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HIGHER DIMENSIONAL APOSYNDETIC DECOMPOSITIONS

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ABSTRACT. Let X be a homogeneous, decomposable continuum that is not aposyndetic. The Aposyndetic Decomposition Theorem yields a cell-like decomposition of X into homogeneous continua with quotient space Y being an aposyndetic, homogeneous continuum.

Assume the dimension of X is greater than one. About 20 years ago the author asked the following questions:

Can this aposyndetic decomposition raise dimension? Can it lower dimension? We answer these questions by proving the following theorem.

Theorem. *The dimension of the quotient space Y is one.*

1. INTRODUCTION

The Aposyndetic Decomposition Theorem [J1] of F. Burton Jones is essential to the study of homogeneous continua. It goes like this.

Theorem 1. *If X is a homogeneous, decomposable continuum that is not aposyndetic, then X admits a continuous decomposition into mutually homeomorphic, indecomposable, homogeneous continua such that the quotient space Y is an aposyndetic, homogeneous continuum.*

The author has strengthened this result by showing that (1) the elements of this decomposition must be cell-like continua [R1], and (2) the elements of this decomposition have the same dimension as X [R6].

In 1983 the author wrote a survey paper [R4] exposing the state of the art in the study of homogeneous continua. He raised a number of questions, about half of which have been answered. One of the unanswered ones is the following [R4, Question 11, p. 224]:

Question 2. Can this aposyndetic decomposition raise dimension? lower dimension?

In this paper we answer this question by proving the following theorem:

Theorem. *If X is a homogeneous, decomposable continuum that is not aposyndetic, then the dimension of the quotient space Y of the aposyndetic decomposition of X is one.*

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2. Results

A continuum is a compact, connected, nonvoid metric space. A continuum is indecomposable if it is not the union of two of its proper subcontinua. A continuum is hereditarily indecomposable if each of its subcontinua is indecomposable. R. H. Bing [B] has constructed hereditarily indecomposable continua of dimension n for $1 \leq n \leq \infty$.

Let x and y be points of the continuum X. If X contains an open set G and a continuum H such that $x \in G \subset H \subset X - \{y\}$, then X is aposyndetic at x with respect to y. If X is aposyndetic at each of its points with respect to every other point, then X is aposyndetic.

A continuum X is cell-like if each map of X into a polyhedron is homotopic to a constant map. A continuum is tree-like if it is cell-like and one-dimensional.

A continuum X is homogeneous if for each pair of points p and q belonging to X, there exists a homeomorphism $h : X \to X$ such that $h(p) = q$. The author [R5] has shown that each homogeneous, hereditarily indecomposable, nondegenerate continuum is tree-like, and hence one-dimensional.

A subcontinuum Z of the continuum X is terminal if each subcontinuum W of X that intersects Z satisfies either $W \subset Z$ or $Z \subset W$. For example, if X is the topologist's $\sin 1/x$ curve and Z is the "limit bar," then Z is a terminal subcontinuum of X.

A decomposition of X into continua is terminal if each element of the decomposition is a terminal subcontinuum of X. Jones [J2] has shown that his aposyndetic decomposition is terminal.

If $f : X \to Y$ is a map and y is a point of Y, then the set $f^{-1}(y)$ is a fiber of f. If each fiber of the map f is a cell-like continuum, then f is a cell-like map.

We use reduced Čech cohomology with integral coefficients. A space is acyclic if each of its cohomology groups is trivial. Note that a cell-like continuum is acyclic.

A nondegenerate continuum Y has cohomological dimension one if $H^q(Y, B) = 0$ for every closed subset B of Y and for every $q > 1$. It is known that a continuum is one-dimensional if and only if it has cohomological dimension one [W, p. 109].

Theorem 3. *Let X be a decomposable, homogeneous continuum that is not aposyndetic, and let $f : X \to Y$ be the quotient map of the aposyndetic decomposition of X. Then $dim Y = 1$.*

Proof. Since X is decomposable, Y is a nondegenerate continuum. Hence $\dim Y \geq 1$. If $\dim X = 1$, then Y, being the cell-like image of X, is also one-dimensional [W, p. 113].

