PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 10, Pages 3043–3051
S 0002-9939(03)07130-2
Article electronically published on May 5, 2003

ON THE ALGEBRA RANGE OF AN OPERATOR ON A HILBERT C^*-MODULE OVER COMPACT OPERATORS

RAJNA RAJIĆ

(Communicated by David R. Larson)

Abstract. Let X be a Hilbert C^*-module over the C^*-algebra $K(H)$ of all compact operators on a complex Hilbert space H. Given an orthogonal projection $p \in K(H)$, we describe the set $V^n(A) = \{\langle Ax, x\rangle \ : \ x \in X, \ \langle x, x\rangle = p\}$ for an arbitrary adjointable operator $A \in B(X)$. The relationship between the set $V^n(A)$ and the matricial range of A is established.

1. Introduction and preliminaries

A left Hilbert C^*-module X over a C^*-algebra $\mathcal{A}$ is by definition (see [6]) a linear space which is a left $\mathcal{A}$-module, together with an $\mathcal{A}$-valued inner product $\langle \cdot\,,\,\cdot\rangle$ on $X \times X$ that is linear in the first and conjugate-linear in the second variable. X is also a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$.

Let $B(X)$ be the set of all maps $A : X \to X$ for which there is a map $A^* : X \to X$ such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in X$. Furthermore, let $K(X)$ be the closed linear subspace of $B(X)$ spanned by $\{\theta_{x,y} : x, y \in X\}$ where $\theta_{x,y}$ is a map in $B(X)$ defined by $\theta_{x,y}(z) = \langle z, y\rangle x$. It is well known that $B(X)$ is a C^*-algebra containing $K(X)$ as a two-sided ideal (for details see [6]).

By $B(H)$ and $K(H)$ we denote the algebra of all bounded operators and the ideal of all compact operators acting on a fixed complex Hilbert space H, respectively.

In the sequel X will denote a left Hilbert C^*-module over the C^*-algebra $K(H)$. X is assumed to be a full Hilbert $K(H)$-module which means that the ideal spanned by all inner products $\langle x, y\rangle$, $x, y \in X$, is dense in $K(H)$. (Otherwise, X would be trivial, since $K(H)$ is a simple C^*-algebra.) It was shown in [3, Theorem 2] that X possesses an orthonormal basis (i.e., an orthogonal system (x_λ) that generates a dense submodule of X such that $\langle x_\lambda, x_\lambda\rangle$ is an orthogonal projection in $K(H)$ of rank 1). Furthermore, X contains a Hilbert space X_e with respect to the inner product $(\cdot\,,\,\cdot) = \mathrm{tr}(\langle \cdot\,,\,\cdot\rangle)$ where 'tr' means the trace. More precisely, for a fixed orthogonal projection e in $K(H)$ of rank 1, X_e is given as the set of all ex, $x \in X$. It is known that X and the Hilbert space X_e have the same dimension (i.e., the cardinality of any orthonormal basis). (For all this see [3].) We shall assume that X (and therefore X_e) is infinite dimensional.

Received by the editors June 20, 2001 and, in revised form, January 22, 2002.
2000 *Mathematics Subject Classification.* Primary 47A12, 46L08.
Key words and phrases. C^*-algebra, Hilbert C^*-module, adjointable operator, matricial range of an operator.

©2003 American Mathematical Society

It was proved in [3, Remark 4(b), Theorem 5] that X_e is an invariant subspace for each A in $B(X)$ and that the map $A \mapsto A|X_e$ establishes an isomorphism between C^*-algebras $B(X)$ and $B(X_e)$ where $B(X_e)$ denotes the algebra of all bounded operators on X_e. This isomorphism enables us to describe the relationship between the matricial range of an operator A and the set $V_p^n(A) = \{\langle Ax, x\rangle : x \in X, \langle x, x\rangle = p\}$ where p is a fixed projection of rank n. This is done in Section 3. In Section 2, $V_p^n(A)$ is described in terms of isometries mapping the range of p into X_e.

Before stating the results we establish some more notation as follows. First, a positive integer n is fixed and it is supposed that H has dimension greater than or equal to n. The inner product in H will be denoted by $(\cdot\,|\,\cdot)$. Then let us fix an n-dimensional orthogonal projection p in $K(H)$. H_n will designate the n-dimensional range of p. We now choose an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ for H_n which is to be held fixed for the rest of this paper. For $\xi, \eta \in H$, $e_{\xi,\eta}$ in $B(H)$ is defined by $e_{\xi,\eta}(\nu) = (\nu|\eta)\xi$. From now on we denote $e_i = e_{\xi_i,\xi_i}$ for $i = 1, \dots, n$. Evidently, $p = e_1 + \cdots + e_n$. In the rest of the paper let us also fix a unit vector ξ in H and denote by e the orthogonal projection $e_{\xi,\xi}$ to the one-dimensional subspace spanned by ξ.

