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CHARACTERIZATION OF THE MOD 3 COHOMOLOGY OF E_7

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Dedicated to Professor Shôrô Araki on his 70th birthday

Abstract. It is shown that the mod 3 cohomology of a homotopy associative mod 3 H-space which is rationally equivalent to the Lie group E_7 and which has integral 3-torsion is isomorphic to that of E_7 as a Hopf algebra over the mod 3 Steenrod algebra.

1. Introduction

The purpose of this paper is to show the following theorem.

Theorem 1. *If a homotopy associative mod* 3 *H-space X is rationally equivalent to E_7 and has integral* 3*-torsion, then $H^*(X;\mathbb{F}_3)$ is isomorphic to $H^*(E_7;\mathbb{F}_3)$ as a Hopf algebra over the mod* 3 *Steenrod algebra.*

For the detail of $H^*(E_7;\mathbb{F}_3)$, see Araki[2] and Kono-Mimura [11]. This theorem is known to be true if one replaces E_7 with F_4 or E_8 (see [14] and Kane [8], [9]).

The starting point of our proof of Theorem 1 is the loop space theorem for odd primes [14]. We recall some results and then we claim that $\wp^3\wp^1 H^3(X;\mathbb{F}_3) \neq 0$. Its proof, which is given in Section 3, is a combination of applications of an obstruction theory about lifting a_n-maps to a_{n-1}-maps in 2-stage Postnikov systems ([15]) with analyses of the $H_*(X;\mathbb{F}_3)$-module Hopf algebra structures of $H_*(X;\mathbb{F}_3)$ and $H_*(\Omega X;\mathbb{F}_3)$ which are given by the adjoint actions $\mathrm{ad}\colon X\times X\to X$ and $\mathrm{Ad}\colon X\times \Omega X\to\Omega X$, respectively. (For a Hopf algebra A, we say B has an A-module Hopf algebra structure if B has an A-module structure as well as a Hopf algebra structure such that the four structure maps of the Hopf algebra B, that is, the multiplication, the unit, the comultiplication, and the counit, are maps of A-modules where A-module structures of the ground ring and $B\otimes B$ are given as in Milnor-Moore [17]. See Kono-Kozima [10] and Hamanaka-Hara [5].) Then, we prove Theorem 1 by further analyses of the adjoint actions and by a factorization of $\wp^3$ through secondary operations ([16], [19]).

The most difficult part in determining the coalgebra structure of the mod 3 cohomology of E_7 is to determine the coproduct of x_{35}, the generator of degree 35. (See [11].) Indeed, Mimura and the first-named author used the inclusion $E_7\hookrightarrow E_8$ to determine it, but no methods for determining it without using E_8 were known. Our proof of Theorem 1 gives such a method.

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2. Proof

We use the following notation. Given a vector space V and its subset S, let $\langle S\rangle$ denote the subspace generated by S. The subscript of an element of a graded algebra designates the degree. Given a Hopf algebra A, let $\mathrm{P}A$ and $\mathrm{Q}A$ denote the primitives and the indecomposables respectively, and let $\bar{x} \in \mathrm{Q}A$ denote the class of an element x of A. The coefficient for homology and cohomology is $\mathbb{F}_3$.

Let X satisfy the hypothesis of Theorem 1. By [14], we have

$$H^*(X) = \mathbb{F}_3[x_8]/(x_8^3) \otimes \bigwedge(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35})$$

as an algebra and we have $x_7 = \wp^1 x_3$ and $x_8 = \beta x_7$. Let $B = \mathbb{F}_3[x_8]/(x_8^3)$ and $R = \langle 1\rangle \oplus \{x \in H^*(X) \mid \bar{\Delta}x \in B \otimes H^*(X), |x| > 0\}$ where $\bar{\Delta}$ is the reduced coproduct of $H^*(X)$. By Baum-Browder [3] and Kane [9], R is a coalgebra over $\mathcal{A}(3)$, the mod 3 Steenrod algebra. Further $R = B$ in even degrees and R is (naturally) isomorphic to $\mathrm{Q}H^*(X)$ in odd degrees. Hence we can choose x_j for $j = 11, 15, 19, 27, 35$ so that $R = \langle x_j \mid j \text{ is odd}\rangle$ in odd degrees. Let $a_j \in \mathrm{P}H_*(X)$ be the dual element of $\bar{x}_j \in \mathrm{Q}H^*(X)$. By Kane [8], we may assume that $a_{11} = a_8 * a_3 = a_{15}\wp^1$ and that $a_{15} = a_8 * a_7$ where $*$ indicates the adjoint action. (Note that $a * b = [a, b]$ if a is primitive. See Kono-Kozima [10] and Hamanaka-Hara [5].) Moreover, we can choose x_{19} to satisfy

Proposition 2. $x_{19} = \wp^3 x_7$.

