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ALMOST PERIODIC SETS AND SUBACTIONS IN TOPOLOGICAL DYNAMICS

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Abstract. Let Γ be a group with subgroup Λ. We show, under certain conditions, that an almost periodic point for the action of Λ is also almost periodic for Γ. This is applied to obtain a theorem of Glasner.

We will be considering flows (X, S) where X is a compact Hausdorff space and S is a semigroup. In many cases of interest to us, the elements of S define homeomorphisms of X.

The point $x \in X$ is said to be an almost periodic point of (X, S) if $\overline{xS}$ is a minimal set and $x \in \overline{xS}$.

The following elementary lemma will be used frequently in the sequel.

Lemma 1. *Let (X, S) and (Y, S) be flows, let $x \in X$ be almost periodic, and let $y \in Y$. Then there is an almost periodic point of the form $(x, y') \in \overline{(x, y)S}$.*

Proof. Let $(x_1, y_1) \in \overline{(x, y)S}$ be almost periodic. Let $\{s_i\}$ be a net in S such that $x_1 s_i \to x$. Then (a subnet of) $(x_1, y_1)s_i \to (x, y')$ for some y'. Since (x, y') is in the orbit closure of an almost periodic point, it is almost periodic.

An *almost periodic set* for (X, S) is a subset A of X such that if $z \in X^{|A|}$ with $range(z) = A$, then z is an almost periodic point of the flow $(X^{|A|}, S)$. ($|A|$ denotes the cardinality of A.) If $x \in X$ is an almost periodic point, then it is easy to see that there is a maximal almost periodic set which contains x.

The following lemma lists elementary properties of almost periodic sets.

Lemma 2. (*i*) *Any non-empty subset of an almost periodic set is an almost periodic set. (In particular, every point of an almost periodic set is almost periodic.)*

(*ii*) *Let A be a maximal almost periodic set for (X, S), and let $x \in X$. Then there is an $x' \in A \cap \overline{xS}$ such that x and x' are proximal. (That is, if W is any neighborhood of the diagonal Δ, then there is an $s \in S$ such that $(xs, x's) \in W$.)*

(*iii*) *Let $\pi : (X, S) \to (Y, S)$ be an onto flow homomorphism, and let B be an almost periodic set in Y. Then there is an almost periodic set A in X such that $\pi(A) = B$.*

(*iv*) *If A is a maximal almost periodic set in X, then $\pi(A)$ is a maximal almost periodic set in Y.*

(*v*) *Suppose $range(z)$ is a maximal almost periodic set. Let $z' \in \overline{zS}$. Then $range(z')$ is a maximal almost periodic set.*

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Proof. (i) follows from the well known (and easily proved) fact that a homomorphic image of an almost periodic point is almost periodic.

(ii) Let $z \in X^{|A|}$ such that $range(z) = A$, and let $(z, x') \in \overline{(z,x)S}$ be an almost periodic point. Let $\{s_i\}$ be a net in S with $(z,x)s_i \to (z,x')$. Since $A = range(z)$ is maximal, $x' \in A$, so we have $(x',x)s_i \to (x',x')$.

(iii) Let $w \in Y^{|B|}$ with $range(w) = B$. Then w is an almost periodic point in $(Y^{|B|}, S)$. Let z be an almost periodic point in $X^{|B|}$ with $\pi(z) = w$, and let $A = range(z)$. Then A is an almost periodic set and $\pi(A) = B$.

(iv) follows immediately from (iii).

(v) Clearly $range(z')$ is an almost periodic set. Suppose (z', x') is an almost periodic point. Let $\{s_i\}$ be a net such that $(z',x')s_i \to (z,x)$. Since (z,x) is almost periodic, and $range(z)$ is maximal, $x \in range(z)$. It follows that $x' \in range(z')$. (If not, there would be a $y' \in range(z')$ with x' and y' proximal and $x' \neq y'$ which would contradict (x', y') almost periodic.)

Let (X, S) be a flow. Then $x \in X$ is said to be a distal point if for every $y \in \overline{xS}$ with $y \neq x$, x and y are distal (not proximal). A minimal flow is called point distal if it contains a distal point.

Corollary 1. *Let x be a distal point for a flow (X, S). Then x is an almost periodic point.*

Proof. This follows from (ii) of Lemma 2.

Let Γ be a group which acts minimally on a space X, and let S be a subsemigroup of Γ which generates Γ. It is not in general the case that (X, S) is minimal. (An example is given at the end of the paper.) Two cases where (X, S) is in fact minimal follow.

