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A NONSTANDARD PROOF OF THE EBERLEIN-ŠMULIAN THEOREM

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ABSTRACT. The Eberlein-Šmulian theorem on the equivalence of weak compactness and the finite intersection property of bounded closed convex sets is given a short elementary proof by applying Abraham Robinson's nonstandard characterization of compactness.

In this short paper we provide a nonstandard proof (in the sense of nonstandard analysis) of the Eberlein-Šmulian theorem on the equivalence of weak compactness and the finite intersection property of bounded closed convex sets in a normed linear space. The current proof is inspired by the proof of the same result provided in [2] inasmuch it makes use of a construction similar to that appearing in the proof of *Day's Lemma* (see the second edition of [2] or [8]). However, contrary to the proof presented in [2], we completely avoid any reference to the second dual and to its weak-star-topology.

In our opinion, the novelty in our approach lies in the use of Abraham Robinson's nonstandard characterization of compactness and of the properties of the *nonstandard hull* of a normed space (introduced by Luxemburg in [7], a construction that we will briefly review). In this way we are able to provide a nonstandard proof which is not just a mere reformulation of a standard one.

Furthermore, since the nonstandard hull construction extends that of *ultraproduct* of normed spaces (well known to functional analysts), our approach suggests that it may be possible to reformulate the current proof in the language of ultraproducts while avoiding second dual and weak-star-topology.

Finally, it may worth investigating whether there are other situations in which one can replace second dual and weak-star-topology with nonstandard hull.

Let $\mathfrak{X}$ be a normed linear space and let ${}^*\mathfrak{X}$ be a sufficiently saturated nonstandard extension so that Robinson's compactness criterion applies. $\mathfrak{B}_{\mathfrak{X}}$ denotes the unit ball of $\mathfrak{X}$. For $\phi \in \mathfrak{X}'$, we often write ϕ instead of ${}^*\phi$ as the extended element in ${}^*(\mathfrak{X}')$. The pairing between a dual element θ and an element c is denoted by $\theta(c)$.

We denote by $\hat{\mathfrak{X}}$ the nonstandard hull of ${}^*\mathfrak{X}$. We recall that $\hat{\mathfrak{X}}$, as a real vector space, is the quotient of the norm-finite elements of ${}^*\mathfrak{X}$ with respect to the subspace of vectors of infinitesimal norm. The vector space $\hat{\mathfrak{X}}$ becomes a Banach space

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with respect to the norm defined by $\|\hat{a}\| = {}^\circ\|a\|$, where $\hat{a}$ is the element in $\hat{\mathfrak{X}}$ corresponding to the norm-finite element $a \in {}^*\mathfrak{X}$ and $^\circ$ is the standard part map. (We use the same notation for the norm on $\mathfrak{X}$ and on $\hat{\mathfrak{X}}$.) There is a natural embedding of $\mathfrak{X}$ into $\hat{\mathfrak{X}}$ that allows us to regard $\mathfrak{X}$ as a subspace of $\hat{\mathfrak{X}}$.

Every linear functional $\phi \in \mathfrak{X}'$ can be extended to a linear functional of $(\hat{\mathfrak{X}})'$ via $\phi(\hat{a}) = {}^\circ({}^*\phi(a))$, where $^*\phi$ is the nonstandard extension of ϕ. Therefore, we view $\mathfrak{X}'$ as a subspace of $(\hat{\mathfrak{X}})'$.

Background material on nonstandard analysis in Banach spaces can be found in [6] and [3]. For various proofs of the Eberlein-Šmulian theorem, see [1], [2], [4] and [5] and [9].

Definition 1. The Šmulian condition holds for a subset $X \subset \mathfrak{X}$ if for every decreasing sequence $(C_n)_{n\in\omega}$ of closed convex subsets in $\mathfrak{X}$ such that $C_n \cap X \neq \emptyset$, then $\bigcap_{n<\omega} C_n \cap X \neq \emptyset$.

Definition 2. Let $x, y \in {}^*\mathfrak{X}$, $\phi \in \mathfrak{B}_{{}^*(\mathfrak{X}')}$ and $G \subset \mathfrak{B}_{{}^*(\mathfrak{X}')}$.

- $x \approx_\phi y$, means $\phi(x) \approx \phi(y)$, i.e., $\phi(x) - \phi(y)$ is an infinitesimal;
- $x \approx_G y$ means $x \approx_\psi y$, for all $\psi \in G$;

We write $\|x\| = \sup\{ {}^\circ\theta(x) \mid \theta \in \mathfrak{B}_{\mathfrak{X}'}\}$.

