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PHELPS' LEMMA, DANEŠ' DROP THEOREM AND EKELAND'S PRINCIPLE IN LOCALLY CONVEX SPACES

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Abstract. A generalization of Phelps' lemma to locally convex spaces is proven, applying its well-known Banach space version. We show the equivalence of this theorem, Ekeland's principle and Daneš' drop theorem in locally convex spaces to their Banach space counterparts and to a Pareto efficiency theorem due to Isac. This solves a problem, concerning the drop theorem, proposed by G. Isac in 1997.

We show that a different formulation of Ekeland's principle in locally convex spaces, using a family of topology generating seminorms as perturbation functions rather than a single (in general discontinuous) Minkowski functional, turns out to be equivalent to the original version.

1. Introduction

The famous 1972 Ekeland theorem called the "Variational Principle" is one of the most frequently applied results of nonlinear functional analysis. For example, the progress of variational calculus and optimal control over the last 25 years is unthinkable without Ekeland's principle.

Around the same time, but independently of each other, a list of theorems was discovered in which all of the theorems characterize the completeness of the underlying space and are equivalent to Ekeland's principle in metric spaces, namely the Krasnosel'skii-Zabrjeko theorem on normal solvability of operator equations [23], the Kirk-Caristi fixed point theorem [4], and the Daneš drop theorem [6].

Over the last three decades, a great deal of effort has gone into looking for another equivalent formulation or generalization of Ekeland's principle; see e.g. [18], [10], [16], [17].

In this paper we continue this effort by showing that a 1974 result of Phelps (Lemma 1.2 of [20] with precursor Lemma 1 in [2], called Phelps' lemma in the following) can be generalized to sequentially closed sets of locally convex spaces, that the locally convex variant can be proven using only the corresponding Banach space version and that this procedure is applicable for proving Ekeland's principle and the drop theorem in locally convex spaces.

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Additionally we show the equivalence of these theorems with each other, surprisingly to their Banach space counterparts and to a second variant of Ekeland's principle due to Fang [9] specialized to locally convex spaces.

Finally, solving a problem stated by G. Isac in [14], we point out the equivalence of the drop theorem in locally convex spaces to one of Isac's theorems which relates Ekeland's principle to Pareto efficiency.

We close the paper with some conclusions concerning the relationships of different versions of the represented theorems. The appendix contains the Banach space versions of Phelps' lemma, Ekeland's principle and Daneš' drop theorem as well as a separation theorem.

2. Preparatory results

Let us start with some notational arrangements. Let X be a real vector space. A functional $p: X \to (-\infty, +\infty)$ is called a seminorm if it satisfies

(i) $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$, $\alpha \in (-\infty, +\infty)$,

(ii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Following Treves [22] a pair $\left(X, \{p_i\}_{i \in I}\right)$ is said to be a locally convex topological vector space (short: locally convex space) if $\{p_i\}_{i \in I}$ is a family of seminorms satisfying

(iii) $i \in I$, $\alpha > 0$ implies the existence of $j \in I$ such that $p_j = \alpha p_i$,

(iv) $i \in I$, q is a seminorm such that $q(x) \leq p_i(x)$ for every $x \in X$ implies the existence of $j \in I$ such that $p_j = q$,

(v) $i_1, i_2 \in I$ implies the existence of $j \in I$ such that $p_j = \sup\{p_{i_1}, p_{i_2}\}$.

The locally convex space X is said to be Hausdorff (or separated) if

(vi) $x, y \in X$ and $x \neq y$ implies the existence of $j \in I$ such that $p_j(x) \neq p_j(y)$.

The family $\{p_i\}_{i \in I}$ of seminorms satisfying (iii) - (v) is called the spectrum of X and is denoted by *spec* (X). The set *spec* (X) generates a topology on X such that addition and multiplication by scalars are continous operations and $p \in \mathit{spec}(X)$ if and only if p is uniformly continuous on X. A subfamily $\{p_\lambda\}_{\lambda \in \Lambda}$ of *spec* (X) is said to be a base of continuous seminorms if for every $p \in \mathit{spec}(X)$ on X there are $\alpha > 0$ and p_λ, $\lambda \in \Lambda$, such that $p(x) \leq \alpha p_\lambda(x)$ for every $x \in X$. A Hausdorff locally convex topological vector space always has a base of continuous seminorms. It can be constructed in the following way. Starting with $\mathit{spec}(X) = \{p_i\}_{i \in I}$ we define $\Lambda = \{\lambda \subset I, \lambda \text{ finite}\}$ and

$$p_\lambda(x) = \sup_{j \in \lambda} p_j(x) \quad \text{for } x \in X,\ \lambda \in \Lambda.$$

As usual, we define the Minkowski functional of a set $A \subset X$ to be

$$\mu_A(x) := \inf\{\alpha > 0:\ x \in \alpha A\}.$$

The main idea of our proofs is to reduce the situation to the Banach space case. We present some preparatory results containing the core of the method.