The proof is given in Section 3. By Proposition 2, we have $(a_8 * a_{19})\wp^3 = a_8 * a_7 = a_{15}$ and hence we may assume that $a_{27} = a_8 * a_{19}$ and that $a_{27}\wp^3 = a_{15}$.

Let $\sigma\colon H_*(\Omega X) \to \mathrm{P}H_*(X)$ be the homology suspension. By Clark [4], Kane [7], and Kraines [13], we have

$$H_*(\Omega X) = \mathbb{F}_3[t_2, t_6, t_{10}, t_{14}, t_{18}, t_{22}, t_{26}, t_{34}]/(t_2^3) \tag{2.1}$$

as an algebra where $\sigma(t_j) = a_{j+1}$ for $j \neq 22$. Since $H_*(\Omega X)$ is bicommutative, we may assume that $t_j \in \mathrm{P}H_*(\Omega X)$ for $j \neq 6, 18$. Moreover, we can easily see that $\bar{\phi}(t_6) = -t_2^2 \otimes t_2 - t_2 \otimes t_2^2$ where $\bar{\phi}$ is the reduced coproduct of $H_*(\Omega X)$.

The relations $a_8 * a_3 = a_{11}$ and $a_8 * a_7 = a_{15}$ are converted into $a_8 * t_2 = t_{10}$ and $a_8 * t_6 = t_{14} - t_{10}t_2^2$, respectively. (See Hamanaka-Hara [5].) Put $v = a_8 * t_{10} \in \mathrm{P}H_*(\Omega X)$. Since $a_8^3 = 0$, we have $a_8 * v = a_8^2 * t_{10} = a_8^3 * t_2 = 0$ and

$$0 = a_8^3 * t_6 = a_8^2 * (t_{14} - t_{10}t_2^2) = \cdots = a_8^2 * t_{14} + t_{10}^3.$$

Hence we have $a_8^2 * t_{14} = -t_{10}^3 \neq 0$.

The relation $a_{27}\wp^3 = a_{15}$ is converted into $t_{26}\wp^3 = t_{14}$. Then, we have

$$(a_8^2 * t_{26})\wp^3 = a_8^2 * t_{14} \neq 0$$

and hence we have $a_8 * t_{26} \neq 0$. Thus we may assume that $t_{34} = a_8 * t_{26}$. This is converted into $a_{35} = a_8 * a_{27}$. Then, we can easily see that $H^*(X)$ is isomorphic to $H^*(E_7)$ as a Hopf algebra over $\mathbb{F}_3$. (Recall that $H_*(X)$ is the universal enveloping Hopf algebra of the restricted Lie algebra $\mathrm{P}H_*(X)$ over $\mathcal{A}(3)$. See Milnor-Moore [17] and Kane [9].)

For the Steenrod module structure, we are left to show that $a_{19}\wp^1 = \pm a_{15}$. Note that $\wp^3 x_{15} = x_{27}$. Let $X\langle 3\rangle$ be the 3-connective cover of X and $g\colon X\langle 3\rangle \to X$

the covering projection. By the Serre spectral sequence, we have $H^*(X\langle 3\rangle) = \bigwedge(\hat{x}_{11}, \hat{x}_{15}, \hat{x}_{27}, \hat{y}_{19}, \hat{y}_{23})$ for $* < 35$ where $\hat{x}_j = g^*(x_j)$ and $\hat{y}_j \notin \mathrm{Im}\ g^*$. Note that $\beta\hat{x}_{15} = \wp^1\hat{x}_{15} = 0$ and that $\wp^3\hat{x}_{15} = \hat{x}_{27}$. According to Liulevicius [16] or Shimada-Yamanoshita [19], the primary operation $\wp^3\hat{x}_{15} = \hat{x}_{27}$ decomposes into $\Phi\hat{x}_{15} = \hat{y}_{23}$ and $\wp^1\hat{y}_{23} = \pm\hat{x}_{27}$ where Φ is a secondary operation. Then, we can easily show that $t_{26}\wp^1 = \pm t_{22}$ by the homotopy fiberings $\Omega(X\langle 3\rangle) \to * \to X\langle 3\rangle$ and $S^1 \to \Omega(X\langle 3\rangle) \xrightarrow{\Omega g} \Omega X$.

