$ and $y' \in X$ with $\langle y' , y \rangle^*_B = 0$ for all $y \in Y$. Then we have only to show $\langle y_0 , y \rangle^*_B = 0$. Since $\langle y' , y \rangle^*_B = 0$, we have

$$\langle y_0 , y \rangle^*_B = \langle y_0 , y \rangle^*_B \cdot \langle y_0 + y' , y \rangle^*_B = \langle y_0 , y \rangle^*_B \cdot \langle x , y \rangle^*_B = \langle x , y_0 \rangle^*_B.$$

Thus we see that $\langle x , y \rangle^*_B = 0$, so that $\langle x , y \rangle_B = (y_0 , y)_B = 0$. \qed

Let $A$ be a $C^*$-algebra and let $X$ be a left Hilbert $A$-module or a right Hilbert $A$-module. When we do not need to specify exactly whether $X$ is a left Hilbert $A$-module or a right one, we shall say that $X$ is a Hilbert $C^*$-module over $A$, or, simply a Hilbert $A$-module.

**Lemma 2.2.** Let $A$ be a $C^*$-algebra and let $X$ be a Hilbert $C^*$-module over $A$. For any subset $Y$ of $X$, $Y^\perp$ is a closed $A$-submodule of $X$.

**Proof.** Without loss of generality, we may assume that $X$ is a left $A$-module with $A$-valued inner product $\langle \cdot , \cdot \rangle$. It follows from continuity of $\langle \cdot , \cdot \rangle$ that $Y^\perp$ is a closed subset, and it is routine to verify that $Y^\perp$ is a vector subspace of $X$. For $a \in A, x \in Y^\perp$ and $y \in Y$, we have $\langle ax , y \rangle = a(x , y) \neq 0$, from which it follows that $AY^\perp \subseteq Y^\perp$. Thus $Y^\perp$ is a left $A$-submodule. \qed

Now we recall the Rieffel correspondence (see [9, Theorem 3.22]). Let two $C^*$-algebras $A$ and $B$ be Morita equivalent and let $X$ be an $A$-$B$-imprimitivity bimodule. We denote by $\mathcal{I}(A)$ (resp. $\mathcal{I}(B)$) the set of all closed (two-sided) ideals of $A$.
(resp. \( B \)), and by \( S(X) \) the set of closed \( A-B \)-submodules of \( X \). Note that \( \mathcal{I}(A) \), \( \mathcal{I}(B) \) and \( S(X) \) can be partially ordered by inclusion, and are then lattices. Then there are natural lattice isomorphisms among \( \mathcal{I}(A) \), \( \mathcal{I}(B) \) and \( S(X) \) given by

\[
\mathcal{I}(A) \ni I \mapsto iX \in S(X), \quad \text{where } iX = \{ y \in X | \langle y, x \rangle \in I \text{ for all } x \in X \}
\]

\[
\mathcal{S}(X) \ni Y \mapsto I_Y \in \mathcal{I}(A), \quad \forall I \in \mathcal{I}(B),
\]

where

\[
I_Y = \text{span}\{ \langle y, x \rangle \mid y \in Y \text{ and } x \in X \}
\]

and

\[
yI = \text{span}\{ \langle x, y \rangle_n \mid y \in Y \text{ and } x \in X \};
\]

\[
\mathcal{I}(B) \ni J \mapsto X_J \in S(X), \quad \text{where } X_J = \{ y \in X | \langle x, y \rangle_n \in J \text{ for all } x \in X \}.
\]

We refer to such lattice isomorphisms among \( \mathcal{I}(A) \), \( \mathcal{I}(B) \) and \( S(X) \) as the Rieffel correspondences (see [9, 3.3]).

**Theorem 2.3.** Let two \( C^* \)-algebras \( A \) and \( B \) be Morita equivalent and let \( X \) be an \( A \)-imprimitivity bimodule. Consider the following conditions:

1. The spectrum \( \widehat{A} \) of \( A \) is discrete in the Jacobson topology.
2. The spectrum \( \widehat{B} \) of \( B \) is discrete in the Jacobson topology.
3. Every closed \( A-B \)-submodule of \( X \) is complemented in \( X \).

Then we have (1) \( \iff \) (2) \( \Rightarrow \) (3). If either \( \widehat{A} \) or \( \widehat{B} \) is a \( T_1 \)-space, then conditions (1) \( \iff \) (2) are equivalent.

**Proof.** (1) \( \iff \) (2). Since \( \widehat{A} \) and \( \widehat{B} \) are homeomorphic by the Rieffel homeomorphism ([9, Corollary 3.33]), the equivalence of (1) and (2) is obvious.

