PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 11, Pages 3329–3334
S 0002-9939(03)06875-8
Article electronically published on February 20, 2003

A $\bar{\partial}\partial$–POINCARÉ LEMMA FOR FORMS NEAR AN ISOLATED COMPLEX SINGULARITY

ADAM HARRIS AND YOSHIHIRO TONEGAWA

(Communicated by Mei-Chi Shaw)

ABSTRACT. Let X be an analytic subvariety of complex Euclidean space with isolated singularity at the origin, and let η be a smooth form of type (1.1) defined on $X \setminus \{0\}$. The main result of this note is a criterion for solubility of the equation $\bar{\partial}\partial u = \eta$. This implies a criterion for triviality of a Hermitian–holomorphic line bundle $(L, h) \to X \setminus \{0\}$ in a neighbourhood of the origin.

1. INTRODUCTION

Let $X \subset \mathbb{C}^N$ be a reduced and irreducible n–dimensional complex analytic variety with isolated singularity at the origin, and write $X_0 := X \setminus \{0\}$ for the set of smooth points. It will be assumed without loss of generality that 0 is the unique singular point of X. Let $\rho : X \to [0, \infty)$, $\rho(0) = 0$, be a strongly plurisubharmonic function, and $\Omega \subset X$ a neighbourhood of the origin with smooth compact boundary $\Sigma = \{x \in X \mid \rho(x) = \varepsilon < \infty\}$. Let $\Omega_0 := \Omega \setminus \{0\}$ be a complex manifold with Kähler form $\omega = i\bar{\partial}\partial\rho$ and associated metric g satisfying the following conditions:

$$(i) \int_{\Omega_0} |R_g|^n < \infty \quad ,$$

where R_g denotes the canonical curvature form associated with g. It will further be assumed that the Sobolev inequality holds with respect to this metric, i.e.,

$$(ii) \left(\int_{\Omega_0} |f|^{\frac{2n}{n-1}} \omega^n\right)^{\frac{n-1}{n}} \leq c(n) \int_{\Omega_0} |\nabla f|^2 \omega^n$$

for smooth compactly supported functions f. In addition, (iii) let $\delta(0, x)$ denote the Riemannian metric distance function on Ω_0, and let $B_\delta(0, r)$ be the associated ball of radius r. For sufficiently small $0 < c < \varepsilon$ it will be assumed that there exists a positive constant K such that

$$\int_{B_\delta(0,r)} \omega^n \leq K r^{2n}, \quad \text{for all} \quad 0 < r \leq c.$$

In fact, when $X \subset \mathbb{C}^N$ is an affine analytic variety, with ρ corresponding to the restriction of the Euclidean norm–squared function, conditions (ii) and (iii) hold in general from the geometric measure theory of analytic varieties (cf. [4]). In particular, property (iii) is fundamental to the definition of the Lelong number for

Received by the editors September 18, 2001 and, in revised form, June 1, 2002.
2000 *Mathematics Subject Classification.* Primary 14J17, 32B15, 32S05, 32W05.

©2003 American Mathematical Society

plurisubharmonic functions, of which the multiplicity of analytic singularities is a special case (cf., e.g., [5]). Moreover, the curvature form R_g in this situation corresponds to $\beta \wedge \beta^*$, where β denotes the second fundamental form of the embedded variety, and it is an easy computation to show that

$$\int_{\Omega_0} |R_g|^n \leq \int_{\Omega_0} |\beta|^{2n} < \infty$$

when, for example, $X : z_{n+1}^k = f(z_1, ..., z_n)$ is an analytic hypersurface in $\mathbb{C}^{n+1}$ with isolated singularity at the origin, such that $2 \leq k < ord_f(0)$. On the other hand surfaces in $\mathbb{C}^3$ defined by an equation of the form $z^k = xy$, which constitute a special class of orbifold singularities (cf., e.g., [3]), do not satisfy condition (i) with respect to the restricted ambient metric. For any singular space X of this type, corresponding to the quotient of $\mathbb{C}^n$ by a finite subgroup of $\mathbb{SU}(n)$, the most natural choice of ρ is that induced by $|z|^2$ on the Euclidean covering space, since the associated orbifold metric is flat.