2. Main result

Definition 2.1. For an operator $A \in B(X)$ we define the set

$$V_p^n(A) = \{\langle Ax, x\rangle : x \in X,\ \langle x, x\rangle = p\}.$$

Remark 2.2. Note that for $n = 1$, $V_p^1(A)$ coincides (up to natural identification) with the classical numerical range of an operator $A|X_{e_1}$ in $B(X_{e_1})$. Namely, $\langle x, x\rangle = e_1$ if and only if x is a unit vector in a Hilbert space X_{e_1}. Furthermore, according to [3, Remark 4(b),(c)] we get $\langle Ax, x\rangle = (Ax, x)e_1$.

The following lemma shows that $V_p^n(A)$ is always non-empty.

Lemma 2.3. *There exists a vector x in X such that $\langle x, x\rangle = p$.*

Proof. Since X is infinite dimensional we can choose an orthogonal set $\{y_1, \dots, y_n\}$ in X such that $\langle y_i, y_i\rangle = e$ for all $i = 1, \dots, n$ (see [3, Remark 4(d)]). Let us denote $x_i = e_{\xi_i,\xi} y_i$ for $i = 1, \dots, n$. Then we have

$$\begin{aligned}\langle x_i, x_j\rangle &= \langle e_{\xi_i,\xi} y_i\,,\, e_{\xi_j,\xi} y_j\rangle = e_{\xi_i,\xi}\langle y_i, y_j\rangle e_{\xi,\xi_j} \\ &= e_{\xi_i,\xi}\delta_{i,j} e e_{\xi,\xi_j} = \delta_{i,j} e_{\xi_i,\xi} e_{\xi,\xi} e_{\xi,\xi_j} = \delta_{i,j} e_{\xi_i,\xi_j} = \delta_{i,j} e_i\end{aligned}$$

for all $i, j = 1, \dots, n$. It remains to define $x = x_1 + x_2 + \cdots + x_n$. □

Remark 2.4. It is clear that each $x \in X$ such that $\langle x, x\rangle = p$ satisfies $\langle x - px, x - px\rangle = 0$, so $x = px$. Further, for every such vector x we have

$$p\langle Ax, x\rangle = \langle A(px), x\rangle = \langle Ax, x\rangle = \langle Ax, px\rangle = \langle Ax, x\rangle p.$$

Hence, the subspace H_n reduces $\langle Ax, x\rangle$ for all $A \in B(X)$. Moreover, for $\eta \perp H_n$ we have $\langle Ax, x\rangle\eta = \langle Ax, x\rangle p\eta = 0$. This shows that the operator $\langle Ax, x\rangle$ acts trivially on $H_n^\perp$, so that, without loss of generality, $\langle Ax, x\rangle$ can be regarded as an operator acting on the n-dimensional space H_n.

Let A be an arbitrary operator in $B(X)$. We shall see that the set $V_p^n(A)$ basically does not depend on the choice of the rank n projection $p \in K(H)$. Namely, if $q \in K(H)$ is an arbitrary projection of rank n, then the sets $V_p^n(A)$ and $V_q^n(A)$ can be naturally identified, as shown in the following proposition.

Proposition 2.5. *Let $A \in B(X)$ and let $p, q \in K(H)$ be projections of finite rank n. Let $\{\xi_1, \dots, \xi_n\}$ and $\{\eta_1, \dots, \eta_n\}$ be orthonormal bases for the ranges of p and q, respectively. Then a map $\Phi : V_p^n(A) \to V_q^n(A)$ defined by $\Phi(\langle Ax, x\rangle) = \langle A(\sum_{i=1}^n e_{\eta_i,\xi_i} x), \sum_{i=1}^n e_{\eta_i,\xi_i} x\rangle$ is a bijection.*

Proof. Let Φ be as in the statement of the proposition. Since

$$\langle \sum_{i=1}^n e_{\eta_i,\xi_i} x, \sum_{i=1}^n e_{\eta_i,\xi_i} x\rangle = \sum_{i,j=1}^n e_{\eta_i,\xi_i} \langle x, x\rangle e_{\xi_j,\eta_j} = \sum_{i,j=1}^n e_{\eta_i,\xi_i} p e_{\xi_j,\eta_j}$$
$$= \sum_{i,j=1}^n \delta_{i,j} e_{\eta_i,\xi_i} e_{\xi_j,\eta_j} = \sum_{i=1}^n e_{\eta_i,\xi_i} e_{\xi_i,\eta_i} = \sum_{i=1}^n e_{\eta_i,\eta_i} = q,$$

we conclude that Φ is a well-defined map.