Proposition 1. *Let X be a locally convex Hausdorff space and $T \subset X$ a convex balanced and bounded set. Then the Minkowski functional $\mu_T(\cdot)$ defines a norm on the linear space $\mathit{span}\, T$.*

Proof. Since μ_T is positive homogeneous and T balanced we may conclude $\mu_T(\alpha x) = |\alpha| \mu_T(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathit{span}\, T$. The convexity of T implies the convexity of μ_T. This fact, together with the positive homogeneity of μ_T, implies the validity of the triangle inequality for μ_T. Obviously, $\mu_T(0) = 0$. It remains to show that $\mu_T(x) = 0$ implies $x = 0$. Since T is bounded in X, for every neighborhood U of $0 \in X$ there exists $\sigma > 0$ such that $\frac{1}{\sigma}T \subset U$. Since $\mu_T(x) = 0$ we have $x \in \frac{1}{\sigma}T$ for each $\sigma > 0$, hence $x \in U$ for every neighborhood U of $0 \in X$ which implies $x = 0$. $\square$

If the assumptions of Proposition 1 are satisfied we denote the normed space $(span\, T, \mu_T)$ by $(X_T, \|\cdot\|_T)$.

Proposition 2. *Let X be a locally convex Hausdorff space and $T \subset X$ a sequentially complete convex balanced and bounded subset of X. Then $(X_T, \|\cdot\|_T)$ is a Banach space.*

Proof. It remains to show that $(X_T, \|\cdot\|_T)$ is complete with respect to the topology induced by $\|\cdot\|_T$.

First, we show that T is complete with respect to $\|\cdot\|_T$. Let $\{x_n\} \subset T$ be a Cauchy sequence with respect to $\|\cdot\|_T$. Then it is Cauchy with respect to the locally convex topology: Because of the boundedness of T for every $i \in I$ there exists $\alpha_i > 0$ such that $T \subset \alpha_i U_i$ where $U_i := \{x \in X : \ p_i(x) \leq 1\}$ and

$$\frac{1}{\alpha_i} p_i(x_m - x_n) = \frac{1}{\alpha_i} \mu_{U_i}(x_m - x_n) \leq \mu_T(x_m - x_n).$$

Hence there exists $x \in X$ such that $x_n \to x$ in X. Moreover, $x \in T$ since T is sequentially closed in X. Next, we shall show $x_n \to x$ with respect to $\|\cdot\|_T$. Fix $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy in X_T we have $x_m - x_n \in \varepsilon T$ for all m, n sufficiently large. Letting $n \to \infty$ we conclude $x_m - x \in \varepsilon T$ for all m sufficiently large since T is sequentially closed in X. This implies

$$\mu_T(x_m - x) = \|x_m - x\|_T \leq \varepsilon$$

for all m sufficiently large which gives $x_n \to x$ with respect to $\|\cdot\|_T$.

Finally, we show that an arbitrary Cauchy sequence $\{x_n\} \subset X_T$ has a limit point $y \in X_T$ with respect to $\|\cdot\|_T$. Note that $\{\|x_n\|_T\} \subset \mathbb{R}$ is Cauchy since

$$|\|x_n\|_T - \|x_m\|_T| \leq \|x_n - x_m\|_T,$$

hence convergent to some $\alpha \in \mathbb{R}$. Assuming $\|x_n\|_T \geq 1$ for all n we claim that $\left\{\frac{x_n}{\|x_n\|_T}\right\}$ is Cauchy in T with respect to $\|\cdot\|_T$. This follows from

$$\begin{aligned}
\left\| \frac{x_n}{\|x_n\|_T} - \frac{x_m}{\|x_m\|_T} \right\|_T &= \left\| \frac{x_n}{\|x_n\|_T} - \frac{x_n}{\|x_m\|_T} + \frac{x_n}{\|x_m\|_T} - \frac{x_m}{\|x_m\|_T} \right\|_T \\
&\leq \left(\frac{1}{\|x_n\|_T} - \frac{1}{\|x_m\|_T} \right) \|x_n\|_T + \frac{1}{\|x_m\|_T} \|x_n - x_m\|_T \\
&\leq \left(1 - \frac{\|x_n\|_T}{\|x_m\|_T} \right) + \|x_n - x_m\|_T \\
&\leq \frac{1}{\|x_m\|_T} (\|x_m\|_T - \|x_n\|_T) + \|x_n - x_m\|_T \\
&\leq 2\|x_n - x_m\|_T.
\end{aligned}$$

Because T is complete we conclude that $\frac{x_n}{\|x_n\|_T} \to z \in T$. Moreover, we have $x_n \to \alpha z \in X_T$ with respect to $\|\cdot\|_T$ because

$$\begin{aligned} \|x_n - \alpha z\|_T &\leq \|(x_n - \|x_n\|_T z) + (\|x_n\|_T z - \alpha z)\|_T \\ &\leq \|(x_n - \|x_n\|_T z)\|_T + \|(\|x_n\|_T z - \alpha z)\|_T \\ &\leq \|x_n\|_T \cdot \left\| \frac{x_n}{\|x_n\|_T} - z \right\|_T + (\|x_n\|_T - \alpha) \cdot \|z\|_T . \end{aligned}$$