The relation $a_8 * a_{19} = a_{27}$ is converted into $a_8 * \bar{t}_{18} = \bar{t}_{26}$. (Note that $\mathrm{Q}H_*(\Omega X)$ has an $H_*(X)$-module structure which is induced from the $H_*(X)$-module Hopf algebra structure of $H_*(\Omega X)$ which is defined by the adjoint action. We also use the symbol $*$ to denote this structure.) Hence we have

$$a_8 * (\bar{t}_{18}\wp^1) = (a_8 * \bar{t}_{18})\wp^1 = \bar{t}_{26}\wp^1 = \pm\bar{t}_{22} \neq 0.$$

Thus we have $\bar{t}_{18}\wp^1 \neq 0$ and hence $\bar{t}_{18}\wp^1 = \pm\bar{t}_{14}$. This is converted into $a_{19}\wp^1 = \pm a_{15}$.

3. Proof of Proposition 2

Note that $\wp^1 x_{15} \in R^{19} = \langle x_{19}\rangle$.

3.1. **Case (i).** First, assume that $\wp^3 x_7 = 0$ and that $\wp^1 x_{15} \neq 0$. We will show a contradiction. We may assume that $\wp^1 x_{15} = x_{19}$ and then we have $a_{19}\wp^1 = a_{15}$.

By the same argument as that in Section 2, we know that $H_*(\Omega X)$ is as in (2.1) where $\sigma(t_j) = a_{j+1}$ for $j \neq 22$, $t_j \in \mathrm{P}H_*(\Omega X)$ for $j \neq 6, 18$, and $\bar{\phi}(t_6) = -t_2^2 \otimes t_2 - t_2 \otimes t_2^2$. Also we have $a_8 * t_2 = t_{10}$, $a_8 * t_6 = t_{14} - t_{10}t_2^2$, and $a_8^2 * t_{14} = -t_{10}^3 \neq 0$. Thus we have $0 \neq a_8 * t_{14} \in \mathrm{P}H_{22}(\Omega X)$ and we may assume that $a_8 * t_{14} = t_{22}$.

The relation $a_{19}\wp^1 = a_{15}$ is converted into $\bar{t}_{18}\wp^1 = \bar{t}_{14}$. It follows that

$$(a_8 * \bar{t}_{18})\wp^1 = a_8 * (\bar{t}_{18}\wp^1) = a_8 * \bar{t}_{14} = \bar{t}_{22}. \tag{3.1}$$

Thus, we have $a_8 * \bar{t}_{18} \neq 0$ and hence we may assume that $\bar{t}_{26} = a_8 * \bar{t}_{18}$.

Here we again consider $g\colon X\langle 3\rangle \to X$. The Serre spectral sequence implies that

$$H^*(X\langle 3\rangle) = \bigwedge(\hat{x}_{11}, \hat{x}_{15}, \hat{x}_{19}, \hat{x}_{27}, \hat{x}_{35}) \otimes \mathbb{F}_3[\hat{y}_{18}] \otimes \bigwedge(\hat{y}_{19}, \hat{y}_{23}) \tag{3.2}$$

where $\hat{x}_i = g^*(x_i)$. Further, by the Serre spectral sequence for the homotopy fibering $X\langle 3\rangle \xrightarrow{g} X \to K(\mathbb{Z}, 3)$, we may assume that $\beta\hat{y}_{18} = \hat{y}_{19}$ and that $\wp^1\hat{y}_{19} = \hat{y}_{23}$. By the homotopy fiberings $\Omega(X\langle 3\rangle) \to * \to X\langle 3\rangle$ and $S^1 \to \Omega(X\langle 3\rangle) \xrightarrow{\Omega g} \Omega X$, we can easily show that (3.1) implies $\wp^1\hat{y}_{23} = \pm\hat{x}_{27}$. It follows that

$$\pm\hat{x}_{27} = \wp^1\wp^1\beta\hat{y}_{18} = (-\beta\wp^1\wp^1 - \wp^1\beta\wp^1)\hat{y}_{18}$$

by the Adem relation. Since $\wp^1\hat{y}_{18} \in H^{22}(X\langle 3\rangle) = 0$, this is a contradiction.