To prove that Φ is injective, suppose that $\Phi(\langle Ax, x\rangle) = \Phi(\langle Ay, y\rangle)$ for some $x, y \in X$, $\langle x, x\rangle = \langle y, y\rangle = p$. It follows that

$$\langle A(\sum_{i=1}^n e_{\eta_i,\xi_i} x), \sum_{i=1}^n e_{\eta_i,\xi_i} x\rangle = \langle A(\sum_{i=1}^n e_{\eta_i,\xi_i} y), \sum_{i=1}^n e_{\eta_i,\xi_i} y\rangle,$$

which implies

$$\sum_{i,j=1}^n e_{\eta_i,\xi_i} \langle Ax, x\rangle e_{\xi_j,\eta_j} = \sum_{i,j=1}^n e_{\eta_i,\xi_i} \langle Ay, y\rangle e_{\xi_j,\eta_j}.$$

Multiplying the above equality on its left side by e_{ξ_i,η_i} and on its right side by e_{η_j,ξ_j} we obtain

$$e_i \langle Ax, x\rangle e_j = e_i \langle Ay, y\rangle e_j$$

for all $i, j = 1, \dots, n$. Thus we have

$$\langle Ax, x\rangle = p\langle Ax, x\rangle p = \sum_{i,j=1}^n e_i \langle Ax, x\rangle e_j$$
$$= \sum_{i,j=1}^n e_i \langle Ay, y\rangle e_j = p\langle Ay, y\rangle p = \langle Ay, y\rangle.$$

It remains to show that Φ is surjective. To see this, take any $\langle Ay, y\rangle \in V_q^n(A)$. We define $x = \sum_{i=1}^n e_{\xi_i,\eta_i} y$. Then $\langle Ax, x\rangle \in V_p^n(A)$ since

$$\langle x, x\rangle = \sum_{i,j=1}^n e_{\xi_i,\eta_i} \langle y, y\rangle e_{\eta_j,\xi_j} = \sum_{i,j=1}^n e_{\xi_i,\eta_i} q e_{\eta_j,\xi_j}$$
$$= \sum_{i,j=1}^n \delta_{i,j} e_{\xi_i,\eta_i} e_{\eta_j,\xi_j} = \sum_{i=1}^n e_{\xi_i,\eta_i} e_{\eta_i,\xi_i} = \sum_{i=1}^n e_{\xi_i,\xi_i} = p.$$

Finally,

$$\Phi(\langle Ax, x\rangle) = \langle A(\sum_{i=1}^{n} e_{\eta_i,\xi_i} x), \sum_{i=1}^{n} e_{\eta_i,\xi_i} x\rangle$$
$$= \langle A(\sum_{i=1}^{n} e_{\eta_i,\xi_i} \sum_{j=1}^{n} e_{\xi_j,\eta_j} y), \sum_{i=1}^{n} e_{\eta_i,\xi_i} \sum_{j=1}^{n} e_{\xi_j,\eta_j} y\rangle$$
$$= \langle A(\sum_{i=1}^{n} e_{\eta_i,\eta_i} y), \sum_{i=1}^{n} e_{\eta_i,\eta_i} y\rangle = \langle Ay, y\rangle.$$

This completes the proof. □

If an arbitrary rank n projection p is fixed, then according to the identification from the above proposition, we can write $V_p^n(A) = V^n(A)$. Remark 2.4 shows us now that the set $V^n(A)$ can be considered as a subset of $B(H_n)$ where H_n denotes, as before, the range of p.

In the following theorem we give an alternative description of the set $V^n(A)$. To do this, we have to introduce a "transposed" operator on $B(H_n)$.

