

UNIFORM PERFECTNESS OF SELF-AFFINE SETS

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ABSTRACT. Let $f_i(x) = A_i x + b_i$ ($1 \leq i \leq n$) be affine maps of Euclidean space \mathbb{R}^N with each A_i nonsingular and each f_i contractive. We prove that the self-affine set K of $\{f_1, \dots, f_n\}$ is uniformly perfect if it is not a singleton.

1. INTRODUCTION

For the semi-group G acting on a metric space (X, d) , the *discontinuous set* $\Omega \subset X$ of G consists of points x such that there is an open ball B centered at x such that there are only finitely many $g \in G$ satisfying $gB \cap B \neq \emptyset$. Its complement $\Lambda = X \setminus \Omega$ is called the *limit set* of G .

Let $f_i : X \rightarrow X$ ($1 \leq i \leq n$) be contractive maps of complete metric space (X, d) and let $G = \langle f_1, \dots, f_n \rangle$ be the semigroup generated by $\{f_1, \dots, f_n\}$ and $G_k = \{f_{j_1} \circ \dots \circ f_{j_k} \mid 1 \leq j_i \leq n, 1 \leq i \leq k\}$ ($k \geq 0$), where we mean $G_0 = \{\text{Identity map}\}$. Then there exists a unique nonempty compact set K such that $K = \bigcup_{i=1}^n f_i(K)$, which is called the *attractor* of the iterated function system (IFS, for short) $\{f_1, \dots, f_n\}$ and is the limit of $\bigcup_{g \in G_k} gX_0$ as $k \rightarrow \infty$ for any nonempty compact subset $X_0 \subset X$ with respect to the Hausdorff metric of X . It can be shown that K coincides with the limit set Λ of G (Lemma 2.1).

We are concerned with the geometric properties of the attractor K . They often exhibit rich structure and give examples of fractals. Such systems have been extensively studied in regards to Hausdorff measure and the dimension of their attractor. If each map in the system is of one-to-one fashion, the attractor is perfect unless it is a singleton (see Theorem 2.1). An example (see Example 1) is given to show that this is not generally true. We expect better geometric properties of the attractor for a more specific family of maps.

Let $f_i(x) = A_i x + b_i$ ($1 \leq i \leq n$) be affine maps of Euclidean space \mathbb{R}^N with each $N \times N$ matrix A_i nonsingular and each f_i contractive in the sense $|f_i(x) - f_i(y)| \leq a_i |x - y|$ for all $x, y \in \mathbb{R}^N$ and some $0 < a_i < 1$. The attractor for such a system will be called a *self-affine set*.

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A compact subset E of Euclidean space \mathbb{R}^N is *uniformly perfect* if E contains at least two points and there exists a constant $c > 0$ such that for any point $x_0 \in E$ and $0 < r < \text{diam}E$, the annulus $\{x \in \mathbb{R}^N | cr \leq |x - x_0| \leq r\}$ meets E (see [Po]). For a compact set E of the complex plane \mathbb{C} , A. F. Beardon and Ch. Pommerenke [BP] relate uniform perfectness to the notion of the hyperbolic metric of $\overline{\mathbb{C}} \setminus E$ being comparable to the reciprocal of the distance to the boundary. There are many other characterizations of uniformly perfect planar sets (see [Po]). In the analytical settings, it has been proved that the Julia sets of rational maps, conformal attractors, and Julia sets of rational semigroups with a common Lipschitz constant are uniformly perfect. We refer the reader to [MC], [HM], [S2], and [S1] for details.

Our question is whether the self-affine sets are uniformly perfect. The main result of this paper gives an affirmative answer to this question when the attractor is not a singleton. In particular, if each f_i is a similitude, then the self-similar set K is either uniformly perfect or a singleton.

Now we come to our main result.

Main Theorem. *The self-affine set containing at least two points is uniformly perfect.*

Combining this and Corollary 4.2 in [JV], one immediately has the following statement.

Corollary 1.1. *The self-affine set containing at least two points has positive Hausdorff dimension.*

2. PERFECTNESS AND PROPERTIES OF AFFINE MAPS

In this section, we give the proof of the perfectness of attractors of one-to-one contractive maps and some necessary preparations for the proof of our Main Theorem.

2.1. Perfectness of attractors. Let (X, d) be a metric space. The ϵ -neighborhood of the subset $A \subset X$, denoted by $N_\epsilon(A)$, is defined to be the set $\{x \in X | d(x, A) < \epsilon\}$. We need the following lemma to prove the perfectness of attractors of one-to-one contractive maps.

