THE MINIMUM NUMBER OF ACUTE DIHEDRAL ANGLES OF A SIMPLEX

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Abstract. For any $n$-dimensional simplex $\Omega \subset \mathbb{R}^n$, we confirm a conjecture of Klamkin and Pook (1988) that there are always at least $n$ acute dihedral angles in $\Omega$.

Let $\Omega$ be an $n$-simplex in $\mathbb{R}^n$ with vertices $A_1, A_2, \cdots, A_{n+1}$ (i.e., $\Omega = \langle A_1, A_2, \cdots, A_{n+1} \rangle$), let $\Omega_i = \langle A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_{n+1} \rangle$ denote its facet which lies in a hyperplane $\pi_i$, and let $e_i$ be the unit outer normal vector of $\pi_i$ ($i = 1, 2, \cdots, n+1$). Denote by $\theta_{ij}$ the supplement of the angle between $e_i$ and $e_j$; we call $\theta_{ij}$ the dihedral angle between $i$ and $j$.

Obviously, for any $n$-simplex, there exist $\binom{n+1}{2}$ dihedral angles. In [1], Klamkin shows that there are always at least three acute dihedral angles in any tetrahedron. Further, he posed the question: can one generalize this result to an $n$-simplex? Pook [1] conjectures that $n$ is the required minimum number of acute dihedral angles in any $n$-simplex.

In this paper, we confirm this supposition.

Theorem 1. There exist at least $n$ acute dihedral angles in any $n$-simplex, and there exists an $n$-simplex which has only $n$ acute dihedral angles.

It is clear that Theorem 1 can be replaced equivalently by the following statement.

Theorem 1'. There exist at most $\frac{1}{2}n(n-1)$ obtuse dihedral angles in any $n$-simplex, and there exists an $n$-simplex which has only $\frac{1}{2}n(n-1)$ obtuse dihedral angles.

To prove Theorem 1', we need the following three lemmas.

Lemma 1. Let $\{x_1, x_2, \cdots, x_n\}$ be a given linearly independent set of vectors from $\mathbb{R}^n$. Then there exists an $n$-simplex $\Omega$ which takes $\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \cdots, \frac{x_n}{\|x_n\|}$ as $n$ unit outer normal vectors of facets.

Proof. Since $x_1, x_2, \cdots, x_n$ are linearly independent, there exist $n$ linearly independent vectors $v_1, v_2, \cdots, v_n$ such that

$$(x_i, v_j) = \delta_{ij} \|x_i\|^2 \ (i, j = 1, 2, \cdots, n),$$

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Lemma 3. \(\delta_{ij}\) is the Kronecker delta symbol and \(\langle , \rangle\) denotes the ordinary inner product of \(R^n\).

Without loss of generality, we may assume that \(v_i = \overrightarrow{A_0 A_i} (i = 1, \ldots, n)\). We now consider the \(n\)-simplex \(\Omega = \{A_1, A_2, \ldots, A_{n+1}\}\). Since \(x_i \perp v_j (i \neq j)\), it follows that
\[x_i \perp \pi_i (i = 1, 2, \ldots, n),\]
where \(\pi_i\) is the hyperplane spanned by the facet \(\Omega_i\). This completes the proof of Lemma 1.

Lemma 2 (Yang and Zhang [3]). Let \(\Omega\) and \(\Omega'\) be two \(n\)-simplices with dihedral angles \(\theta_{ij}\) and \(\theta'_{ij}\) \((1 \leq i < j \leq n + 1)\), respectively. If
\[\theta_{ij} \leq \theta'_{ij} (1 \leq i < j \leq n + 1),\]
then \(\Omega\) and \(\Omega'\) are similar, namely,
\[\theta_{ij} = \theta'_{ij} (1 \leq i < j \leq n + 1).\]

Let \(\{u_1, \ldots, u_n\}\) and \(\{s_1, \ldots, s_n\}\) be two bases of \(R^n\). They are called dual if
\[\langle u_i, s_j \rangle = \delta_{ij} (1 \leq i \leq n)\]
It is well known that there exists a unique dual basis for a given basis of \(R^n\).