3.2. **Case (ii).** Now, assume that $\wp^3 x_7 = 0$ and that $\wp^1 x_{15} = 0$. We can show a contradiction and thus, we have Proposition 2.

First, note that we have $\bar{\Delta}x_{11} = x_8 \otimes x_3$, $\bar{\Delta}x_{15} = x_8 \otimes x_7$, and $\wp^1 x_{11} = x_{15}$. Moreover, since $R^{23} = R^{31} = 0$, we have $\wp^1 x_{19} = \wp^3 x_{19} = 0$. We may put

$$\bar{\Delta}x_{19} = a x_8^2 \otimes x_3 + b x_8 \otimes x_{11} \qquad a, b \in \mathbb{F}_3.$$

Applying $\wp^1$, we have

$$0 = \bar{\Delta}\wp^1 x_{19} = a x_8^2 \otimes x_7 + b x_8 \otimes x_{15}.$$

Hence $a = b = 0$. So x_{19} is primitive. Similarly, we have $\wp^1 x_{27} = 0$ and we can show that $\bar{\Delta} x_{27} \in \langle x_8 \otimes x_{19} \rangle$.

Let i_k denote the fundamental class of $K(\mathbb{F}_3, k)$ and put $K(\mathbb{F}_3, k, l) = K(\mathbb{F}_3, k) \times K(\mathbb{F}_3, l)$. Let $w\colon K(\mathbb{F}_3, 19) \to K(\mathbb{F}_3, 23, 31)$ be defined by $w^*(i_{23}) = \wp^1 i_{19}$ and $w^*(i_{31}) = \wp^3 i_{19}$. Let $\alpha\colon K(\mathbb{F}_3, 23, 31) \to K(\mathbb{F}_3, 23, 47)$ be defined by $\alpha^*(i_{23}) = i_{23}$ and $\alpha^*(i_{47}) = \wp^4 i_{31}$ and put $w_1 = \alpha \circ w$. Let E and E_1 be the fibres of w and w_1, respectively. Then we have a homotopy commutative diagram

$$\begin{tikzcd}
 & & K(\mathbb{F}_3, 22, 30) \arrow[r, "\Omega\alpha"] \arrow[d, "j"'] & K(\mathbb{F}_3, 22, 46) \arrow[d, "j_1"] & \\
 & & E \arrow[r, "h"] \arrow[d, "q"'] & E_1 \arrow[r, "v_1"] \arrow[d, "q_1"] & K(\mathbb{F}_3, 58) \\
X\langle 3\rangle \arrow[r, "g"] & X \arrow[r, "f"'] \arrow[ur, "\tilde{f}"] & K(\mathbb{F}_3, 19) \arrow[r, equal] \arrow[d, "w"'] & K(\mathbb{F}_3, 19) \arrow[d, "w_1"] & \\
 & & K(\mathbb{F}_3, 23, 31) \arrow[r, "\alpha"] & K(\mathbb{F}_3, 23, 47) &
\end{tikzcd}$$

where the vertical sequences are homotopy fibre sequences, $f^*(i_{19}) = x_{19}$, and $\tilde{f}$ is a lift of f. There is an Adem relation

$$\wp^{10} = \wp^9 \wp^1 + \wp^3 \wp^4 \wp^3.$$

Hence there is an H-map $v_1\colon E_1 \to K(\mathrm{F}_3, 58)$ with $(v_1 \circ j_1)^*(i_{58}) = \wp^9 i_{22} + \wp^3 i_{46}$. By Hemmi [6], we have $[a_3(v_1)] = u_1 \otimes u_1 \otimes u_1 \in H^{57}(E_1 \wedge E_1 \wedge E_1)$ where $a_3(v_1)$ is the a_3-deviation of v_1 and $u_1 = q_1^*(i_{19})$.

Step 1. We will prove that $f \circ g$ is an a_3-map and that $\tilde{f}$ can be chosen so that $h \circ \tilde{f} \circ g$ is an H-map. Note that $[a_3(f)] \in H^{18}(X \wedge X \wedge X)$ must be a sum of terms involving x_3, x_7, x_8. Put $A = \langle x_3, x_7, x_8 \rangle$, which is a subspace of $H^*(X)$. Note that A is invariant under $\mathcal{A}(3)$ and that $g^*(A) = 0$. So $a_3(f \circ g) \simeq a_3(f) \circ (g \wedge g \wedge g) \simeq *$ and thus $f \circ g$ is an a_3-map.