Lemma 2.1. *If $\{f_1, \dots, f_n\}$ is a finite family of contractive maps of complete metric space X , then $\Lambda = K$.*

Proof. For any point $x \in \Omega$, there is an open ball B centered at x such that there are only finitely many $g \in G$ satisfying $gB \cap B \neq \emptyset$.

Denote $X_0 = \overline{B}$, $X_k = \bigcup_{g \in G_k} gX_0$. Then $X_k \rightarrow K$ in the Hausdorff metric as $k \rightarrow \infty$. We conclude that $X_k \cap B = \emptyset$ for large k . This implies $x \notin K$.

On the other hand, if $x \notin K$, there is an open ball B centered at x such that B is disjoint with an ϵ_0 -neighborhood $N_{\epsilon_0}(K)$ of K for some $\epsilon_0 > 0$. Denote $X_0 = \overline{B}$, $X_k = \bigcup_{g \in G_k} gX_0$. Then $X_k \subset N_{\epsilon_0}(K)$ for large k . This yields $X_k \cap B = \emptyset$ for large k and $\#\{g \in G | gB \cap B \neq \emptyset\} < \infty$. Hence $x \in \Omega$.

We proved that $\Omega = X \setminus K$, i.e., $\Lambda = K$. □

Theorem 2.1. *Let $f_i : X \rightarrow X$ ($1 \leq i \leq n$) be one-to-one contractive maps of complete metric space (X, d) and K be the attractor. Then K is perfect if it is not a singleton.*

Proof. Suppose that K has an isolated point x . Choose an open ball $B(x, r)$ centered at x with radius $r > 0$ such that $B(x, r) \cap K = \{x\}$. From Lemma 2.1, there exists an element $g \in G$ such that $gB(x, \frac{1}{2}r) \cap B(x, \frac{1}{2}r) \neq \emptyset$. Since $gB(x, \frac{1}{2}r) \subset B(g(x), \frac{1}{2}r)$, we have $g(x) \in B(x, r)$ and $g(x) = x$. It follows from g being a one-to-one contractive map that if there is a point in K different from x , say y , then $g^k(y) \rightarrow x$ in K as $k \rightarrow \infty$. This is a contradiction because x is an isolated point. \square

Theorem 2.1 does not hold if we draw off the one-to-one condition.

Example 1. Let $X = \mathbb{R}^1$, and let $f_1(x) \equiv 0$ and $f_2(x) = \frac{1}{2}x + \frac{1}{2}$. Then f_1 and f_2 are contractive, and $K = \{0, 1, a_1, \dots, a_n, \dots\}$, where $a_n = \sum_{j=1}^n \frac{1}{2^j}$.

2.2. Properties of affine maps. Let $A = (a_{ij})$ be an $N \times N$ real matrix. The norm of A , denoted by $\|A\|$, is defined to be $\sqrt{\sum a_{ij}^2}$. Denote by e^j the j^{th} member of the standard basis of \mathbb{R}^N . It is easy to check that

- (1) $\|AB\| \leq \|A\|\|B\|$,
- (2) $\|H\| = \sqrt{N}$ for orthogonal matrix H , and
- (3) $\|A\| = \sqrt{\sum_{j=1}^N |A(e^j)|^2}$.

The following lemma plays a crucial role in the proof of our Main Theorem.

Lemma 2.2. *Let $f(x) = Ax + a$ be a one-to-one linear map of \mathbb{R}^N . Then there exists $\alpha_A > 0$ depending only on A such that*

$$\text{diam } \phi \circ f(E_0) \geq \alpha_A \text{diam } \phi(B_0),$$

for any closed ball $B_0 = \overline{B(0, R_0)}$, subset $E_0 = \{0, R_0e^1, \dots, R_0e^N\}$, and any map $\phi(x) = Bx + b$ of \mathbb{R}^N .

Proof. First note that there is an orthogonal transformation $h(x) = Hx$ depending on ϕ so that $|\phi \circ h(x) - b|$, or equivalently $|BHx|$, assumes the maximum for $x \in B_0$ at $R_0e^1 \in E_0$. Therefore,

$$\begin{aligned} \text{diam } \phi \circ f(E_0) &= \text{diam } BA(E_0) \\ &\geq \max_{1 \leq j \leq N} \{|BA(R_0e^j)|\} \geq \frac{R_0}{\sqrt{N}} \|BA\| \\ &= \frac{R_0 \|BA\| \|A^{-1}H\|}{\sqrt{N} \|A^{-1}H\|} \geq \frac{R_0 \|BH\|}{N \|A^{-1}\|} \\ &\geq \frac{\max_{1 \leq j \leq N} \{|BH(R_0e^j)|\}}{N \|A^{-1}\|} = \frac{|BH(R_0e^1)|}{N \|A^{-1}\|} = \frac{\max_{x \in B_0} \{|BHx|\}}{N \|A^{-1}\|} \\ &= \frac{\text{diam } \phi \circ h(B_0)}{2N \|A^{-1}\|} = \frac{1}{2N \|A^{-1}\|} \text{diam } \phi(B_0). \end{aligned}$$