Definition 2.6. Let $\{\xi_1, \dots, \xi_n\}$ be the fixed orthonormal basis for H_n. We define a "transposed" operator $\tau : B(H_n) \to B(H_n)$ by the formula $t \mapsto \tau(t)$ where $\tau(t)$ is given by its action on the basis $\{\xi_1, \dots, \xi_n\}$:

$$\tau(t)\xi_j = \sum_{i=1}^{n} (t\xi_i \mid \xi_j)\xi_i.$$

Remark 2.7. According to the above definition, for $t \in B(H_n)$ and $\eta = \sum_{j=1}^{n} \alpha_j \xi_j \in H_n$, it follows that

$$\tau(t)\eta = \sum_{j=1}^{n} \alpha_j \tau(t)\xi_j = \sum_{j=1}^{n} \alpha_j (\sum_{i=1}^{n} (t\xi_i \mid \xi_j)\xi_i).$$

Further, let us denote by $[t_{ij}]$ and $[\tau(t)_{ij}]$ the matrix representations of the linear operators t and $\tau(t)$ with respect to the orthonormal basis $\{\xi_1, \dots, \xi_n\}$. Then we get

$$\tau(t)_{kj} = (\tau(t)\xi_j|\xi_k) = (\sum_{i=1}^{n} (t\xi_i|\xi_j)\xi_i|\xi_k) = (t\xi_k|\xi_j) = t_{jk}$$

for all $k, j = 1, \dots, n$. This shows that the matrix of $\tau(t)$ is obtained by transposing the matrix of t, hence the map τ is a linear operator on $B(H_n)$.

In our next proposition some elementary properties of the map τ are collected.

Proposition 2.8. *The operator τ from Definition* 2.6 *has the following properties*:

(i) $\tau(t^*t) \geq 0$ *for every* $t \in B(H_n)$,
(ii) $\tau^2(t) = t$ *for every* $t \in B(H_n)$,
(iii) $\tau(t^*) = \tau(t)^*$ *for every* $t \in B(H_n)$,
(iv) $\tau(ts) = \tau(s)\tau(t)$ *for all* $t, s \in B(H_n)$.

Since all assertions are clear, the proof is omitted.

We now state our theorem.

Theorem 2.9. *Let A be an operator in $B(X)$. Then*

$$\begin{aligned}\tau(V^n(A)) &:= \{\tau(\langle Ax,x\rangle) \ : x\in X,\ \langle x,x\rangle = p\}\\ &= \{v^*A|X_e v : \ v: H_n\to X_e \text{ is an isometry}\}.\end{aligned}$$

Remark 2.10. Note that in the assertion of this theorem we use the fact that X_e is an invariant subspace for each A in $B(X)$ (see [3, Remark 4(b)]).

Proof of Theorem 2.9. Given an isometry $v: H_n\to X_e$, we define for $i=1,\dots,n$ the vector $x_i = e_{\xi_i,\xi}v\xi_i$. Then $\{x_1,\dots,x_n\}$ is an orthogonal set in X such that $\langle x_i,x_j\rangle = \delta_{i,j}e_i$. Indeed, for $1\le i,j\le n$ we have

$$\langle x_i,x_j\rangle = e_{\xi_i,\xi}\langle v\xi_i,v\xi_j\rangle e_{\xi,\xi_j} = e_{\xi_i,\xi}(v\xi_i,v\xi_j)e_{\xi,\xi}e_{\xi,\xi_j} = \delta_{i,j}e_{\xi_i,\xi_j} = \delta_{i,j}e_i,$$

since the equality $\langle y,z\rangle = (y,z)e$ is satisfied for all y,z from X_e (see [3, Remark 4(c)]). The vector $x = x_1+\cdots+x_n$ clearly satisfies $\langle x,x\rangle = p$. Furthermore, for $1\le i,j\le n$, we obtain

$$\begin{aligned}(\tau(\langle Ax,x\rangle)\xi_j\,|\,\xi_i) &= (\langle Ax,x\rangle\xi_i\,|\,\xi_j)\\ &= \sum_{k,l=1}^n(\langle Ax_k,x_l\rangle\xi_i\,|\,\xi_j) = \sum_{k,l=1}^n(\langle A(e_{\xi_k,\xi}v\xi_k),e_{\xi_l,\xi}v\xi_l\rangle\xi_i\,|\,\xi_j)\\ &= \sum_{k,l=1}^n(e_{\xi_k,\xi}\langle Av\xi_k,v\xi_l\rangle e_{\xi,\xi_l}\xi_i\,|\,\xi_j) = \sum_{k,l=1}^n(e_{\xi_k,\xi}(Av\xi_k,v\xi_l)e_{\xi,\xi}e_{\xi,\xi_l}\xi_i\,|\,\xi_j)\\ &= \sum_{k,l=1}^n(v^*A|X_ev\xi_k\,|\,\xi_l)(e_{\xi_k,\xi_l}\xi_i\,|\,\xi_j) = (v^*A|X_ev\xi_j\,|\,\xi_i).\end{aligned}$$

Therefore, we have shown that $v^*A|X_ev = \tau(\langle Ax,x\rangle)$.