(1) \( \Rightarrow \) (3). Let \( Y \) be any closed \( A-B \)-submodule of \( X \). Put

\[
Y^\perp = \{ x \in X | \langle x, y \rangle = \langle x, y \rangle_n = 0 \text{ for all } y \in Y \}.
\]

Then Lemma 2.2 shows that \( Y^\perp \) is also a closed \( A-B \)-submodule of \( X \). By the Rieffel correspondence ([9, Theorem 3.22]), there corresponds a closed ideal \( I_Y \) of \( A \) to \( Y \). Then condition (1) yields that \( A = I_Y \oplus (I_Y)^\perp \) for some closed ideal \( (I_Y)^\perp \) of \( A \) with \( I_Y = (I_Y)^\perp = \{ 0 \} \). In fact, let \( p \) be an open projection in the second dual \( A^{**} \) with \( A^{**}p \cap A = I_Y \) ([8, 3.11.9]). Since \( \widehat{A} \) is discrete, it hence follows from [4, Theorem 2.3] that \( p \) is a multiplier for \( A \), so that \( Ap = I_Y \). Put \( q = 1 - p \). Then \( q \) is a multiplier for \( A \), and so it is also an open projection ([8, 3.12.9, 3.11.9]). Hence we see that \( Aq = (I_Y)^\perp \).

We claim that \( (I_Y)^\perp = I_{Y^\perp} \). Take any \( \langle y, x \rangle \in I_{Y^\perp} \) with \( y \in Y^\perp \) and \( x \in X \). For any \( x' \in X \) and \( y' \in Y \), \( \langle y', x' \rangle \) belongs to \( I_Y \) ([9, Theorem 3.22]) and

\[
\langle y, y' \rangle = \langle y, y' \rangle_n = 0 \text{ by the definition of } Y^\perp \text{ above. Then we have}
\]

\[
\langle x, y \rangle^* \langle y', x' \rangle = \langle x, y \rangle \langle y', x' \rangle = \langle x, \langle y, y' \rangle_n, x' \rangle = \langle x, y, x' \rangle = 0,
\]

which shows that \( \langle y, x \rangle \in (I_Y)^\perp \). Since \( I_{Y^\perp} = \text{span}\{ \langle y, x \rangle \mid y \in Y^\perp \text{ and } x \in X \} \), we see that \( I_{Y^\perp} \subset (I_Y)^\perp \). Next we must show that \( (I_Y)^\perp \subset I_{Y^\perp} \). Let \( Y' \) be the closed \( A-B \)-submodule of \( X \) corresponding to the closed ideal \( (I_Y)^\perp \) of \( A \) under the Rieffel correspondence. Then we see that \( (I_Y)^\perp = I_{Y'} \) and that \( Y' = \{ y \in X | \langle y, x \rangle \in (I_Y)^\perp \text{ for all } x \in X \} \) ([9, Theorem 3.22]). Thus \( (I_Y)^\perp = I_{Y'} = \text{span}\{ \langle y', x \rangle \mid y' \in X' \text{ and } x \in X \} \). Take any \( y' \in Y' \) and fix it. We assert that \( y' \in Y^\perp \). Take any \( x, z \in X \) and any \( y \in Y \). We then see that
\[ \lambda(y', x) \in I_{Y'} = (I_Y)^\perp \text{ and } \lambda(y, z) \in I_Y. \] We therefore have
\[ 0 = \lambda(y', x)^* \lambda(y, z) = \lambda(x, y')^* \lambda(y', z) \]
\[ = \lambda(x, y') \cdot y, z = \lambda(x, y')_B, z). \]
Since \( z \) is arbitrary in \( X \), we see that \( x \cdot (y', y)_b = 0 \) for all \( x \in X \). Since \( X \) is full, we obtain that \( (y', y)_b = 0 \). On the other hand, since we have \( \lambda(y', y) \in I_Y \cap I_{Y'} = I_Y \cap (I_Y)^\perp = \{0\} \), we obtain that \( \lambda(y', y) = 0 \) for all \( y \in Y \). Hence we conclude that \( y' \subseteq Y^\perp \), so that \( Y' \subseteq Y^\perp \). We thus see that \( I_{Y'} \subseteq I_{Y'}^\perp \), i.e.,
\[ (I_Y)^\perp = I_{Y'}^\perp. \]
Since the Rieffel correspondences are natural lattice isomorphisms among \( \mathcal{I}(A), \mathcal{I}(B) \) and \( S(X) \), a least upper bound \( Y \vee Y^\perp(= Y \oplus Y^\perp) \) of \( Y \) and \( Y^\perp \) corresponds to a least upper bound \( I_Y \vee I_{Y^\perp} = I_Y \vee (I_Y)^\perp = I_Y \oplus (I_Y)^\perp = \alpha \) of \( I_Y \) and \( I_{Y^\perp} \).
Since \( A \) and \( X \) correspond by the Rieffel correspondence, we see that \( X = Y \oplus Y^\perp \).