Since $\|x_n\|_T \to \alpha$, the very right-hand side of the last inequality chain tends to zero, which completes the proof of the proposition. □

Proposition 3. *Let X be a locally convex Hausdorff space and $M \subset X$ a sequentially closed subset of X. Then $M \cap X_T$ is closed with respect to $\|\cdot\|_T$.*

Proof. Let $\{x_n\} \subset M \cap X_T$ be a convergent sequence with respect to $\|\cdot\|_T$. Then it is Cauchy and the completeness of $(X_T, \|\cdot\|_T)$ implies $x \in X_T$ for the limit point x. The sequence $\{x_n\}$ is Cauchy with respect to the locally convex topology as well (see the proof of Proposition 2), hence convergent in X to the same limit point x. Since M is sequentially closed in X we have $x \in M \cap X_T$. □

For a given set $B \subset X$ we define the cone

$$K := \mathbb{R}_+ B = \{x \in X : \exists \alpha \geq 0, b \in B \text{ such that } x = \alpha b\}.$$

By $\operatorname{cl} B$ we denote the closure of the set $B \subset X$ with respect to the locally convex topology.

Proposition 4. *Let X be a locally convex Hausdorff space and $T \subset X$ a sequentially complete convex balanced and bounded subset of X. Let $B \subset T$ be a sequentially closed bounded and convex subset of T such that $0 \notin B$. If $M \subset K$ is bounded with respect to the locally convex topology in X, then M is bounded in X_T with respect to the Banach space topology.*

Proof. Let $x \in M$, $x \neq 0$, be an arbitrary element of M. Since $x \in K$ there exist $t > 0$ and $b \in B$ such that $x = t \cdot b$. Since $0 \notin B$ there exist $\delta > 0$, $\mu \in \Lambda$ such that $p_\mu(b) \geq \delta$. Because M is bounded in X there exists a constant $c = c_\mu > 0$ such that $p_\mu(y) \leq c$ for all $y \in M$. Hence

$$t = \frac{p_\mu(x)}{p_\mu(b)} \leq \frac{c}{\delta}.$$

$B \subset T$ implies $p_\mu(b) \leq 1$ and we may conclude

$$\|x\|_T = t \|b\|_T \leq t \leq \frac{c}{\delta}.$$

Since $x \in M$ is arbitrary the proposition is proven. □

3. Phelps' lemma in locally convex spaces

In this section we shall prove Phelps' lemma in locally convex spaces. The following theorem is a generalization of Lemma 1.2 in [20] (see Theorem 11 of the appendix) to locally convex spaces.

Theorem 1. *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $M \subset X$ be a nonempty sequentially closed subset of X, and $B \subset X$ a nonempty sequentially closed bounded and convex set such that $0 \notin cl B$.*

Then, for each $x_0 \in X$ such that $M \cap (x_0 + K)$ is bounded and nonempty there exists $x^* \in X$ such that

$$x^* \in M \cap (x_0 + K) \quad \text{and} \quad \{x^*\} = M \cap (x^* + K).$$

Proof. Define the set

$$T := s\text{-}cl\, co \{B \cup -B \cup \{\pm x_0\}\}$$

where *s-cl co* denotes the sequential closure of the convex hull, and let $X_T = span\, T$. Then $(X_T, \|\cdot\|_T)$ is a Banach space according to Proposition 2. Define $M_T := M \cap X_T$ which is nonempty and closed in the Banach space topology as well as B because of Proposition 3. Moreover, $x_0 \in M_T$, $K \subset X_T$ and $M_T \cap (x_0 + K)$ is bounded in X_T.

Phelps' lemma in Banach spaces (Theorem 11 of the appendix) yields a point $x^* \in X_T$ such that

$$x^* \in M_T \cap (x_0 + K) \quad \text{and} \quad \{x^*\} = M_T \cap (x^* + K).$$

We claim that even $\{x^*\} = M \cap (x^* + K)$. This is true because $(x^* + K) \subset X_T$. □

Phelps' lemma in Banach spaces turns out to be equivalent to Ekeland's variational principle in metric spaces as it has been proven by Georgiev [10] and later by Attouch/Riahi [1]; see also [21]. Moreover, it is equivalent to Daneš' drop theorem and a long list of theorems all equivalent to Ekeland's principle as well; see [18], [10], [16], for example. The situation remains unchanged in locally convex spaces. Concerning the drop theorem and "linear" variants of Ekeland's principle we show these equivalences in the next sections. The relations for the "nonlinear" case (in uniform spaces) can be found in [15], [13].

4. Ekeland's principle in locally convex spaces

In this section we shall present two versions of Ekeland's principle in locally convex spaces which turn out to be equivalent.

In spite of their equivalence there is also a significant difference. Phelps' classical approach from the beginning of the 1960's uses a Minkowski gauge discontinuous in general of a bounded set as a perturbation function.

The recent approach inspired by Fang's paper [9] employs not a single perturbation function but a family of uniformly continuous seminorms.

We start with "the grandfather of it all" (Ekeland [8]), the following theorem from Phelps [19]. Note that our formulation is slightly more general than the original version due to the fact that we refer to sequential completeness and sequential lower semicontinuity rather than completeness and lower semicontinuity in the sense of the locally convex topology.