Conversely, let x be a vector in X such that $\langle x,x\rangle = p$. We define an operator $v: H_n\to X_e$ on the orthonormal basis $\{\xi_1,\dots,\xi_n\}$ by putting $v\xi_i = e_{\xi,\xi_i}x$ for $i=1,\dots,n$. Observe that the operator v takes its values in X_e, since

$$\langle v\xi_i,v\xi_i\rangle = e_{\xi,\xi_i}\langle x,x\rangle e_{\xi_i,\xi} = e_{\xi,\xi_i}pe_{\xi_i,\xi} = e_{\xi,\xi} = e$$

for $i=1,\dots,n$. Moreover, v is an isometry since

$$(v\xi_i,v\xi_j) = \operatorname{tr}(\langle v\xi_i,v\xi_j\rangle) = \operatorname{tr}(e_{\xi,\xi_i}\langle x,x\rangle e_{\xi_j,\xi}) = \operatorname{tr}(e_{\xi,\xi_i}pe_{\xi_j,\xi}) = \delta_{i,j}$$

for all $i,j=1,\dots,n$. If we put $x_i = e_ix$ for $i=1,\dots,n$, then $\langle x,x\rangle = p$ implies $x = px = (e_1+\cdots+e_n)x = x_1+\cdots+x_n$. Observe that $\langle x_i,x_j\rangle = \delta_{i,j}e_i$ and also that $x_i = e_ix = e_{\xi_i,\xi}e_{\xi,\xi_i}x = e_{\xi_i,\xi}v\xi_i$ for $i,j=1,\dots,n$. Thus, as in the proof of the first part, we conclude that $\tau(\langle Ax,x\rangle) = v^*A|X_ev$. This completes the proof. □

3. Relation between $V^n(A)$ and $W^n(A)$

Let $\mathcal{A}$ be a unital C^*-algebra. Given an element a in $\mathcal{A}$, we shall denote by $C^*(a)$ the C^*-algebra generated by a and the identity. Let $CP(C^*(a),\mathbf{C}^n,1)$ be the set of all completely positive maps of $C^*(a)$ into $B(\mathbf{C}^n)$ which preserve the identity. (The reader is referred to [1] or [7] for the definition and more details about completely positive maps.)

Furthermore, given a subset S of a unital C^*-algebra $\mathcal{A}$, we denote by $\operatorname{mconv}(S)$ the matricial convex hull of S, i.e., the set of all finite sums of the type $\sum_i t_i^*a_it_i$, where each $a_i\in S$ and where the elements $t_i\in\mathcal{A}$ are such that $\sum_i t_i^*t_i = 1$.

By S^- we denote the topological closure of a set S.

In what follows, let us fix a unitary operator $u : \mathbf{C}^n \to H_n$. (If $\{\epsilon_1, \dots, \epsilon_n\}$ and $\{\xi_1, \dots, \xi_n\}$ denote the standard orthonormal basis for $\mathbf{C}^n$ and our fixed orthonormal basis for H_n, respectively, then u can be chosen by its action on the basis $\{\epsilon_1, \dots, \epsilon_n\}$, i.e., $u\epsilon_j = \xi_j$ for $j = 1, \dots, n$.) In the sequel $\psi : B(H_n) \to B(\mathbf{C}^n)$ will denote an isomorphism between C^*-algebras $B(H_n)$ and $B(\mathbf{C}^n)$ defined by $\psi(t) = u^*tu$, $t \in B(H_n)$.

We begin by recalling the definition of the matricial range of an element in a unital C^*-algebra (see [2], [4] or [9]).

Definition 3.1. Let $\mathcal{A}$ be a unital C^*-algebra. For an element $a \in \mathcal{A}$, the *matricial range* of a is the set

$$W^n(a) = \{\varphi(a) \ : \ \varphi \in CP(C^*(a), \mathbf{C}^n, 1)\}.$$

Remark 3.2. It is clear that the corresponding elements of $*$-isomorphic C^*-algebras have the same matricial range. In particular, for all $A \in B(X)$ we have $W^n(A) = W^n(A|X_e)$.