Suppose $\dim X > 1$. Each fiber $f^{-1}(y)$ is a homogeneous continuum with the property that $\dim f^{-1}(y) = \dim X$ [R6]. Homogeneous, hereditarily indecomposable continua have dimension less than or equal to one [R5], and fibers of f are homogeneous continua, so no fiber of f is hereditarily indecomposable. Since the decomposition is terminal, each hereditarily indecomposable subcontinuum of X is contained in a fiber of f.

M. Levin [L] and J. Krasinkiewicz [Kra] have shown that there exists a map $p : X \to I$ of X onto the unit interval I such that each component of each fiber $p^{-1}(t)$ is hereditarily indecomposable. Let $g : X \to Z$ and $h : Z \to I$ be the monotone-light factorization [N, p. 279] of p. Each fiber $g^{-1}(z)$ is a hereditarily indecomposable continuum (possibly a point), so each fiber of g is contained in a fiber of f. It follows that there exists a monotone map $k : Z \to Y$ satisfying $f = k \circ g$.

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, "g"] & I \\ Y & Z \arrow[l, "k"] \arrow[u, "h"'] \end{tikzcd}$$

Each fiber of h is totally disconnected, so Z is one-dimensional [HW, Theorem VI7, p. 91].

It suffices to show that the cohomological dimension of Y is one. Let B be a closed subset of Y. Consider $H^q(Y, B)$ for $q > 1$. Let $A = f^{-1}(B)$ and $C = k^{-1}(B)$. Since each fiber of f is cell-like, each fiber has trivial cohomology. The Vietoris-Begle Theorem [S, p. 344] implies that $f^* : H^q(Y, B) \to H^q(X, A)$ is an isomorphism. Since $f = k \circ g$, we have $f^* = g^* \circ k^*$. Since Z is one-dimensional, $H^q(Z, C) = 0$. Hence $H^q(Y, B) = 0$ as well. This completes the proof. □

As a consequence of this theorem, we obtain some information on the cohomology groups of homogeneous, decomposable continua that are not aposyndetic. First we need a definition.

A continuum X is unicoherent if each pair of subcontinua of X whose union is X has a connected intersection. A continuum is hereditarily unicoherent if each of its subcontinua is unicoherent.

Corollary 4. *If X is a homogeneous, decomposable continuum that is not aposyndetic, then $H^1(X) \neq 0$, and $H^q(X) = 0$ for $q > 1$.*

Proof. Since $f^* : H^q(Y) \to H^q(X)$ is an isomorphism, it suffices to show these claims for the cohomology groups of Y. The second claim is true because Y is one-dimensional. To prove the first claim, recall [R2, Theorem 1, p. 450] that an acyclic one-dimensional continuum is hereditarily unicoherent. Jones [J2] has shown that a hereditarily unicoherent, homogeneous continuum is indecomposable. □

There is a theorem for indecomposable, homogeneous continua that corresponds to the Aposyndetic Decomposition Theorem. It is called the Terminal Decomposition Theorem [R3], and it goes like this.

Theorem 5. *If X is a homogeneous, indecomposable continuum that contains a nondegenerate, proper, terminal subcontinuum and if $H^1(X) \neq 0$, then X admits a continuous decomposition into mutually homeomorphic, indecomposable, homogeneous continua such that the quotient space Y is a homogeneous, indecomposable continuum that contains no proper, nondegenerate, terminal subcontinuum.*

The condition of nontrivial first cohomology of X is necessary to insure the existence of maximal terminal, proper subcontinua of X; these are the elements of the decomposition. As before, the elements of the decomposition are cell-like continua of the same dimension as X. Hence, by the same proof as Theorem 3, we have the following theorem.

Theorem 6. *If X satisfies the hypotheses of Theorem 5, then the quotient space Y of the terminal decomposition is one-dimensional.*

Question 7. If X satisfies the hypotheses of Theorem 5, is the quotient space Y of the terminal decomposition of X a solenoid?

According to a theorem of Pavel Krupski [Kru, Theorem 3.1, p. 167], the answer is yes, provided that the complement of any subcontinuum of Y has finitely many components.

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