Theorem 2. *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $f : X \to (-\infty, +\infty]$ be an extended-valued proper sequentially lower semicontinuous function, bounded from below. Let $S \subset X$ be a sequentially closed bounded and convex set such that $0 \in S$. Then, for each $\gamma > 0$, $x_0 \in dom f$ there exists $x^* \in X$ such that*

(1) $$f(x^*) + \gamma \mu_S (x^* - x_0) \leq f(x_0),$$

and for all $x \in X$, $x \neq x^*$ *we have*

$$f(x^*) < f(x) + \gamma \mu_S (x - x^*) . \tag{2}$$

Proof. Define the set

$$T := s\text{-}cl\, co \{S \cup -S \cup \{\pm x_0\}\} .$$

Let $X_T = span\, T$. Then $(X_T, \|\cdot\|_T)$ is a Banach space according to Proposition 2. We define the set

$$C := \{x \in X_T : \ f(x) + \gamma \mu_S (x - x_0) \leq f(x_0)\} ,$$

which is closed in X_T, and the function

$$g(x) := \begin{cases} f(x) & : \quad x \in C, \\ +\infty & : \quad \text{otherwise}, \end{cases}$$

which is lower semicontinuous with respect to the Banach space topology.

We can apply Ekeland's variational principle in Banach spaces (Theorem 10 of the appendix), getting a point $x^* \in C$ such that for $x \in X_T$, $x \neq x^*$, we have

$$g(x^*) < g(x) + \gamma \|x - x^*\|_T . \tag{3}$$

We shall consider different cases.

First, if $x \in C$ inequality (3) implies

$$f(x^*) < f(x) + \gamma \mu_S (x - x^*)$$

since, because $S \subset T$, $\|y\|_T = \mu_T(y) \leq \mu_S(y)$.

Second, if $x \in X_T \backslash C$ we have

$$f(x) + \gamma \mu_S (x - x_0) > f(x_0)$$

as well as

$$f(x^*) + \gamma \mu_S (x^* - x_0) \leq f(x_0) ,$$

hence

$$\begin{aligned} f(x^*) + \gamma \mu_S (x^* - x_0) &< f(x) + \gamma \mu_S (x - x_0) \\ &\leq f(x) + \gamma [\mu_S (x^* - x_0) + \mu_S (x - x^*)] \end{aligned}$$

since μ_S is positive homogenous and convex, and hence it satisfies the triangle inequality. Note that $\mu_S (x^* - x_0) < +\infty$ and the last inequality is valid even if $\mu_S (x - x_0) = +\infty$. We may conclude that (2) is true.

Third, if $x \in X \backslash X_T$ inequality (2) is trivially satisfied because $\mu_S(y) = +\infty$ for all $y \in X \backslash X_T$. □

A Banach space version of Theorem 2 is already given in Example 2 of [3]. The Minkowski functional of a bounded set in locally convex spaces is discontinuous in general. A recent result of Fang [9] gives rise to replace the possibly discontinuous perturbation function μ_S by uniformly continuous seminorms. The price we have to pay is that we cannot use a single perturbation function except the family $\{p_\lambda\}_{\lambda \in \Lambda}$. The result reads as follows.

Theorem 3. *Let* X *be a Hausdorff sequentially complete locally convex topological vector space. Let* $f : \ X \to (-\infty, +\infty]$ *be an extended-valued proper and sequentially lower semicontinuous function, bounded from below. Let* $\{p_\lambda\}_{\lambda \in \Lambda}$ *be a base* *of continuous seminorms generating the topology on X and $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ a family of positive numbers. Then, for every $x_0 \in \mathrm{dom} f$ there exists $x^* \in X$ such that*

$$f(x^*) + \gamma_\lambda p_\lambda(x^* - x_0) \le f(x_0) \tag{4}$$

for all $\lambda \in \Lambda$, and for all $x \neq x^*$ there exists $\mu \in \Lambda$ such that

$$f(x^*) < f(x) + \gamma_\mu p_\mu(x - x^*). \tag{5}$$

Moreover, we shall prove the following theorem.

Theorem 4. *Theorem 1 and Theorem 3 are mutually equivalent.*

Proof. First we show that Theorem 1 implies Theorem 3. In order to do this we define the locally convex space $Y = X \times \mathbb{R}$ where the elements of the spectrum are given by $q_i((x,r)) = \max\{p_i(x), |r|\}$ for $(x,r) \in Y$, $i \in I$. Let $M = \mathrm{epi} f$ and

$$B := \{(x,-1) \in Y : \ \gamma_\lambda p_\lambda(x) \le 1 \text{ for all } \lambda \in \Lambda\}.$$

Then M is nonempty and sequentially closed since f is proper and sequentially lower semicontinuous and B is nonempty closed convex and bounded. We note that the cone $K = \mathbb{R}_+ B$ is given by

$$K = \{(y,s) \in Y : \ s + \gamma_\lambda p_\lambda(y) \le 0 \text{ for all } \lambda \in \Lambda\}.$$