Let $E \to X_0$ be a holomorphic vector bundle, with Hermitian metric H, and Kähler metric g on X_0 derived from $i\bar{\partial}\partial\rho$ as above. Assume that the curvatures F_H and R_g both belong to $L^n(\Omega_0)$. If $\Psi \in C^\infty(X_0, \bigwedge^{0.1}(E))$ is a $\bar{\partial}$–closed (0.1)–form also belonging to $L^n(\Omega_0)$, then it is shown in [6], theorem 1, that the equation $\bar{\partial}\mu = \Psi$ admits a smooth solution on Ω_0 such that $\|\mu\| \leq \|\Psi\|$. The concern of the present note is to apply this result, in the special case $E = X_0 \times \mathbb{C}$, to the problem of finding $u \in C^\infty(\Omega_0)$ such that $\bar{\partial}\partial u = \eta$ for a closed form η of type (1.1) on X_0. The main theorem is stated as follows:

Theorem. *Suppose the curvature of the Kähler form ω on X_0 belongs to $L^n(\Omega_0)$. If η is a closed form of type* (1.1) *on X_0, such that $\eta \mid_{T\Sigma}$ is d–exact and the function $|\eta|$ belongs to $L^n(\Omega_0)$, then there exists a smooth function u on Ω_0 such that $\bar{\partial}\partial u = \eta$.*

The proof also makes explicit use of the conical structure theorem of Milnor [8], together with a relative Poincaré lemma, derived in the context of hypersurfaces of contact type, by Weinstein [10]. For simplicity it is assumed that ω is associated with restriction of the ambient Euclidean metric. In conclusion, a corollary to the main theorem provides a criterion for triviality of Hermitian–holomorphic line bundles $(L, h) \to X_0$ when restricted to Ω_0, generalising the analogous theorem of Bando [2] for line bundles over the punctured ball in $\mathbb{C}^2$. Its statement is as follows:

Corollary. *Suppose that the curvature of ω belongs to $L^n(\Omega_0)$. Let $(L, h) \to X_0$ be a Hermitian–holomorphic line bundle with curvature $F_h = \bar{\partial}\partial \log(h)$ such that the first Chern class vanishes in $H^2(\Omega_0, \mathbb{Z})$ and $|F_h|$ also belongs to $L^n(\Omega_0)$. Then L restricted to Ω_0 is trivial.*

In particular, the corollary provides a criterion for triviality of the canonical bundle, $K_{X_0} \mid_{\Omega_0}$, complementing the well–known cases of orbifolds and isolated singularities of complete intersections. We note further that "Bogomolov-type smoothness" theorems for the deformation spaces of complex structure on the intersection of X_0 with the ball of radius ε, and for CR–structure on the boundary, $\mathbb{S}_\varepsilon^{2N-1} \cap X_0$, have been established in the context of K_{X_0} trivial by Miyajima [9], and Akahori and Miyajima [1]. From the algebraic point of view, the above criterion may be compared with the well–known fact that $H^1(X \setminus \{0\}, \mathcal{O}_X) = 0$ if and only if the depth of the structure sheaf is at least three at the singular point. In this case it follows at once that all topologically trivial line bundles are holomorphically trivial in a neighbourhood of the origin.

The authors would like to thank Professor K. Miyajima, Professor M. Eastwood and Dr. N. Buchdahl for their helpful comments at different stages. The author (Harris) gratefully acknowledges the support of a research fellowship from the Japan Society for the Promotion of Science during his stay at Keio University in Tokyo, and the hospitality of Masaryk University in Brno, where the writing of this article was completed.

2. The $\bar{\partial}\partial$–Poincaré lemma

Let $X^n \subset \mathbb{C}^N$ be a reduced and irreducible complex analytic variety with isolated singularity $0 \in X$ and with Kähler form ω on X_0 corresponding to the restriction of the standard Kähler form on Euclidean space. Consider a domain Ω of finite volume, corresponding to $B_\varepsilon \cap X$ with $\iota : \Sigma \hookrightarrow X_0$ an embedded, smooth, compact real hypersurface corresponding to $\mathbb{S}^{2N-1}_\varepsilon \cap X_0$. If $\rho(x) = |\underline{x}|^2$ with respect to the Euclidean inner product for all $x \in X_0$, take ε sufficiently small so that $\nabla\rho$, the gradient vector field of ρ on the smooth tangent bundle TX_0, has no vanishing points (i.e., critical points of ρ) other than the origin (cf. [8], corollary 2.8). Consequently, if ν represents the angle between $\nabla\rho(x)$ and $\underline{x}$, viewed as vectors in $\mathbb{R}^{2N}$, it may be assumed that $|\cos(\nu)|$ is bounded away from zero on Ω_0 (in particular $\lim_{\underline{x}\to 0} \frac{\underline{x}}{|\underline{x}|}$ belongs to the tangent cone to X at the origin, hence $\lim_{\underline{x}\to 0} |\cos(\nu)| = 1$).