(3) \( \Rightarrow \) (1). Suppose that \( \tilde{A} \) is a \( T_1 \)-space. Take any open central projection \( p \) in the second dual \( A^{**} \) of \( A \). In order to obtain condition (1), by \([3] \) Theorem 2.3 \it it suffices to show that \( \tilde{A} \) is a multiplier for \( A \). For this, we shall show that \( \tilde{A} \) is also a closed projection in \( A^{**} \) \([3, 3.12.9, 3.11.9, \text{and } 3.11.10]\), hence, equivalently, that \( q = 1 - p \) is open in \( A^{**} \). Put \( I = A^{**} \cap A \) and let \( jX \) be the closed \( A-B \) submodule of \( X \) corresponding to the closed ideal \( I \) of \( A \) under Rieffel correspondence. Thus condition (3) yields that \( X = jX \oplus (jX)^\perp \), and, taking into account Lemma 2.1, Lemma 2.2 shows that \( jX \) is a closed \( A-B \) submodule of \( X \). Then \( (jX)^\perp = jX \) for some closed ideal \( A \) of \( A \) by the Rieffel correspondence, and we see that \( A = I \vee j \) because \( A \) corresponds to \( X \).

We now assert that \( IJ = I \cap J = \{0\} \), that is, \( I \) and \( J \) are mutually orthogonal. In fact, since a greatest lower bound \( I \wedge J \) of \( I \) and \( J \) corresponds to a greatest lower bound \( I_{\wedge}X(= I_{\wedge}^X \wedge jX) \) and since \( I_{\wedge}X \wedge jX = I_X \cap jX = jX \cap (jX)^\perp = \{0\} \), we see that \( I \cap J = I \wedge J = \{0\} \). Thus we obtain that \( A = I \oplus J \). Then we conclude that \( J = A^{**}q \cap A \); hence \( q \) is open. Thus we see that \( p \) is a multiplier for \( A \), from which it follows that \( \tilde{A} \) is discrete.

\[ \square \]

Remark 2.4. (1) In the proof of the implication (1) \( \Rightarrow \) (3) in Theorem 2.3 above, we have seen therein that \( A = I_Y \oplus I_{Y'} \) and \( I_Y \wedge I_{Y'} = \{0\} \), which shows that \( Y \wedge Y' = Y \) and \( Y \wedge Y' = \{0\} \). Then it is not trivial that \( Y \wedge Y' = \{0\} \) immediately implies \( Y' = Y^\perp \). In fact, this implication is an essential part of the proof.

(2) In the implication (3) \( \Rightarrow \) (1) in Theorem 2.3, the assumption that \( \tilde{A} \) or \( \tilde{B} \) be a \( T_1 \)-space is necessary. In addition, in general, condition (3) does not necessarily imply that \( \tilde{A} \) or \( \tilde{B} \) is a \( T_1 \)-space. For example, let \( A \) be a UHF \( C^\ast \) -algebra which is not of type I. Then \( \tilde{A} \) is not a \( T_1 \)-space, hence, not discrete. If it were so, \( \tilde{A} \) and \( \text{Prim}(A) \) must be homeomorphic. Since \( A \) is separable, it must be of type I, which is a contradiction. Let \( X \) be a Hilbert \( A \)-module. Then \( X \) is full because \( A \) is simple. By the Rieffel correspondence, \( X \) does not have any non-trivial closed \( A-B \) submodules because \( A \) is simple. Hence every closed \( A-B \) submodule of \( X \) is complemented in \( X \). But, \( \tilde{A} \) is not a \( T_1 \)-space.

Let \( A \) be a \( C^\ast \)-algebra and let \( X \) be a right Hilbert \( A \)-module with \( A \)-valued inner product \( \langle \cdot, \cdot \rangle \). Define \( (X, X) \) to be the linear span of the set \( \{ \langle x, y \rangle | x, y \in X \} \). We define the linear operator \( \theta_{x,y} \) by \( \theta_{x,y}(z) = x \cdot y, z \) for all \( x, y, z \in X \). We denote by \( K(X) \) the \( C^\ast \)-algebra generated by the set \( \{\theta_{x,y} | x, y \in X \} \) (see \([9] \) Proposition 2.21 and Lemma 2.25)). Now we are in a position to establish the main
result as a corollary to Theorem 2.3. From now on, without loss of generality, we will assume that a Hilbert $A$-module $X$ is a right module.