Let $(x_0, f(x_0))$ be an element of M and $m = \inf f$. Then $M_0 = M \cap ((x_0, f(x_0)) + K)$ is bounded because for $(x,r) \in M_0$ and for each $\lambda \in \Lambda$ we have

$$m - f(x_0) + \gamma_\lambda p_\lambda(x - x_0) \le r - f(x_0) + \gamma_\lambda p_\lambda(x - x_0) \le 0,$$

hence

$$p_\lambda(x - x_0) \le \frac{f(x_0) - m}{\gamma_\lambda}.$$

Applying Theorem 1 to Y, M and K as defined above we get $(x^*, r^*) \in M_0$ such that

$$\{(x^*, r^*)\} = M \cap ((x^*, r^*) + K).$$

It is an easy task to derive (4) and (5) from these relationships.

Now, we prove the converse, namely Theorem 3 implies Theorem 1 returning to the notation of Theorem 1. The first step is to prove the existence of a linear continuous functional $l \in X^*$ and numbers $0 < \gamma_\lambda < 1$ such that

$$K \subset K_\Lambda := \{x \in X : \ \gamma_\lambda p_\lambda(x) \le l(x) \text{ for all } \lambda \in \Lambda\}.$$

Since $0 \notin \mathrm{cl}\, B$ there exists a neighborhood U of $0 \in X$ such that $U \cap B = \emptyset$. The separation theorem (Theorem 13 of the appendix) yields $l \in X^*$, $\|l\| = 1$ such that

$$\delta := \sup l(U) \le \inf l(B).$$

Since B is bounded there exists for each $\lambda \in \Lambda$ a constant $\eta_\lambda > 0$ such that $p_\lambda(x) \le \eta_\lambda$ for all $x \in B$. Then

$$\frac{\delta}{\eta_\lambda} p_\lambda(x) \le \delta \le l(x) \quad \text{for all } x \in B, \ \lambda \in \Lambda,$$

hence taking $\gamma_\lambda = \frac{\delta}{\eta_\lambda}$ we see that

$$\gamma_\lambda p_\lambda(x) \le l(x) \quad \text{for all } x \in B, \ \lambda \in \Lambda$$

which means $B \subset K_\Lambda$. For the next step, we define the function $f = l + \chi_{M_0}$ where $M_0 = M \cap (x_0 + K)$ and χ_{M_0} denotes the indicator function of the set M_0. Then f

is bounded from below and sequentially lower semicontinuous. Applying Theorem 3 we get $x^* \in M_0$ such that

$$l(x^*) + \gamma_\lambda p_\lambda (x^* - x_0) \leq l(x_0) \quad \text{for all } \lambda \in \Lambda, \tag{6}$$

and for all $x \in M_0$, $x \neq x^*$ there exists $\mu \in \Lambda$ such that

$$l(x^*) < l(x) + \gamma_\mu p_\mu (x - x^*) . \tag{7}$$

Note that since $x^* \in M_0$ we have $x^* \in x_0 + K$. Next, we claim $\{x^*\} = M_0 \cap (x^* + K)$. This is true since from (7) we may conclude for $x \in M_0$, $x \neq x^*$,

$$l(x - x^*) + \gamma_\mu p_\mu (x - x^*) > 0$$

which implies $x \notin x^* + K_\Lambda$, all the more $x \notin x^* + K$. At last we show by contradiction $\{x^*\} = M \cap (x^* + K)$. Assume there exists $z \in M \cap (x^* + K)$, $z \notin M_0 \cap (x^* + K) = M \cap (x_0 + K) \cap (x^* + K)$. This would imply $z \notin x_0 + K$. Since $x^* \in x_0 + K$ we have $z \in x^* + K \subset x_0 + K + K = x_0 + K$, a contradiction. The proof is complete. □

Almost the same proof can be used to show the equivalence of Phelps' lemma and the variational principle with Minkowski functional.

Theorem 5. *Theorem 1 and Theorem 2 are mutually equivalent.*

Proof. We indicate the minor modifications of the proof of Theorem 4.

First, let Theorem 1 be valid. Define $Y = X \times \mathbb{R}$, $M = \operatorname{epi} f$, $B = \{(x, -1) \in Y \colon \gamma\mu_S(x) \leq 1\}$ and $K = \mathbb{R}_+ B = \{(y, s) \in Y :\ s + \gamma\mu_S(y) \leq 0\}$. Then the set $M_0 = M \cap ((x_0, f(x_0)) + K)$ is bounded since S is bounded. The remaining part of the proof goes along the lines of the first part of the proof of Theorem 4.

Second, let Theorem 2 be true. Define $S = \{y \in X :\ y = \alpha x,\ 0 \leq \alpha \leq 1,\ x \in B\} = co\{\{0\}, B\}$. Find by means of the separation theorem (Theorem 13 of the appendix) $l \in X^*$ and $0 < \gamma < 1$ such that $K \subset \{x \in X :\ \gamma\mu_S(x) \leq l(x)\}$ and proceed as in the second part of the proof of Theorem 4. □

Since both versions of Ekeland's principle in locally convex spaces are equivalent to Phelps' lemma they are equivalent to each other.