Lemma 3. Let \(U = \{u_1, \ldots, u_n\}\) be a basis of \(R^n\) and \(\{s_1, \ldots, s_n\}\) be the dual basis of \(U\). If \(\langle u_i, u_j \rangle < 0 (i \neq j)\), then \(\langle s_i, s_j \rangle > 0\) for all \(i, j\).

Proof. We first consider \(n = 2\). Suppose that \(\langle u_1, u_2 \rangle < 0\) and \(\{s_1, s_2\}\) is the dual basis. Then \(s_1 = \langle s_1, s_1 \rangle u_1 + \langle s_1, s_2 \rangle u_2\). Hence
\[0 = \langle s_1, u_2 \rangle = \langle s_1, s_1 \rangle \langle u_1, u_2 \rangle + \langle s_1, s_2 \rangle \|u_2\|^2.\]
Thus \(\langle s_1, s_2 \rangle > 0\).

Turning to the case of general \(n\), let
\[H_1 = \{x \in R^n \mid \langle u_1, x \rangle = 0\}\]
Then the vectors \(u_2, \ldots, u_n\) are on the opposite side of \(H_1\) from \(u_1\). For \(j \geq 2\), let \(u_j = y_j + z_j, y_j \perp H_1, z_j \in H_1\); then
\[y_j = \frac{\langle u_1, u_j \rangle}{\|u_1\|^2} u_1, \quad z_j = u_j - \frac{\langle u_1, u_j \rangle}{\|u_1\|^2} u_1.\]
If \(i \neq j\) and \(i, j \geq 2\), then
\[\langle z_i, z_j \rangle = \langle u_i - \frac{\langle u_1, u_i \rangle}{\|u_1\|^2} u_1, u_j - \frac{\langle u_1, u_j \rangle}{\|u_1\|^2} u_1 \rangle = \langle u_i, u_j \rangle - 2 \frac{\langle u_1, u_i \rangle \langle u_1, u_j \rangle}{\|u_1\|^2} + \frac{\langle u_1, u_1 \rangle \langle u_1, u_j \rangle}{\|u_1\|^2} = \langle u_i, u_j \rangle - \frac{\langle u_1, u_2 \rangle \langle u_1, u_j \rangle}{\|u_1\|^2} < 0.\]
Note that \(s_k \in H_1\) for \(k \geq 2\), and
\[\langle z_j, s_k \rangle = \langle u_j - \frac{\langle u_1, u_i \rangle}{\|u_1\|^2} u_1, s_k \rangle = \langle u_j, s_k \rangle = \delta_{jk}\]
for \(k, j \geq 2\).

Since \(\{z_2, \ldots, z_n\}\) and \(\{s_2, \ldots, s_n\}\) are biorthogonal sets of \(n - 1\) vectors in the \((n - 1)\)-dimensional subspace \(H_1\), they must be dual bases in \(H_1\), and \(\langle z_i, z_j \rangle < 0\)
where \( i \neq j \). If \( \text{dim } H_1 > 2 \), then we can continue projecting to lower-dimensional subspaces until \( \{ u_1, \ldots, u_n \} \) is mapped to a basis \( G = \{ g_{n-1}, g_n \} \) in a two-dimensional subspace with \( \langle g_{n-1}, g_n \rangle < 0 \) and with \( \{ s_{n-1}, s_n \} \) the dual basis of \( G \). But then in dimension 2, \( \langle s_{n-1}, s_n \rangle > 0 \), completing the proof.

Proof of Theorem \( \text{I}' \). We will first construct an \( n \)-simplex that has exactly \( \frac{1}{2}n(n-1) \) obtuse dihedral angles. Denote by \( \overline{\Omega} \) an \( n \)-simplex all of whose dihedral angles are acute (obviously, such an \( \overline{\Omega} \) exists, e.g., regular simplex).