**Corollary 2.5.** Let $A$ be a $C^*$-algebra and let $X$ be a Hilbert $C^*$-module over $A$. We denote by $I$ the closed ideal of $A$ generated by $\langle X, X \rangle$. Consider the following conditions:

1. The spectrum $\hat{A}$ of $A$ is discrete in the Jacobson topology.
2. The spectrum $\hat{I}$ of $I$ is discrete in the Jacobson topology.
3. Every closed $K(X)$-$I$-submodule of $X$ is complemented in $X$.

Then we have $(1) \implies (2) \implies (3)$. If $\hat{I}$ is a $T_1$-space, then conditions (2) and (3) are equivalent.

**Proof.** $(1) \implies (2)$. Since $I$ is a closed ideal of $A$, $\hat{I}$ is identified with an open subset of $\hat{A}$ ([8, 4.1.11]). Thus $\hat{I}$ is discrete.

$(2) \implies (3)$. Since $I$ is the closed ideal of $A$ generated by $\langle X, X \rangle$, $X$ can be considered as a full Hilbert $C^*$-module over $I$. Then $X$ becomes a $K(X)$-$I$-imprimitivity bimodule (see [9, Proposition 3.8]). Hence condition (3) follows from Theorem 2.3.

Assume that $\hat{I}$ is a $T_1$-space. Then the implication $(3) \implies (2)$ follows from Theorem 2.3.

The use of [4, Theorem 2.3] is essential in the proof of Theorem 2.3. The following theorem can be proved along lines similar to those in the proof of Theorem 2.3 above, using [4, Theorem 2.5] instead of [4, Theorem 2.3]. We will leave the details to the reader.

**Theorem 2.6.** Let two $C^*$-algebras $A$ and $B$ be Morita equivalent and let $X$ be an $A$-$B$-imprimitivity bimodule. Consider the following conditions:

1. The primitive spectrum $\text{Prim}(A)$ of $A$ is discrete in the Jacobson topology.
2. The primitive spectrum $\text{Prim}(B)$ of $B$ is discrete in the Jacobson topology.
3. Every closed $A$-$B$-submodule of $X$ is complemented in $X$.

Then we have $(1) \iff (2) \implies (3)$. If either $\text{Prim}(A)$ or $\text{Prim}(B)$ is a $T_1$-space, then conditions $(1)-(3)$ are equivalent.

**Corollary 2.7.** Let $A$ be a $C^*$-algebra and let $X$ be a Hilbert $C^*$-module over $A$. We denote by $I$ the closed ideal of $A$ generated by $\langle X, X \rangle$. Consider the following conditions:

1. The primitive spectrum $\text{Prim}(A)$ of $A$ is discrete in the Jacobson topology.
2. The primitive spectrum $\text{Prim}(I)$ of $I$ is discrete in the Jacobson topology.
3. Every closed $K(X)$-$I$-submodule of $X$ is complemented in $X$.

Then we have $(1) \implies (2) \implies (3)$. If $\text{Prim}(I)$ is a $T_1$-space, then conditions (2) and (3) are equivalent.

**Proof.** Note that $\text{Prim}(I)$ is an open subset in $\text{Prim}(A)$ ([8, 4.1.10 or 4.1.11]). We assume that $X$ is a right Hilbert $A$-module. Since $X$ is then a $K(X)$-$I$-imprimitivity bimodule, the corollary follows from Theorem 2.6.

**Example 2.8.** If a $C^*$-algebra $A$ is finite-dimensional, then it is isomorphic to the direct sum of some matrix algebras. Hence the spectrum $\hat{A}$ of $A$ is discrete. Let $X$ be a full right Hilbert $A$-module. Then, by Corollary 2.5, every closed $K(X)$-$A$-submodule of $X$ is complemented in $X$. 

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More generally, if $A$ is a dual $C^*$-algebra, then the spectrum $\hat{A}$ of $A$ is discrete. Recall here that a $C^*$-algebra is said to be dual if it is isomorphic to a $C^*$-subalgebra of the $C^*$-algebra of all compact operators on some Hilbert space (cf. [11, 4.7.20]). Note that a separable $C^*$-algebra is dual if and only if its spectrum is discrete in the Jacobson topology ([4, p. 706]). Let $A$ be a dual $C^*$-algebra and let $X$ be a full right Hilbert $A$-module. Then every closed $\mathcal{K}(X)$-$A$-submodule of $X$ is complemented in $X$.

We end by stating a remark. Let $A$ be a $C^*$-algebra. Then $A$ can be regarded as an $A$-$A$-imprimitivity bimodule, for the bimodule structure given by the multiplication in $A$, with $\langle a, b \rangle = ab^*$ and $\langle a, b \rangle_A = a^*b$ for $a, b \in A$ (see [9, Example 3.5]). Hence Theorem 2.3 (resp. Theorem 2.6) generalizes [4, Theorem 2.3] (resp. [4, Theorem 2.5]).

References