Theorem 6. *Theorem 3 and Theorem 2 are mutually equivalent.*

Phelps' lemma in the product space $X \times \mathbb{R}$ is often called the maximal (or minimal) point lemma because of the underlying ordering argument; see [21]. Very general versions of the maximal point lemma (in spaces $X \times Z$, X a complete metric space, Z a locally convex Hausdorff space) can be found for example in [11], [12].

Note that it is also possible to derive Theorem 3 from Fang's Theorem 3.2 in [9].

5. The drop theorem in locally convex spaces

Mizoguchi [15] as well as Cheng et al. [5] extended Daneš' drop theorem to locally convex spaces. We shall give a reformulation of Cheng's version which is at the same time an improvement of Mizoguchi's result. We prove it only using the drop theorem in Banach spaces (Theorem 12 of the appendix).

The drop $D(x, B)$ generated by $x \in X$ and $B \subset X$ is defined to be the set

$$D(x, B) := co\{\{x\}, B\} = \{tx + (1 - t) b :\ b \in B, t \in [0, 1]\} .$$

Note that the simple definition of drops does not require any topological notion. The crucial point in formulating the drop theorem is to say when two closed sets are of positive distance. Here we use a single element of the base of seminorms to make available a substitute for the distance of sets in metric spaces.

Theorem 7. *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $A \subset X$ be a nonempty sequentially closed set and $B \subset X$ a nonempty sequentially closed bounded and convex set. Let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms on X and $\mu \in \Lambda$, $\delta > 0$ such that*

$$\forall a \in A, b \in B : \quad p_\mu (a - b) \geq \delta. \tag{8}$$

Then, for each $x_0 \in A$ there exists $x^* \in X$ such that

$$x^* \in A \cap D (x_0, B) \quad \text{and} \quad \{x^*\} = A \cap D (x^*, B).$$

Proof. Without loss of generality we may assume $0 \in B$. Define the set

$$T := s\text{-}cl\, co \{B \cup -B \cup \{\pm x_0\}\}$$

and let $X_T = span\, T$. Then T is sequentially complete convex balanced and bounded, hence $(X_T, \|\cdot\|_T = \mu_T (\cdot))$ is a Banach space; cf. Proposition 2. Define $A_T := A \cap X_T$ which is nonempty and closed in the Banach space topology as well as B.

We shall apply the drop theorem in Banach spaces, i.e. Theorem 12 of the appendix. This is possible because the sets A_T and B are of positive distance in X_T: Since T is bounded there exists $\alpha > 0$ such that $T \subset \alpha U$ where $U = \{x \in X : p_\mu (x) \leq 1\}$. This implies $p_\mu (x) \leq \mu_{\alpha U} (x) \leq \mu_T (x)$ for all $x \in X$. Hence

$$\forall a \in A, b \in B : \quad \mu_T (a - b) \geq \frac{1}{\alpha}\mu_U (a - b) \geq \frac{\delta}{\alpha}$$

and it follows that

$$\forall a \in A_T, b \in B : \quad \|a - b\|_T := \mu_T (a - b) \geq \frac{\delta}{\alpha} > 0.$$

According to the drop theorem in Banach spaces there exists $x^* \in X_T$ such that

$$x^* \in A_T \cap D (x_0, B) \quad \text{and} \quad \{x^*\} = A_T \cap D (x^*, B).$$

We claim that even $\{x^*\} = A \cap D (x^*, B)$. This is true because $D (x^*, B) \subset X_T$. $\square$

Next, we clarify the relation of the above theorem to Cheng's variant in [5]. Cheng et al. introduced the following definition.

Definition 1. Two nonempty subsets of the locally convex space X are said to be strongly Minkowski separated iff there exists a seminorm $p \in spec (X)$ and $z \in X$ such that either

$$\inf \{p (a + z) : \ a \in A\} > \sup \{p (b + z) : \ b \in B\} \tag{9}$$

or

$$\sup \{p (a + z) : \ a \in A\} < \inf \{p (b + z) : \ b \in B\}. \tag{10}$$

The following lemma is an extension of Proposition 2 of [5]. It shows that our Theorem 7 is equivalent to Cheng's version.

Lemma 1. *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $A \subset X$ be a nonempty sequentially closed set and $B \subset X$ a nonempty sequentially closed bounded and convex set. Then A and B are strongly Minkowski separated if and only if relation (8) is satisfied for some $\mu \in \Lambda$, $\delta > 0$.*

Proof. First, let (8) be satisfied for some $\mu \in \Lambda$, $\delta > 0$. Define

$$\eta = \inf\{p_\mu(a-b) : a \in A, b \in B\} \geq \delta,$$

$$S = \left\{s \in X : \inf\{p_\mu(s-a) : a \in A\} \leq \frac{\eta}{2}\right\}.$$

Then $\operatorname{int} S \neq \emptyset$ and

$$\inf\{p_\mu(s-b) : s \in S, b \in B\} \geq \frac{\eta}{2} \tag{11}$$

since

$$p_\mu(s-b) = p_\mu((a-b)-(a-s)) \geq p_\mu(a-b) - p_\mu(a-s) \geq \frac{\eta}{2}.$$