Lemma 3. *Let (X, Γ) be a minimal flow, with Γ an abelian group. Let S be a subsemigroup of Γ which generates Γ (equivalently $\Gamma = SS^{-1}$). Then (X, S) is minimal.*

Proof. Let X_0 be a minimal set for S. Let $x \in X_0$ and let $s \in S$. Then xs^{-1} is an almost periodic point for S, so xs^{-1} is in X_1, an S minimal set. Then $x = xs^{-1}s \in X_1$ so $X_1 = X_0$. Therefore $xS^{-1} \subset X_0$, and it follows that $x\Gamma \subset X_0$, so $X_0 = X$.

Theorem 1. *Let Γ be a group and let S be a semigroup which generates Γ. Let (X, Γ) be a point distal minimal flow. Then (X, S) is minimal.*

Proof. Let x be a distal point for (X, Γ), and let $X_0 = \overline{xS}$. It follows from Corollary 1 that X_0 is an S minimal set. It is sufficient to show that $x\Gamma \subset X_0$. For this, in turn, it is sufficient to show that $xs\sigma^{-1} \in X_0$, for σ, $s \in S$. (Note that $xs\sigma^{-1}$ is also a distal point, and each $t \in \Gamma$ is a finite product of elements of S and S^{-1}.) Clearly $x\sigma \in X_0$. Let Γ_1 be the cyclic subgroup generated by s, and let S_1 be the semigroup generated by s. Now $x\sigma$ is a distal point, hence an almost periodic point for (X, Γ_1). By Lemma 3 we have $x\sigma s^{-1} \in \overline{x\sigma\Gamma_1} = \overline{x\sigma S_1} \subset \overline{xS} = X_0$.

Lemma 4. *Let Γ be a group, and let S be a subsemigroup of Γ which generates Γ. Let (X, Γ) be a minimal flow. Suppose there is an $x \in X$ such that xt is an S almost periodic point for every $t \in \Gamma$. Then (X, S) is minimal.*

Proof. Let $X_0 = \overline{xS}$. Then X_0 is an S minimal set. We show $X_0 = X$. It is sufficient to show that $x\sigma s^{-1} \in X_0$, for all $s, \sigma \in S$. Clearly $x\sigma \in X_0$. Now $x\sigma s^{-1}$ is an S almost periodic point, so $X_1 = \overline{x\sigma s^{-1}S}$ is an S minimal set. Then $x\sigma = x\sigma s^{-1}s \in X_1$. But $x\sigma \in X_0$ so $X_1 = X_0$, and $x\sigma s^{-1} \in X_0$.

Theorem 2. *Let Γ be a group, let S be a semigroup which generates Γ, and let (X, Γ) be a minimal flow. Suppose that either Γ is abelian, or (X, Γ) is point distal. Then (X, S) is minimal.*

Proof. We show, in both cases, that there is an $x \in X$ such that xt is an S almost periodic point, for all $t \in T$, so the conclusion follows from Lemma 4.

If Γ is abelian, then $t \in \Gamma$ defines a flow automorphism of (X, S). Hence if x is S almost periodic, so is xt.

Suppose (X, Γ) is point distal, and let x be a Γ distal point. Then, if $t \in \Gamma$, xt is a Γ distal point, hence an S distal point, and therefore an S almost periodic point.

Let Γ be a group and let (X, Γ) be a flow. Let $\tilde{\Gamma}(X)$ be the group of homeomorphisms of X defined by Γ.

Theorem 3. *Let (X_i, Γ) $(i \in \mathcal{I})$ be a family of flows, and let Λ be a subgroup of Γ. Suppose, for each $i \in \mathcal{I}$, $\tilde{\Gamma}(X_i) = \tilde{\Lambda}(X_i)$. Let (x_i) be an almost periodic point of $(\bigtimes_{i\in\mathcal{I}} X_i, \Lambda)$. Then (x_i) is an almost periodic point of $(\bigtimes_{i\in\mathcal{I}} X_i, \Gamma)$.*

Proof. Recall that a point in a product flow is almost periodic if and only if its restriction to all finite products is almost periodic. Therefore it suffices to consider finitely many flows (X_i, Γ) $(i = 1, \ldots, n)$ and an almost periodic point $(x_1, \ldots, x_n)$ in $(X_1 \times \cdots \times X_n, \Lambda)$. We show that $(x_1, \ldots, x_n)$ is almost periodic for the action of Γ.