Remark 3. (1) $x \approx_{\mathfrak{B}_{\mathfrak{X}'}} y$ if and only if $\|x - y\| = 0$.
(2) $\|x\| \leq 1$ whenever $x \in \mathfrak{B}_{{}^*\mathfrak{X}}$;
(3) (Robinson's criterion for weak compactness) A subset $X \subset \mathfrak{X}$ is weakly compact if and only if $\forall x \in {}^*X\ \exists x_0 \in X\ \|x - x_0\| = 0$.
(4) For $x \in \mathfrak{B}_{{}^*\mathfrak{X}}$, $\|x\|$ equals the norm of the element in $\mathfrak{X}''$ given by $\theta \mapsto {}^\circ\theta(x)$.

The following is an easy remark. Although its conclusion is not needed later, it is important because it contains some arguments that will play a crucial role in our main result (Theorem 5).

Remark 4. Assume the Šmulian condition for the unit ball $\mathfrak{B}_{\mathfrak{X}}$ and let $a \in \mathfrak{B}_{{}^*\mathfrak{X}}$. Then for any countable $S \subset \mathfrak{B}_{\mathfrak{X}'}$ there is $b \in \mathfrak{B}_{\mathfrak{X}}$ such that $b \approx_S a$. In order to prove this we write S as the union of an increasing family $(S_n)_{n\geq 1}$ of finite subsets.

Let $r_\phi = \phi(\hat{a})$, for $\phi \in S_n$. The statement

$$\alpha_n : \qquad \exists x \left(\|x\| \leq 1 \wedge \bigwedge_{\phi \in S_n} |\phi(x) - r_\phi| < \frac{1}{n} \right)$$

is true in $^*\mathfrak{X}$ (the existential quantifier is witnessed by the element a). So, by applying the Transfer Principle to α_n, we get that, for each n, there is $a_n \in \mathfrak{B}_{\mathfrak{X}}$ such that $|\phi(a) - \phi(a_n)| \leq n^{-1}$ for all $\phi \in S_n$.

By using the Šmulian condition, let $b \in \bigcap_{n\geq 1} \overline{\mathrm{conv}}\{a_i \mid i \geq n\}$. We claim that element b has the required property: fix a positive natural number n and $\phi \in S$. Let $m > 2n$ be such that $\phi \in S_m$. By choice of b, there exists a convex linear combination $c = \sum_{i=m}^{m+k} \lambda_i a_i$ such that $\|b - c\| < \frac{1}{2n}$. Therefore,

$$|\phi(b) - \phi(a)| \leq |\phi(b) - \phi(c)| + |\phi(a) - \phi(c)| < \frac{1}{2n} + \sum_{i=m}^{m+k} \lambda_i |\phi(a) - \phi(a_i)| < \frac{1}{n}.$$

Being n and ϕ arbitrary, we get $b \approx_S a$.

Hence if $\mathfrak{X}'$ is separable and the Šmulian condition for $\mathfrak{B}_{\mathfrak{X}}$ holds, then $\mathfrak{B}_{\mathfrak{X}}$ is weakly compact.

We know that a stronger result holds, for which we want to give a nonstandard proof:

Theorem 5. *Assume the Šmulian condition for the unit ball $\mathfrak{B}_{\mathfrak{X}}$ of a normed linear space $\mathfrak{X}$. Then $\mathfrak{B}_{\mathfrak{X}}$ is weakly compact.*

Proof. Let $a \in \mathfrak{B}_{{}^*\mathfrak{X}}$ and let $\hat{a} \in \mathfrak{B}_{\hat{\mathfrak{X}}}$ be the corresponding element in the nonstandard hull. We want to find $b \in \mathfrak{B}_{\mathfrak{X}}$ so that $\|a - b\| = 0$. Define a seminorm ν on $\mathrm{Lin}(\mathfrak{X} \cup \{\hat{a}\})$ by $\nu(x) = \sup_{\phi \in \mathfrak{B}_{\mathfrak{X}'}} \phi(x)$. Note that $\nu(y) \le \|y\|$ for all $y \in \mathrm{Lin}(\mathfrak{X} \cup \{\hat{a}\})$ and $\nu(y) = \|y\|$ for all $y \in \mathfrak{X}$.