Without loss of generality we may assume $0 \in \operatorname{int} S$. To complete the proof it suffices to show

$$r = \inf\{\mu_S(b) - \mu_S(s) : b \in B,\ s \in S\} > 0.$$

Suppose the contrary, i.e. $r = 0$. Then there exist, for each $\varepsilon > 0$, $b_\varepsilon \in B$ and $s_\varepsilon \in S$ such that $0 < \mu_S(b_\varepsilon) - \mu_S(s_\varepsilon) < \varepsilon$. Since $\mu_S(b_\varepsilon) > 1$, $\mu_S(s_\varepsilon) \leq 1$ we get $1 < \mu_S(b_\varepsilon) \leq 1 + \varepsilon$, hence $\mu_S(b_\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\beta_\varepsilon = \mu_S(b_\varepsilon)^{-1}$. Then $\mu_S(\beta_\varepsilon b_\varepsilon) = 1$, i.e. $\beta_\varepsilon b_\varepsilon \in S$. Since B is bounded we have

$$p_\mu(b_\varepsilon - \beta_\varepsilon b_\varepsilon) = (1-\beta_\varepsilon)\, p_\mu(b_\varepsilon) \to 0$$

as $\varepsilon \to 0$. This contradicts (11) since $b_\varepsilon \in B$, $\beta_\varepsilon b_\varepsilon \in S$.

Second, assume that A and B are strongly Minkowski separated. Let us assume (9). Define

$$\eta := \inf\{p(a+z) : a \in A\} - \sup\{p(b+z) : b \in B\}.$$

Then we have for each $\bar{b} \in B$

$$\eta \leq \inf\left\{p\left(a - \bar{b} + \bar{b} + z\right) : a \in A\right\} - p\left(\bar{b} + z\right) \leq \inf\left\{p\left(a - \bar{b}\right) : a \in A\right\}.$$

This implies $\eta \leq \inf\{p(a-b) : a \in A,\ b \in B\}$. Since p is a seminorm there exist $\mu \in \Lambda$, $\alpha > 0$ such that $p(x) \leq \alpha p_\mu(x)$ for all $x \in X$. Setting $\delta = \frac{\eta}{\alpha}$ we arrive at (8). A similiar proof is possible starting from (10). □

Mizoguchi [15] in 1990 presented a generalization of Daneš' drop theorem to locally convex spaces which turns out to be equivalent to Ekeland's principle and Caristi's fixed point theorem in uniform spaces. A stronger separation property is assumed, i.e. relation (8) has to be valid not for a single seminorm but for members of a base of seminorms. A careful analysis of the proof shows that it goes through assuming the separation property (8) only. We omit the details.

6. The Pareto efficiency theorem due to Isac

In a recent paper [14] G. Isac presented a version of the variational principle in locally convex spaces which relates it to vectorial optimization.

Following Isac, we shall call a point $x \in A \subset X$ an efficient point of A with respect to the pointed convex cone $K \subset X$ iff $A \cap (x+K) = \{x\}$. The set of efficient points of A with respect to the cone K is denoted by $\mathit{Eff}(A;K)$. The notion of efficiency replaces minimality if one deals with vector-valued functions.

As before, let $\{p_\lambda\}_{\lambda\in\Lambda}$ be a base of continuous seminorms generating the topology on X. Let $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ be a family of positive numbers. We define the cone

$$K = \{(x,r) \in Y : \ r + \gamma_\lambda p_\lambda(x) \le 0 \text{ for all } \lambda \in \Lambda\}$$

which is a nuclear cone in the product space $Y = X \times \mathbb{R}$; see Isac [14].

The following theorem is a slightly simplified version of Theorem 3 of [14] for sequentially complete locally convex spaces.

Theorem 8. *Let X be a Hausdorff sequentially complete locally convex topological vector space. Let $f : \ X \to (-\infty, +\infty]$ be an extended-valued proper sequentially lower semicontinuous function, bounded from below. Then $\mathit{Eff}(\mathrm{epi}\, f; K)$ is nonempty and for each $x_0 \in X$ there exists $x^* \in X$ such that*

$$(x^*, f(x^*)) \in (x_0, f(x_0)) + K, \tag{12}$$

$$(x^*, f(x^*)) \in \mathit{Eff}(\mathrm{epi}\, f; K). \tag{13}$$

Moreover, we shall prove the following theorem.

Theorem 9. *Theorem 8 and Theorem 3 are mutually equivalent.*

Proof. First, let us note that the relations (12) and (4) are in fact equivalent.

Let $(x^*, f(x^*)) \in \mathit{Eff}(epi f; K)$. Then, by definition

$$\mathrm{epi} f \cap (x^*, f(x^*)) + K = \{(x^*, f(x^*))\}.$$

Thus, if $(x, f(x)) \in \mathrm{epi} f$, $(x, f(x)) \ne (x^*, f(x^*))$ we have $(x, f(x)) - (x^*, f(x^*)) \notin K$, hence

$$f(x) - f(x^*) + \gamma_\lambda p_\mu(x - x^*) > 0$$

for some $\mu \in \Lambda$ which proves (5).