The proof is by induction on n. The case $n = 1$ is an immediate consequence of the hypothesis. Suppose the result holds for $n - 1$. Let $(x_1, x_2, \ldots, x_n)$ be a Λ almost periodic point, and let C be a maximal almost periodic set (with respect to Λ) containing $(x_1, \ldots, x_n)$.

Let $z = (z_1, \ldots, z_n) \in (X_1 \times \cdots \times X_n)^{|C|}$ with $range(z) = C$. Then z is a Λ almost periodic point.

It follows from the induction hypothesis that $(z_1, \ldots, z_{n-1})$ is a Γ almost periodic point.

(Note that whenever $\tilde{\Gamma}(X) = \tilde{\Lambda}(X)$, then $\tilde{\Gamma}(X^a) = \tilde{\Lambda}(X^a)$ for any cardinal number a.)

Let $A_i = \pi_i(C)$ (where π_i is the projection of $X_1 \times \cdots \times X_n$ onto X_i). Then A_i is a maximal almost periodic set for both the Λ and Γ actions.

Let $(z_1, \ldots, z_{n-1}, z'_n) \in \overline{(z_1, \ldots, z_n)\Gamma}$ be an almost periodic point for both the Γ and Λ actions. Consider the point $(z_1, \ldots, z_{n-1}, z'_n, x_n)$. Note that $(z_1, \ldots, z_{n-1}, x_n)$ is a Λ almost periodic point. Therefore, there is a Λ almost periodic point of the form $(z_1, \ldots, z_{n-1}, z''_n, x_n) \in \overline{(z_1, \ldots, z_{n-1}, z'_n, x_n)\Lambda}$. Now $range(z''_n)$ is a maximal almost periodic set in X_n so $x_n \in range(z''_n)$. Also, $(z_1, \ldots, z_{n-1}, z''_n)$ is a Γ almost periodic point (since it is in the orbit closure of the Γ almost periodic point $(z_1, \ldots, z'_n)$). Since $x_i \in range(z_i)$ for $i = 1, \ldots, n-1$, it follows that $(x_1, \ldots, x_n)$ is Γ almost periodic.

With appropriate modifications, Theorem 2 can be stated and proved in case Γ and Λ are semigroups.

The following is a (slight) generalization of a theorem of Glasner [G]. (Glasner assumed minimality of the action of T.)

Theorem 4. *Let T be a homeomorphism of the compact space X, let n be a positive integer, and let θ and τ be the homeomorphisms of $X^n = X \times \cdots \times X$ defined by $\theta = T \times \cdots \times T$, and $\tau = T \times T^2 \times \cdots \times T^n$. Let $x = (x_1, \ldots, x_n)$ be an almost periodic point for θ. Then x is an almost periodic point for the group generated by θ and τ.*

Proof. Let $\Lambda = \mathbf{Z}$, and let Γ be the direct product of Λ and $\mathbf{Z}^n = \mathbf{Z} \times \cdots \times \mathbf{Z}$. We may regard Λ as a subgroup of Γ. Let $X_i = X$ for $i = 1, \ldots, n$. The Λ action on X_i is by the homeomorphism T, and the action of $\mathbf{Z}^n$ on X_i is defined by $(k_1, \ldots, k_n)(x) = T^{ik_i}(x)$. It is clear that $\tilde{\Lambda}(X_i) = \tilde{\Gamma}(X_i)$ and the result follows from Theorem 3.

We are indebted to Benjamin Weiss for pointing out the following example of a minimal action of a group with a generating subsemigroup whose action is not minimal. Let $\Gamma = F_2$, the free group on two generators a and b, and let X be the space of infinite one-sided sequences $x = x_1x_2\ldots$ with $x_i \in \{a, b, a^{-1}, b^{-1}\}$ such that x_ix_{i+1} is not of the form aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$. Let Γ act on X (on the left) by concatenation, and cancellation if required. (For example, if $x = a^{-1}a^{-1}bba\ldots$, $bx = ba^{-1}a^{-1}bba\ldots$ and $ax = a^{-1}bba\ldots$.) It is easy to see that this action of F_2 is minimal. Now let S be the subsemigroup of Γ generated by a and b, and E the closed subset of X consisting of points in which only a and b appear. Then S generates Γ, and E is invariant under the S action, so (X, S) is not minimal.

A slight modification of this example shows that even for a semigroup of homeomorphisms, it is possible for the orbit closure of a point to be minimal, although the point is not in the minimal set. Let X, E, and S be as above, and let $R = Sa$. Then E is minimal under the action of the semigroup R. If $x = a^{-1}bbbb\ldots$, then $\overline{Rx} = E$ but $x \notin E$.

References

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