Lemma 1. *Let η be a smooth, closed 2–form on X_0, such that $\eta\,|_{T\Sigma}$ is exact. Then there exists a smooth form $\psi \in C^\infty(\Omega_0, T^*X_0)$ satisfying $d\psi = \eta$.*

Proof. Given $\eta\,|_{T\Sigma} = d\lambda$, let $\bar{\lambda}$ denote any smooth extension of λ, compactly supported in a neighbourhood of Σ, and write $\vartheta = \eta - d\bar{\lambda}$. Following Milnor ([8], theorem 2.10) define a smooth vector field

$$\mathbf{v}(x) = \frac{-\nabla\rho(x)}{\langle \nabla\rho(x), \frac{\underline{x}}{|\underline{x}|}\rangle}$$

with respect to the standard inner product on $\mathbb{R}^{2N}$. From the system of ordinary differential equations $\frac{dx}{dt} = \mathbf{v}$ (†), it follows that

$$\frac{d\rho(x)}{dt} = 2\langle \underline{x}, \mathbf{v}(x)\rangle = -2\sqrt{\rho}\ .$$

Hence $\frac{d\sqrt{\rho}}{dt} = -1$, $\sqrt{\rho} = \varepsilon - t$, implies that there exists, for each $p \in \Sigma$, a unique solution of (†) on $[0, \varepsilon)$ such that $x(0) = p \in \Sigma$. Let $\varphi : \Omega_0 \cup \Sigma \to [0, \varepsilon) \times \Sigma$ be the natural diffeomorphism such that $\varphi^{-1}(t, p) = \underline{x}_p(t)$. The remaining argument is simply a modified version of the relative Poincaré lemma established in [10]. Specifically, let $\pi_s : [0, \varepsilon) \times \Sigma \to [0, \varepsilon) \times \Sigma$ be the dilation $(t, p) \mapsto (st, p)$, $0 \le s \le 1$, and define

$$\varphi_s(x) = \varphi^{-1} \circ \pi_s \circ \varphi(x) = \underline{x}_p(st)\ .$$

Now

$$\frac{d}{ds}\,|_{s=s_0}\ \varphi_s(x) = \frac{d}{ds}\,|_{s=s_0}\ \underline{x}_p(st) = t\mathbf{v}(\underline{x}_p(s_0 t))\ ,$$

and hence (cf., e.g., Hermann [7])

$$\begin{aligned}\frac{d}{ds}\,|_{s=s_0}\ \varphi_s^*(\vartheta) &= d(\varphi_{s_0}^*(t\mathbf{v}(\underline{x}_p(s_0t))\rfloor\vartheta)) + \varphi_{s_0}^*(t\mathbf{v}(\underline{x}_p(s_0t))\rfloor d\vartheta)\\ &= d(t\varphi_{s_0}^*(\mathbf{v}(\varphi_{s_0}(x))\rfloor\vartheta))\ ,\end{aligned}$$

since $d\vartheta = 0$ (here the symbol "$\rfloor$" denotes contraction of a form by a vector). But $\varphi_0^*(\vartheta) = \vartheta\ |_{T\Sigma} = 0$, and $\varphi_1^*(\vartheta) = \vartheta$ implies

$$d((\varepsilon - \sqrt{\rho})\int_0^1 \varphi_s^*(\mathbf{v}(\varphi_s(x))\rfloor\vartheta)ds) = \vartheta\ ,$$

and hence

$$\psi = \bar{\lambda} + (\varepsilon - \sqrt{\rho})\int_0^1 \varphi_s^*(\mathbf{v}(\varphi_s(x))\rfloor\vartheta)ds\ .$$

□

We remark that for the case of X_0 an orbifold singularity, there is no essential change in the argument if ρ corresponding to the restriction of $|\underline{x}|^2$ in the ambient $\mathbb{C}^N$ is replaced by ρ corresponding to the push–forward of the Euclidean norm–squared on the covering $\mathbb{C}^n$.