Conversely, let $(x,r) \in \mathrm{epi} f$, $(x,r) \ne (x^*, f(x^*))$ and

$$(x,r) \in (x^*, f(x^*)) + K. \tag{14}$$

Then, for each $\lambda \in \Lambda$ we have

$$f(x) - f(x^*) + \gamma_\lambda p_\lambda(x - x^*) \le r - f(x^*) + \gamma_\lambda p_\lambda(x - x^*) \le 0,$$

which contradicts (5). Hence an element $(x,r) \in \mathrm{epi} f$ such that $(x,r) \ne (x^*, f(x^*))$ satisfying (14) does not exist, which proves (13). □

7. Remarks and comments

We summarize the considerations above to the following remarks.

Remark 1. Each of Phelps' lemma (Theorem 1), Ekeland's principle with Minkowski functional (Theorem 2) and the drop theorem (Theorem 7) in locally convex spaces is equivalent to its Banach space counterpart (Theorems 10 - 12 of the appendix, respectively).

This is due to the method of proof. Of course, the locally convex version implies the Banach space variant.

Remark 2. Phelps' lemma (Theorem 1), both versions of Ekeland's principle (Theorems 2, 3) and the drop theorem in locally convex spaces are equivalent to each other.

This is due to Theorems 4 – 6.

Remark 3. The drop theorem in locally convex spaces and Isac's efficiency theorem (Theorem 8) are equivalent.

This is due to Theorem 9 and Remark 2 and solves a problem stated by Isac in [14].

Remark 4. Phelps' lemma (Theorem 1), both versions of Ekeland's principle (Theorems 2, 3) and the drop theorem in locally convex spaces are equivalent to Ekeland's principle in metric spaces.

It is well known that Ekeland's principle in metric spaces is equivalent to the drop theorem in Banach spaces (Theorem 12 of the appendix); see e.g. [7], [18]. Considering this fact, Remarks 1 and 2 imply Remark 4.

We conclude that Lemma 1 in [19] – essentially our Theorem 1 – is in fact the first of the long list of theorems equivalent to Ekeland's variational principle in metric spaces and to each other characterizing the property of (sequential) completeness of the space in which they play. We only mention Kirk-Caristi's fixed point theorem [4], Penot's flower petal theorem [18] and the equilibirum version of Ekeland's principle due to Oettli and Théra [16].

It is possible to put (and to show its equivalence) the above-mentioned theorems into the context of locally convex and even more general topological spaces, namely the $\mathcal{F}$-type spaces of Fang [9]. This has been done elsewhere [13].

8. Appendix

For the convenience of the reader we give the Banach space versions of Ekeland's principle, Phelps' lemma and Daneš' drop theorem.

Theorem 10 (Ekeland's variational principle in Banach spaces [1], [8])**.** *Let $(X, \|\cdot\|)$ be a Banach space and $f : X \to (-\infty, +\infty]$ an extended-valued proper lower semicontinuous function, bounded from below. Then, for each $\gamma > 0$, $x_0 \in domf$ there exists $x^* \in X$ such that*

$$f(x^*) + \gamma \|x^* - x_0\| \leq f(x_0),$$
$$\forall x \in X \setminus \{x^*\} : \quad f(x^*) < f(x) + \gamma \|x - x^*\|.$$

Theorem 11 (Phelps' lemma in Banach spaces [20])**.** *Let $(X, \|\cdot\|)$ be a Banach space. Let $A \subset X$ be a nonempty closed set and $B \subset X$ a nonempty closed bounded and convex set such that $0 \notin B$. Let K be the cone defined by*

$$K := \mathbb{R}_+ B = \{x = \alpha b : \ \alpha \geq 0, \ b \in B\}.$$

Then, for each $x_0 \in A$ such that $A \cap (x_0 + K)$ is bounded and nonempty there exists $x^* \in X$ such that

$$x^* \in A \cap (x_0 + K) \quad \text{and} \quad \{x^*\} = A \cap (x^* + K).$$

Theorem 12 (Daneš' drop theorem in Banach spaces [6])**.** *Let $(X, \|\cdot\|)$ be a Banach space. Let $A \subset X$ be a nonempty closed set and $B \subset X$ a nonempty closed bounded and convex set such that*

$$d(A, B) = \inf \{\|a - b\| : \ a \in A, \ b \in B\} > 0.$$

Then, for each $x_0 \in A$ there exists $x^* \in X$ such that

$$x^* \in A \cap D(x_0, B) \quad \text{and} \quad \{x^*\} = A \cap D(x^*, B).$$

Finally, we quote the separation theorem (Bishop/Phelps [2]) used for the proofs of Theorems 4 and 5.

Theorem 13. *Suppose that A and B are convex subsets of a real Hausdorff topological vector space, and that the interior of B is nonempty and disjoint from A. Then there exists a continuous linear functional $l \neq 0$ on X such that $\sup l\,(A) \leq \inf l\,(B)$.*

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