Let $\gamma\colon X \wedge X \to K(\mathbb{F}_3, 22, 30)$ be a lift of $D_{\tilde{f}}\colon X \wedge X \to E$ where $D_{\tilde{f}}$ is the H-deviation of $\tilde{f}$. By Zabrodsky [20], $a_3(w \circ f) \simeq (\Omega w) \circ a_3(f)$ represents $(\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})[\gamma]$ under identifications

$$[X \wedge X, K(\mathbb{F}_3, 22, 30)] = H^{22}(X \wedge X) \oplus H^{30}(X \wedge X)$$

and

$$[X \wedge X \wedge X, \Omega K(\mathbb{F}_3, 23, 31)] = H^{22}(X \wedge X \wedge X) \oplus H^{30}(X \wedge X \wedge X).$$

Hence

$$\wp^1[a_3(f)] = (\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})\gamma^*(i_{22}),$$
$$\wp^3[a_3(f)] = (\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})\gamma^*(i_{30}).$$

We can check that $\wp^1[a_3(f)]$ must be a sum of terms involving x_7, x_7, x_8 and that $\wp^3[a_3(f)] = 0$. We can find an element

$$\xi \in \langle x_7 \otimes x_7x_8, x_7x_8 \otimes x_7, x_7 \otimes x_{15}, x_{15} \otimes x_7 \rangle$$

such that $(\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})(\xi) = \wp^1[a_3(f)]$. Thus $\gamma^*(i_{22}) - \xi$ and $\gamma^*(i_{30})$ lie in $\mathrm{Ker}(\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})$.

Here we think about $\mathrm{Cotor}^{2,j}_{H^*(X)}(\mathbb{F}_3, \mathbb{F}_3)$ for $j = 22, 30$. Recall that $\bar{\Delta} x_{27} \in \langle x_8 \otimes x_{19} \rangle$.

If x_{27} is primitive, we have $H^*(X) \cong C \otimes C' \otimes C''$ for $* < 35$ as a Hopf algebra over $\mathbb{F}_3$ where C, C', and C'' are Hopf algebras over $\mathbb{F}_3$ which are isomorphic to $H^*(F_4), H^*(S^{19})$, and $H^*(S^{27})$, respectively. We know the module structure of $\mathrm{Cotor}^{*,*}_{C}(\mathbb{F}_3, \mathbb{F}_3)$ by Kono-Mimura-Shimada [12]. Then we can easily show that

$$\begin{aligned} \mathrm{Cotor}^{2,22}_{H^*(X)}(\mathbb{F}_3, \mathbb{F}_3) &= \langle \{x_3 \otimes x_{19}\} \rangle, \\ \mathrm{Cotor}^{2,30}_{H^*(X)}(\mathbb{F}_3, \mathbb{F}_3) &= \langle \{x_3 \otimes x_{27}\} \rangle. \end{aligned} \tag{3.3}$$

If x_{27} is not primitive, then we have $H^*(X) \cong H^*(E_7)$ for $* < 35$ as a Hopf algebra over $\mathbb{F}_3$. We know the module structure of $\mathrm{Cotor}^{*,*}_{H^*(E_7)}(\mathbb{F}_3, \mathbb{F}_3)$ by Mimura-Sambe [18]. Then we can easily show that $\mathrm{Cotor}^{2,22}_{H^*(X)}(\mathbb{F}_3, \mathbb{F}_3)$ is as in (3.3) and $\mathrm{Cotor}^{2,30}_{H^*(X)}(\mathbb{F}_3, \mathbb{F}_3) = \langle \{\eta\} \rangle$ where

$$\eta = x_3 \otimes x_{27} - x_{19} \otimes x_{11} + x_3x_8 \otimes x_{19} + x_8 \otimes x_3x_{19} - x_8x_{19} \otimes x_3.$$