Since $\bar{\lambda}$ is compactly supported, for any $\gamma \geq 1$ we may write

$$\begin{aligned}\int_{\Omega_0}|\psi|^\gamma &\leq C + \int_{\Omega_0}|\varepsilon - \sqrt{\rho}|^\gamma|\int_0^1 \varphi_s^*(\mathbf{v}(\varphi_s(x))\rfloor\vartheta)ds|^\gamma\\ &\leq C + \int_{\Omega_0}\int_0^1 \varepsilon^\gamma|\varphi_s^*(\mathbf{v}(\varphi_s(x))\rfloor\vartheta)|^\gamma ds\ ,\end{aligned}$$

via Jensen's inequality. Now

$$\int_{\Omega_0}|\mathbf{v}(x)\rfloor\vartheta|^\gamma \leq \int_{\Omega_0}|\mathbf{v}(x)|^\gamma|\vartheta|^\gamma\ .$$

Recall, moreover, that $\mathbf{v}(x) = \frac{-\nabla\rho}{\langle\nabla\rho, \frac{\underline{x}}{|\underline{x}|}\rangle}$ implies

$$|\mathbf{v}(x)| = \frac{1}{\cos(\nu)} \leq c\ ,$$

due to the initial assumption concerning critical points of ρ. Thus

$$\int_{\Omega_0}|\mathbf{v}(x)\rfloor\vartheta|^\gamma \leq c\int_{\Omega_0}|\vartheta|^\gamma < \infty$$

if $|\vartheta| \in L^\gamma(\Omega_0)$, i.e., if $|\eta| \in L^\gamma(\Omega_0)$. Note that $|\varphi_s^*(\mathbf{v}(\varphi_s(x))\rfloor\vartheta)|$ is bounded on Ω_0 for all $s < 1$, hence $|\eta| \in L^\gamma(\Omega_0)$ implies

$$\int_0^1\int_{\Omega_0}|\varepsilon - \sqrt{\rho}|^\gamma|\varphi_s^*(\mathbf{v}(\varphi_s(x))\rfloor\vartheta)|^\gamma ds < \infty\ .$$

It now follows at once from Fubini's theorem that $\int_{\Omega_0}|\psi|^\gamma < \infty$. We have now proved

Lemma 2. *For any $\gamma \geq 1$, if $|\eta| \in L^\gamma(\Omega_0)$, then $|\psi| \in L^\gamma(\Omega_0)$.*

Now let η be a form of type (1.1), with $\psi^{1.0}$, $\psi^{0.1}$ denoting the projections of ψ onto the holomorphic cotangent space, $(T^{1.0}X_0)^*$, and anti–holomorphic cotangent space $(T^{0.1}X_0)^*$, respectively. Note that $|\psi^{1.0}|$, $|\psi^{0.1}| \leq |\psi|$ implies $|\psi^{1.0}|$, $|\psi^{0.1}| \in L^\gamma(\Omega_0)$ whenever $|\eta| \in L^\gamma(\Omega_0)$ by the argument above. It remains to conclude

Theorem 1. *Suppose the curvature of the Kähler form ω on X_0 belongs to $L^n(\Omega_0)$. If η is a closed form of type* (1.1) *on X_0, such that $\eta \mid_{T\Sigma}$ is exact and $|\eta| \in L^n(\Omega_0)$, then there exists a smooth function u on Ω_0 such that $\bar\partial\partial u = \eta$.*

Remark. Apart from the L^n–assumption for R_g, recall that some care was taken in the Introduction to specify that ω should satisfy a certain volume–growth estimate, together with the Sobolev inequality in a neighbourhood of the origin. Although these conditions are satisfied automatically by ω as defined in this article, it is worth pointing out that they are explicitly used in the solution of the Cauchy–Riemann problem (cf. [6], theorem 1), which we apply below.