Now we inductively define an increasing sequence $(S_n)_{n \ge 1}$ of finite subsets of $\mathfrak{B}_{\mathfrak{X}'}$ and a sequence $(a_n)_{n \ge 1}$ of elements of $\mathfrak{B}_{\mathfrak{X}}$ such that

(1) $|\phi(\hat{a}) - \phi(a_n)| \le n^{-1}$, for all $\phi \in S_n$;

(2) $\forall x \in L_n = \mathrm{Lin}\{a_1, \dots, a_n, \hat{a}\} \quad \nu(x) \le \frac{n+1}{n} \max\{|\phi(x)| : \ \phi \in S_{n+1}\}$.

To begin, we let $S_1 = \{\phi\}$, for some arbitrary $\phi \in \mathfrak{B}_{\mathfrak{X}'}$, and apply α_1 in Remark 4 to get a_1 such that $|\phi(\hat{a}) - \phi(a_1)| \le 1$.

For the induction step, we assume that $(S_i)_{i \le n}$ and $(a_i)_{i \le n}$ are defined and we define S_{n+1} first, followed by a_{n+1}.

A simple argument using the compactness of the unit ball of L_n with respect to the seminorm ν shows that there is a finite set T_{n+1} such that

$$\nu(x) \le \frac{n+1}{n} \max\{|\phi(x)| : \ \phi \in T_{n+1}\},$$

for all $x \in L_n$. We let $S_{n+1} = S_n \cup T_{n+1}$.

To get a_{n+1} we apply the Transfer Principle to α_{n+1}, as in the proof of Remark 4.

Next we use the Šmulian condition to choose an element

$$b \in \bigcap_{n \ge 1} \overline{\mathrm{conv}}\{a_n, a_{n+1}, \dots\}.$$

Let $S = \bigcup_{n \ge 1} S_n$. By the same argument used in Remark 4, we get $a \approx_S b$. So, by definition of extension of a functional from $\mathfrak{X}$ to $\hat{\mathfrak{X}}$, we obtain $\phi(\hat{a}) = \phi(b)$ for all $\phi \in S$.

We claim that $\nu(\hat{a} - b) = 0$, from which $\|a - b\| = 0$ follows immediately. Suppose otherwise, i.e., $\nu(\hat{a} - b) = r > 0$. Since $b \in \overline{\mathrm{conv}}\{a_1, a_2, \dots\}$, there exist a positive natural number m and a convex linear combination $c = \sum_{i=1}^{m} \lambda_i a_i \in L_m$ such that $\nu(b - c) < \frac{r}{4}$. Then, for some $\phi \in S_{m+1}$, we have

$$r - \frac{r}{4} \le \nu(\hat{a} - c) \le \frac{m+1}{m} |\phi(\hat{a} - c)|.$$

Consequently, $|\phi(\hat{a} - b)| \ge |\phi(\hat{a} - c)| - |\phi(c - b)| \ge \frac{3mr}{4(m+1)} - \frac{r}{4} > 0$, contradicting $\phi(\hat{a}) = \phi(b)$. □

Replacing $\mathfrak{B}_{\mathfrak{X}}$ by an arbitrary bounded set in the above arguments leads to the following:

Theorem 6. *A bounded subset Y in a normed linear space is weakly compact if and only if the Šmulian condition holds for Y.*

Notice that Šmulian condition and weak compactness separately imply boundedness (use the fact that a subset Y of a normed space $\mathfrak{X}$ is bounded if and only if $\phi[Y]$ is a bounded set of scalars for all $\phi \in \mathfrak{X}'$). So we may remove the assumption that Y is bounded in the statement of the previous theorem.

We recall the following characterization: a Banach space is reflexive if and only if its unit ball is weakly compact. From this the Šmulian theorem now follows easily.

Theorem 7 (Šmulian)**.** *A Banach space is reflexive if and only if the Šmulian condition holds for its unit ball.*

Proof. ($\Rightarrow$) Recall that a closed convex subset of a normed space is weakly closed (as a consequence of the Hahn-Banach theorem involving separation of convex sets by linear functionals). So, if $\mathfrak{B}_{\mathfrak{X}}$ is weakly compact and $(C_n)_{n\in\omega}$ is as in Definition 1, then $\mathfrak{B}_{\mathfrak{X}} \cap C_n$ is weakly compact for all n and therefore the finite intersection property implies nonempty intersection.

($\Leftarrow$) Use Theorem 5. □

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