In [5, Theorem 3.5] J. Bunce and N. Salinas showed that for a given operator $A|X_e$ in $B(X_e)$ it holds that

$$W^n(A|X_e) = \mathrm{mconv}\{v^*A|X_e v \ : \ v : \mathbf{C}^n \to X_e \text{ is an isometry}\}^-.$$

Theorem 2.9 immediately implies that

$$\begin{aligned} \psi(\tau(V^n(A))) &= \{\psi(v^*A|X_e v) : \ v : H_n \to X_e \text{ is an isometry}\} \\ &= \{v^*A|X_e v : \ v : \mathbf{C}^n \to X_e \text{ is an isometry}\}. \end{aligned}$$

Thus we have the following result:

Theorem 3.3. *If $A \in B(X)$, then $W^n(A) = \mathrm{mconv}(\psi(\tau(V^n(A))))^-$.*

Further, for an operator $T \in B(H)$, recall that the essential matricial range of T is the set

$$W_e^n(T) = \{\varphi(T) \ : \ \varphi \in CP(C^*(T), \mathbf{C}^n, 1), \ \varphi|C^*(T) \cap K(H) = 0\}.$$

We now introduce the definition of the essential matricial range of $A \in B(X)$ as follows:

Definition 3.4. For an operator $A \in B(X)$ the *essential matricial range* of A is the set

$$W_e^n(A) = \{\varphi(A) \ : \ \varphi \in CP(C^*(A), \mathbf{C}^n, 1), \ \varphi|C^*(A) \cap K(X) = 0\}.$$

Remark 3.5. Since $A \in K(X)$ if and only if $A|X_e \in K(X_e)$ (see [3, Theorem 6]), it follows that $W_e^n(A) = W_e^n(A|X_e)$.

Given an operator $A \in B(X)$, there is an interesting relationship between the sets $W^n(A)$, $\psi(\tau(V^n(A)))$ and $W_e^n(A)$, as an immediate consequence of Theorem 2.9 and Theorem 3.7 of [5].

Theorem 3.6. *If $A \in B(X)$, then $W^n(A) = \mathrm{mconv}(\psi(\tau(V^n(A))) \cup W_e^n(A))$.*

Finally, as a consequence of the equivalence of the conditions (a) and (c) in Theorem 3.1 of [5], we get a description of the essential matricial range of an operator A in $B(X)$.

Theorem 3.7. *Let A be in $B(X)$. Then $l \in W_e^n(A)$ if and only if there exists an orthogonal sequence (x_k) in X such that $\langle x_k, x_k\rangle = p$ for all $k \in \mathbf{N}$ and $\lim_{k\to\infty} \psi(\tau(\langle Ax_k, x_k\rangle)) = l$.*

Proof. As in the proof of Theorem 2.9 we conclude that $v : H_n \to X_e$ is an isometry if and only if there exists a vector x in X such that $\langle x, x\rangle = p$. Thereby $v^*A|X_e v = \tau(\langle Ax, x\rangle)$ and the vectors $x_i = e_i x$ satisfy $\langle x_i, x_j\rangle = \delta_{i,j} e_i,\ x_1 + \cdots + x_n = x,\ x_i = e_{\xi_i,\xi} v\xi_i$ and $v\xi_i = e_{\xi,\xi_i} x$ for all $i, j = 1, \ldots, n$. Observe that isometries $vu, v'u : \mathbf{C}^n \to X_e$ have mutually orthogonal ranges if and only if isometries $v, v' : H_n \to X_e$ have mutually orthogonal ranges, that is, if and only if $\langle x, x'\rangle = 0$ is satisfied for the corresponding vectors x and x'. To complete the proof, it remains to apply Theorem 3.1 ((a) $\Leftrightarrow$ (c)) of [5]. □

Remark 3.8. We provide here an alternative proof for the sufficiency part. To do this, we need the following lemma concerning the characterisation of an operator in $K(X)$. Notice that if $n = \mathrm{rank}(p) = 1$, our lemma reduces to Theorem 7 ((a) $\Leftrightarrow$ (c)) from [3].

Lemma 3.9. *For $A \in B(X)$ the following statements are mutually equivalent*:

(i) $A \in K(X)$.