Let \( u_1, u_2, \ldots, u_n \) be unit outer normal vectors of \( n \) facets of \( \overline{\Omega} \). Then

\[
\langle u_i, u_j \rangle < 0, \quad i \neq j,
\]

and \( U = \{ u_1, \ldots, u_n \} \) is a basis of \( R^n \). Further let \( \{ s_1, \ldots, s_n \} \) be the dual basis of \( U \). By Lemma 1, there exists an \( n \)-simplex \( \Omega \) which takes \( e_1, \ldots, e_n \) as unit outer normal vectors of \( n \) facets, where \( e_i = \frac{u_i}{\| u_i \|} \). By Lemma 3, we have

\[
\langle e_i, e_j \rangle = \frac{1}{\| s_i \| \| s_j \|} \langle s_i, s_j \rangle > 0, \quad 1 \leq i, j \leq n.
\]  
(1.1)

Let \( e_{n+1} \) be the unit outer normal vector of the \((n+1)\)-th facet of \( \Omega \), and \( f_i \) the \((n-1)\)-dimensional volume of the facet \( \Omega_i \) of \( \Omega \). By Minkowski’s realization theorem (see [2, p. 390]), we have

\[
\sum_{i=1}^{n+1} f_i e_i = 0.
\]

Therefore, for \( j \in \{1, 2, \ldots, n\} \),

\[
\langle e_{n+1}, e_j \rangle = \langle -\sum_{i=1}^{n} e_i \frac{f_i}{f_{n+1}}, e_j \rangle
\]

\[
= -\sum_{i=1}^{n} \langle e_i, e_j \rangle \frac{f_i}{f_{n+1}} < 0.
\]  
(1.2)

By (1.1) and (1.2), we know that \( \Omega \) has \( \frac{1}{2}n(n-1) \) obtuse dihedral angles and \( n \) acute dihedral angles.

Assume that there exists an \( n \)-simplex whose obtuse dihedral angles are more than \( \frac{1}{2}n(n-1) \). Without loss of generality, we may assume that

\[
\alpha_1, \ldots, \alpha_{\frac{1}{2}n(n-1)-1}, \alpha_{\frac{1}{2}n(n-1)+1}, \ldots, \alpha_{\frac{1}{2}n(n-1)+j}
\]

are its obtuse dihedral angles, while \( \alpha_{\frac{1}{2}n(n-1)+j+1}, \ldots, \alpha_{\frac{1}{2}n(n+1)} \) are acute. Thus, we can construct a rectangular \( n \)-simplex with acute dihedral angles \( \alpha_{\frac{1}{2}n(n-1)+j+1}, \ldots, \alpha_{\frac{1}{2}n(n+1)} \). In fact, let \( \{ \overline{e}_1, \ldots, \overline{e}_n \} \) be a standard base of \( R^n \), and let

\[
\overline{e}_{n+1} = \sqrt{\frac{1-\lambda}{j}} \overline{e}_1 - \cdots - \sqrt{\frac{1-\lambda}{j}} \overline{e}_j - \cos \alpha_{\frac{1}{2}n(n-1)+j+1} \overline{e}_{j+1} - \cdots - \cos \alpha_{\frac{1}{2}n(n+1)} \overline{e}_n,
\]

where

\[
\lambda = \cos^2 \alpha_{\frac{1}{2}n(n-1)+j+1} + \cdots + \cos^2 \alpha_{\frac{1}{2}n(n+1)}.
\]
Then the rectangular simplex that takes $\tau_1, \cdots, \tau_n, \tau_{n+1}$ as $n+1$ unit normal vectors of facets has $n$ acute dihedral angles

$$\arccos \frac{1 - \lambda}{j}, \cdots, \arccos \frac{1 - \lambda}{j}, \alpha_{\frac{j(n-1)+j+1}{2}}, \cdots, \alpha_{\frac{j(n+1)}{2}}.$$ 

By Lemma 2, this is contradictory. This completes the proof. □

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