Thus we may put $\gamma^*(i_{22})$ as $\gamma^*(i_{22}) = \xi + \varepsilon x_3 \otimes x_{19} + l$ and $\gamma^*(i_{30})$ either as $\gamma^*(i_{30}) = \varepsilon' x_3 \otimes x_{27} + l'$ or as $\gamma^*(i_{30}) = \varepsilon' \eta + l'$ where $\varepsilon, \varepsilon' \in \mathbb{F}_3$ and $l, l' \in \mathrm{Im}\bar{\Delta}$. As explained in [15], we can alter γ so that $l = l' = 0$. Note that $\wp^4(x_{19} \otimes x_{11}) = \wp^4(x_3x_8 \otimes x_{19}) = 0$ and hence $\wp^4\eta \in A \otimes H^*(X) + H^*(X) \otimes A$. (In fact, $\wp^4\eta = 0$. Recall that R is invariant under $\mathcal{A}(3)$.) It follows that $((\Omega\alpha) \circ \gamma)^*(i_{22}) = \gamma^*(i_{22})$ and $((\Omega\alpha) \circ \gamma)^*(i_{46}) = \wp^4\gamma^*(i_{30})$ lie in $A \otimes H^*(X) + H^*(X) \otimes A$. Hence $D_{h\circ\tilde{f}\circ g} \simeq h \circ D_{\tilde{f}} \circ (g \wedge g) \simeq h \circ j \circ \gamma \circ (g \wedge g) \simeq j_1 \circ (\Omega\alpha) \circ \gamma \circ (g \wedge g) \simeq *$ and thus $h \circ \tilde{f} \circ g$ is an H-map. This proves Step 1.

Step 2. We will show a contradiction. We know that $H^*(X\langle 3 \rangle)$ is as in (3.2). By Step 1, $h \circ \tilde{f} \circ g$ is an H-map, so $(v_1 \circ h \circ \tilde{f} \circ g)^*(i_{58}) \in \mathrm{P}H^{58}(X\langle 3 \rangle) = 0$. It follows that

$$\begin{aligned} [a_3(v_1 \circ h \circ \tilde{f} \circ g)] &= [a_3(v_1) \circ ((h \circ \tilde{f} \circ g) \wedge (h \circ \tilde{f} \circ g) \wedge (h \circ \tilde{f} \circ g))] \\ &\quad + [(\Omega v_1) \circ a_3(h \circ \tilde{f} \circ g)] \\ &= \hat{x}_{19} \otimes \hat{x}_{19} \otimes \hat{x}_{19} + \wp^9 y + \wp^3 y' \\ &= (\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})(z) \\ &\in H^{57}(X\langle 3 \rangle \wedge X\langle 3 \rangle \wedge X\langle 3 \rangle) \end{aligned} \tag{3.4}$$

for some $y, y' \in H^*(X\langle 3 \rangle \wedge X\langle 3 \rangle \wedge X\langle 3 \rangle)$ and $z \in H^*(X\langle 3 \rangle \wedge X\langle 3 \rangle)$ since $f \circ g$ is an a_3-map. Checking degrees, we have $y = 0$. Since we assume that $\wp^1 x_{15} = 0$, we have $\wp^1 \hat{x}_{15} = 0$. Let $H^{19,19,19}$ be the summand of $H^{57}(X\langle 3 \rangle \wedge X\langle 3 \rangle \wedge X\langle 3 \rangle)$ which corresponds to $H^{19}(X\langle 3 \rangle) \otimes H^{19}(X\langle 3 \rangle) \otimes H^{19}(X\langle 3 \rangle)$, and let $\pi\colon H^{57}(X\langle 3 \rangle \wedge X\langle 3 \rangle \wedge X\langle 3 \rangle) \to H^{19,19,19}$ be the natural projection. Then, we can easily check that $\pi(\wp^3 H^{45}(X\langle 3 \rangle \wedge X\langle 3 \rangle \wedge X\langle 3 \rangle)) = 0$. Accordingly, (3.4) implies that

$$\hat{x}_{19} \otimes \hat{x}_{19} \otimes \hat{x}_{19} = \pi(\bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta})(z).$$

If $t \in \mathrm{P}H_{19}(X\langle 3\rangle)$ with $\langle t, \hat{x}_{19}\rangle \neq 0$, then

$$\begin{aligned} 0 &\neq \langle t\otimes t\otimes t, \hat{x}_{19}\otimes \hat{x}_{19}\otimes \hat{x}_{19}\rangle \\ &= \langle t\otimes t\otimes t, \pi(\bar{\Delta}\otimes 1 - 1\otimes \bar{\Delta})(z)\rangle \\ &= \langle t\otimes t\otimes t, (\bar{\Delta}\otimes 1 - 1\otimes \bar{\Delta})(z)\rangle \\ &= \langle t^2\otimes t - t\otimes t^2, z\rangle. \end{aligned}$$

So $t^2 \in \mathrm{P}H_{38}(X\langle 3\rangle)$ is nonzero. This is a contradiction.

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