(ii) $\lim_{k\to\infty}\langle Ax_k, x_k\rangle = 0$ *for each orthogonal system (x_k) in X such that $\langle x_k, x_k\rangle = p$ for all $k \in \mathbf{N}$.*

Proof. (i) $\Rightarrow$ (ii). Since $A = B + iC$, where $B, C \in K(X)$ are self-adjoint operators, we may assume that A is self-adjoint. First, observe that $\|\langle Ax_k, x_k\rangle\| \le \|A\|\|x_k\|^2 = \|A\|\|p\| = \|A\|$ for all $k \in \mathbf{N}$. If l is a cluster point of the norm bounded sequence $(\langle Ax_k, x_k\rangle)$ in $B(H_n)$, then there exists a subsequence $(\langle Ax_{k_i}, x_{k_i}\rangle)$ of $(\langle Ax_k, x_k\rangle)$ converging to l. Obviously, l is a self-adjoint operator in $B(H_n)$. We shall show that l must be zero. Let $\{\eta_1, \ldots, \eta_n\}$ be an orthonormal basis for H_n consisting of eigenvectors of l. Put $f_j = e_{\eta_j,\eta_j}$ for $j = 1, \ldots, n$. Then $(f_j x_{k_i})_i$ is an orthonormal system in X (i.e., an orthogonal system of vectors whose inner squares are orthogonal projections of rank 1). Indeed,

$$\langle f_j x_{k_s}, f_j x_{k_t}\rangle = f_j\langle x_{k_s}, x_{k_t}\rangle f_j = f_j \delta_{s,t} p f_j = \delta_{s,t} f_j$$

for all $s, t \in \mathbf{N}$. Now, for $j = 1, \ldots, n$, we have

$$f_j l f_j = \lim_{i\to\infty} f_j\langle Ax_{k_i}, x_{k_i}\rangle f_j = \lim_{i\to\infty}\langle Af_j x_{k_i}, f_j x_{k_i}\rangle = 0,$$

where the last equality follows from Theorem 7 of [3]. We conclude that $l = 0$.

(ii) $\Rightarrow$ (i). Let (y_k) be an arbitrary orthonormal system in a Hilbert space X_e. We define

$$x_k = e_{\xi_1,\xi} y_{(k-1)n+1} + e_{\xi_2,\xi} y_{(k-1)n+2} + \cdots + e_{\xi_n,\xi} y_{kn}$$

for $k \in \mathbf{N}$. It is easy to see that (x_k) is an orthogonal system in X such that $\langle x_k, x_k\rangle = p$ for all $k \in \mathbf{N}$. By the hypothesis it follows that

$$\lim_{k\to\infty}\langle Ae_i x_k, e_i x_k\rangle = \lim_{k\to\infty} e_i\langle Ax_k, x_k\rangle e_i = 0$$

for $i = 1, \ldots, n$. Now we use the facts that X_e is invariant for A and that $\langle x, y\rangle = (x, y)e$ for all $x, y \in X_e$ (see [3, Remark 4(b),(c)]) to obtain

$$\begin{aligned}\langle Ae_i x_k, e_i x_k\rangle &= e_{\xi_i,\xi}\langle Ay_{(k-1)n+i}\,,\, y_{(k-1)n+i}\rangle e_{\xi,\xi_i}\\ &= (A|X_e y_{(k-1)n+i}, y_{(k-1)n+i})e_i\end{aligned}$$

for $i = 1, \ldots, n$, $k \in \mathbf{N}$. Hence, $\lim_{k\to\infty}(A|X_e y_{(k-1)n+i}, y_{(k-1)n+i}) = 0$ for $i = 1, \ldots, n$. It obviously follows that $\lim_{k\to\infty}(A|X_e y_k, y_k) = 0$, so by Theorem 1.8.7 of [8], $A|X_e$ is a compact operator on X_e. Therefore, Theorem 6 of [3] implies that $A \in K(X)$. □

An alternative proof of the sufficiency part of Theorem 3.7. For $k \in \mathbf{N}$ we define a map $\varphi_k : C^*(A) \to B(H_n)$ by setting

$$\varphi_k(T) = \tau(\langle Tx_k, x_k\rangle) \quad (T \in C^*(A)).$$

Then, φ_k is a completely positive map which takes I to p. Namely, since every positive element of $M_r(C^*(A))$ is a finite sum of the elements of the type $[T_i^* T_j]$ where $T_i \in C^*(A)$, it is sufficient to show that $[\varphi_k(T_i^* T_j)]$ is a positive element of $M_r(B(H_n))$ for all $T_1, \ldots, T_r \in C^*(A)$. We have

$$[\varphi_k(T_i^* T_j)] = [\tau(\langle T_i^* T_j x_k, x_k\rangle)] = [\tau(\langle T_j x_k, T_i x_k\rangle)] = \tau([\langle T_i x_k, T_j x_k\rangle])$$