Proof. If η is specifically of type (1.1) with respect to the complex structure on X_0, then $\partial\psi^{1.0} = \bar\partial\psi^{0.1} = 0$. It now follows from [6], theorem 1, that there exist $f, g \in C^\infty(\Omega_0)$ such that

$$\bar\partial f = \overline{\psi^{1.0}} \ , \qquad \bar\partial g = \psi^{0.1} \ .$$

But

$$d\bar f - \psi = \overline{\partial f} - \psi^{0.1} = \bar\partial(\bar f - g) \ ,$$

hence $u = \bar f - g$. □

3. Application to line bundles over X_0

Let $(L, h) \to (X_0, \omega)$ be a Hermitian–holomorphic line bundle with curvature $F_h = \bar\partial\partial\log(h)$. As above the Kähler form ω on X_0 will be assumed to be the one induced from the standard Kähler structure on $\mathbb{C}^N$.

Corollary 1. *Suppose that the curvature of ω belongs to $L^n(\Omega_0)$. Let $(L, h) \to X_0$ be such that the first Chern class vanishes in $H^2(\Omega_0, \mathbb{Z})$ and $|F_h|$ belongs to $L^n(\Omega_0)$. Then L restricted to Ω_0 is trivial.*

Proof. The vanishing of $c_1(L)$ implies that a smooth unitary frame may be chosen for $L \mid_{\Omega_0}$, with respect to which the connection form τ of the Hermitian metric connection ∇_h is globally defined such that $d\tau = F_h$. As in Lemma 1, let ψ be a solution of the equation $d\psi = F_h$ such that $|\psi| \in L^n(\Omega_0)$, with $\psi \mid_\Sigma = \tau \mid_\Sigma$. From the main theorem, there exists a function u, smooth up to the boundary Σ, such that $F_h - \bar\partial\partial u = 0$. Note that for any such u the conjugate $\bar u$ is another solution, hence a real–valued solution exists. Moreover $\tau - \partial u$, corresponding to the connection form of $\nabla_{e^{-u}h}$, is flat and represents a cohomology class of $H^1(\Omega_0, \mathbb{C})$. If $u = \bar f - g$ as in Theorem 1, it follows that

$$\tau - \partial u \mid_{T\Sigma} = \psi - \partial u \mid_{T\Sigma} = d(\bar f - u) = dg \mid_{T\Sigma} \ .$$

Since any loop in Ω_0 is homotopy equivalent to a loop in Σ, we conclude that the holonomy of sections covariantly constant with respect to $\nabla_{e^{-u}h}$ is trivial, and hence there exists a global holomorphic frame for $L \mid_{\Omega_0}$. □

References

1. Akahori, T., Miyajima, K. *An analogy of Tian–Todorov theorem on deformations of CR–structures*, Compositio Math. 85 (1993) 57–85 MR **94b:**32028
2. Bando, S., *Removable singularities for holomorphic vector bundles*, Tohoku Math. J. 43 (1991) 61–67 MR **92a:**32037
3. Dimca, A., *Singularities and topology of hypersurfaces*, Springer Universitext 1992 MR **94b:**32058

4. Federer, H., *Geometric measure theory*, Grund. Math. Wiss. 153, Springer Berlin, Heidelberg, New York 1969 MR **41:**1976
5. Griffiths, P., and Harris, J., *Principles of Algebraic Geometry*, Wiley 1978 MR **95d:**14001
6. Harris, A., and Tonegawa, Y. *L^p–curvature and the Cauchy–Riemann equation near an isolated singular point*, Nagoya Math. J. 164 (2001) 35–51
7. Hermann, R., *Differential geometry and the calculus of variations*, Math. Sci. Eng. 49, Academic Press 1968 MR **38:**1635
8. Milnor, J. *Singular points of complex hypersurfaces*, Ann. Math. Stud. 61, Princeton 1968 MR **39:**969
9. Miyajima, K. *A note on Bogomolov–type smoothness on deformations of regular parts of isolated singularities*, Proc. Am. Math. Soc. 125, No. 2 (1997) 485–492 MR **97d:**32027
10. Weinstein, A., *Symplectic manifolds and their Lagrangian submanifolds*, Adv. Math. 6 (1971) 329–346 MR **44:**3351

Department of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia
E-mail address: harris@ms.unimelb.edu.au
Current address: Department of Mathematics & Computer Science, University of New England, Armidale, New South Wales 2351, Australia

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: tonegawa@math.sci.hokudai.ac.jp