Take $\alpha_A = \frac{1}{2N \|A^{-1}\|}$. \square

3. PROOF OF THE MAIN THEOREM

Suppose that K is not a singleton. We may assume that the origin lies in K and is the fixed point of some affine map in $\{f_1, \dots, f_n\}$, say, f_1 . Otherwise, we can apply a translation $T(x) = x + (A_1 - Id)^{-1}b_1$ of \mathbb{R}^N , where we denote by Id the identity matrix, to conjugate the system to a newer one. This does not change the uniform perfectness. Let B_0 be the closed ball centered at the origin and of radius R_0 such that $f_i(B_0) \subset B_0$ for $1 \leq i \leq n$. Denote by E_0 the subset $\{\mathbf{0}, R_0e^1, \dots, R_0e^N\}$.

It follows from Theorem 2.1 that K is perfect and hence contains infinitely many points. If K does not lie in any lower-dimensional subspace of \mathbb{R}^N , or in other words the nonzero vectors in K span \mathbb{R}^N , one can find N distinct points y^j ($1 \leq j \leq N$) in K other than the origin, such that the vectors y^j ($1 \leq j \leq N$) are linearly independent. Let $f_0(x) = A_0x$ be the linear map sending R_0e^j to y^j . Hence A_0 is nonsingular.

On the other hand, if $K \setminus \{\text{zero vector}\}$ spans a lower-dimensional subspace L of \mathbb{R}^N , then L is invariant under each f_i for $1 \leq i \leq n$. Note that a canonical isomorphism between L and some lower-dimensional Euclidean space \mathbb{R}^m ($1 \leq m < N$) conjugates f_i 's to affine maps and keeps uniform perfectness. Thus it suffices to consider the former case.

Let x be any point in K and $B(x, r)$ be the closed ball centered at x and of radius $r < \text{diam } K$. There are $g_k \in G_k$ such that $x \in g_{k+1}(B_0) \subset g_k(B_0)$ and $g_{k+1} = g_k \circ f_{j_{k+1}}$ ($1 \leq j_{k+1} \leq n$) for each $k \geq 0$. Let k_0 be the smallest integer such that $\text{diam } g_{k_0}(B_0) \geq r$. Then $g_{k_0+1}(B_0) \subset B(x, r)$. Let $\alpha = \min_{0 \leq i \leq n} \{\alpha_{A_i}\}$, where α_{A_i} is the constant given for f_i ($0 \leq i \leq n$) in Lemma 2.2. Therefore, $\text{diam } g_{k_0+1} \circ f_0(E_0) \geq \alpha \text{diam } g_{k_0+1}(B_0) \geq \alpha \text{diam } g_{k_0+1}(E_0) \geq \alpha^2 \text{diam } g_{k_0}(B_0) \geq \alpha^2 r$. Let $c = \frac{1}{2}\alpha^2$. Some vertex of $g_{k_0+1} \circ f_0(E_0)$, which lies in K , meets the annulus $\{y \in \mathbb{R}^N | cr \leq |y - x| \leq r\}$. This completes the proof. \square

Example 2. One cannot extend the Main Theorem to the one-to-one contractive maps of complete metric space. Let $f_1(x) = \frac{1}{3}x^2$, $f_2(x) = \frac{1}{3}x^2 + \frac{2}{3}$, and $X = [0, 1]$. Obviously, the attractor of $\{f_1, f_2\}$ is a Cantor set. Define r_n and R_n inductively by $r_1 = \frac{1}{3}$, $R_1 = \frac{2}{3}$ and $r_n = \frac{1}{3}r_{n-1}^2$, $R_n = \frac{1}{3}R_{n-1}^2$ for $n > 1$. Then the sequence of annuli (in this case, one interval or two disjoint intervals with equal length) $A_n = \{x \in X | r_n < |x| < R_n\}$ do not intersect the attractor and all are centered at the origin. Note that $R_{n+1}/r_{n+1} = (R_n/r_n)^2 = \dots = 2^{2^n} \rightarrow \infty$ as $n \rightarrow \infty$. This contradicts the definition of uniform perfectness.

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