where τ also stands for the "transposed" operator on $M_r(B(H_n))$. By Lemma 4.2 of [6] (the assertion is also true for left Hilbert C^*-modules), we have $[\langle T_i x_k, T_j x_k\rangle] \geq 0$, so $[\varphi_k(T_i^* T_j)] \geq 0$. Since the set of all completely positive maps of $C^*(A)$ into $B(H_n)$ which take I to p is BW-compact (the BW-topology is introduced in [1]), there exists a subsequence (φ_{k_i}) of (φ_k) and a completely positive map $\varphi : C^*(A) \to B(H_n)$ such that $\varphi(T) = \lim_{i\to\infty} \varphi_{k_i}(T)$ for all $T \in C^*(A)$. In particular, by the preceding lemma, for $K \in C^*(A) \cap K(X)$ it holds that $\varphi(K) = \lim_{i\to\infty} \tau(\langle Kx_{k_i}, x_{k_i}\rangle) = 0$. Hence, $\psi \circ \varphi$ is a completely positive map of $C^*(A)$ into $B(\mathbf{C}^n)$ satisfying $(\psi \circ \varphi)(I) = \psi(p) = u^* p u = 1$ and such that $(\psi \circ \varphi)(K) = 0$ for all $K \in C^*(A) \cap K(X)$. Therefore,

$$l = \lim_{i\to\infty} \psi(\tau(\langle Ax_{k_i}, x_{k_i}\rangle)) = \lim_{i\to\infty} (\psi \circ \varphi_{k_i})(A) = (\psi \circ \varphi)(A) \in W_e^n(A),$$

as desired. □

In conclusion, let us consider the case when a given operator A in $B(X)$ is normal. W. B. Arveson proved in [2, Proposition 2.4.1] that the matricial range of a normal operator $T \in B(H)$ is the set

$$W^n(T) = \{\sum_{i=1}^{r} \lambda_i k_i : r \geq 1, \lambda_i \in \sigma(T), k_i \in B(\mathbf{C}^n), k_i \geq 0, \sum_{i=1}^{r} k_i = 1\}^-,$$

where $\sigma(T)$ denotes the spectrum of T.

Notice that Proposition 2.8 (i) implies $l \in W^n(T)$ if and only if $\psi(\tau(\psi^{-1}(l))) \in W^n(T)$.

We need only to apply Proposition 2.8, Theorem 3.3, Theorem 3.6 and Theorem 3.7 to obtain

Corollary 3.10. *Let A be a normal operator in $B(X)$. Then the following statements hold*:

(i) $W^n(A) = \mathrm{mconv}(\psi(V^n(A)))^-$.
(ii) $W^n(A) = \mathrm{mconv}(\psi(V^n(A)) \cup W_e^n(A))$.
(iii) $l \in W_e^n(A)$ *if and only if there is an orthogonal sequence* (x_k) *in* X *such that* $\langle x_k, x_k\rangle = p$ *for all* $k \in \mathbf{N}$ *and* $\lim_{k\to\infty} \psi(\langle Ax_k, x_k\rangle) = l$.

Acknowledgement

The author would like to express her gratitude to Professor Damir Bakić for his guidance throughout this work. Thanks are also due to the referee for many useful comments and suggestions essentially improving the text.

References

1. W. B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141–224. MR **40:**6274
2. ———, *Subalgebras of C^*-algebras* II, Acta Math. **128** (1972), 271–308. MR **52:**15035
3. D. Bakić and B. Guljaš, *Hilbert C^*-modules over C^*-algebras of compact operators*, Acta Sci. Math. (Szeged) **68** (2002), 249–269.
4. F. F. Bonsall and J. Duncan, *Numerical ranges* II, London Math. Soc. Lecture Note Series **10**, Cambridge University Press, Cambridge, 1973. MR **56:**1063
5. J. Bunce and N. Salinas, *Completely positive maps on C^*-algebras and the left matricial spectra of an operator*, Duke Math. J. **43** (1976), 747–774. MR **55:**3798
6. C. Lance, *Hilbert C^*-modules*, London Math. Soc. Lecture Note Series **210**, Cambridge University Press, Cambridge, 1995. MR **96k:**46100
7. V. I. Paulsen, *Completely bounded maps and dilations*, Research Notes in Math., vol. 146, Pitman, London, 1986. MR **88h:**46111
8. J. R. Ringrose, *Compact non-selfadjoint operators*, Van Nostrand, Princeton, 1971.
9. R. R. Smith and J. D. Ward, *Matrix ranges for Hilbert space operators*, Amer. J. Math. **102** (1980), 1031–1081. MR **82a:**46067

Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia

E-mail address: rajna.rajic@zg.hinet.hr