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π_1 OF HAMILTONIAN S^1 MANIFOLDS

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ABSTRACT. Let (M, ω) be a connected, compact symplectic manifold equipped with a Hamiltonian S^1 action. We prove that, as fundamental groups of topological spaces, $\pi_1(M) = \pi_1(\text{minimum}) = \pi_1(\text{maximum}) = \pi_1(M_{red})$, where M_{red} is the symplectic quotient at any value in the image of the moment map ϕ.

Let (M, ω) be a connected, compact symplectic manifold equipped with a circle action. If the action is Hamiltonian, then the moment map $\phi : M \to \mathbb{R}$ is a perfect Bott-Morse function. Its critical sets are precisely the fixed point sets M^{S^1} of the S^1 action, and M^{S^1} is a disjoint union of symplectic submanifolds. Each fixed point set has even index. By [1], ϕ has a unique local minimum and a unique local maximum. We will use Morse theory to prove

Theorem 0.1. *Let (M^{2n}, ω) be a connected, compact symplectic manifold equipped with a Hamiltonian S^1 action. Then as fundamental groups of topological spaces, $\pi_1(M) = \pi_1(\mathit{minimum}) = \pi_1(\mathit{maximum}) = \pi_1(M_{red})$, where M_{red} is the symplectic quotient at any value in the image of the moment map ϕ.*

Remark 0.2. The theorem is not true for orbifold π_1 of M_{red}, as shown in the example below. (See [5] or [11] for the definition of orbifold π_1.)

Let $a \in \text{im}(\phi)$, and $\phi^{-1}(a) = \{x \in M \mid \phi(x) = a\}$ be the level set. Define $M_a = \phi^{-1}(a)/S^1$ to be the symplectic quotient.

Note that if a is a regular value of ϕ, and if the circle action on $\phi^{-1}(a)$ is not free, then M_a is an orbifold, and we have an orbi-bundle:

$$\begin{array}{ccc} S^1 & \hookrightarrow & \phi^{-1}(a) \\ & & \downarrow \\ & & M_a \end{array} \tag{0.1}$$

If a is a critical value of ϕ, then M_a is a stratified space ([10]).

Now, let S^1 act on $(S^2 \times S^2, 2\rho \oplus \rho)$ (where ρ is the standard symplectic form on S^2) by $\lambda(z_1, z_2) = (\lambda^2 z_1, \lambda z_2)$. Let 0 be the minimal value of the moment map. Then for $a \in (1, 2)$, M_a is an orbifold which is homeomorphic to S^2 and has two $\mathbb{Z}_2$ singularities. The orbifold π_1 of M_a is $\mathbb{Z}_2$, but the π_1 of M_a as a topological space is trivial.

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Let a be a regular or a critical value of ϕ. Define

$$M^a = \{x \in M \mid \phi(x) \leq a\}.$$

By Morse theory, we have the following lemmas about how M^a and $\phi^{-1}(a)$ change when ϕ doesn't cross or crosses a critical level.

Lemma 0.3 (Theorem 3.1 in [7]). *Assume $[a,b] \subset im(\phi)$ is an interval consisting of regular values. Then $\phi^{-1}(a)$ is diffeomorphic to $\phi^{-1}(b)$.*

Lemma 0.4 (See [7] and [3]). *Let $c \in (a,b)$ be the only critical value of ϕ in $[a,b]$, $F \subset \phi^{-1}(c)$ the fixed point set component, D^- the negative disk bundle of F, and $S(D^-)$ its sphere bundle. Then M^b is homotopy equivalent to $M^a \cup_{S(D^-)} D^-$.*

Lemma 0.5. *Under the same hypothesis of Lemma 0.4, $\phi^{-1}(a) \cup_{S(D^-)} D^-$ has the homotopy type of $\phi^{-1}(c)$.*

Proof. If F is a point, then from the proof of Theorem 3.2 in [7], we can see that the region between $\phi^{-1}(a) \cup_{S(D^-)} D^-$ and $\phi^{-1}(c)$ is homotopy equivalent to both $\phi^{-1}(a) \cup_{S(D^-)} D^-$ and $\phi^{-1}(c)$. (See pp. 18 and 19 in [7].)

The same idea applies when F is a submanifold. □

This lemma immediately implies the following

Lemma 0.6. *Under the same hypothesis of Lemma 0.4, M_c has the homotopy type of $M_a \cup_{S(D^-)/S^1} D^-/S^1$.*

We will also need

Lemma 0.7. *Assume F is a critical set, $\phi(F) \in (a,b)$ and there are no other critical sets in $\phi^{-1}([a,b])$. If* index$(F) = 2$, *then there is an embedding i from F to M_a such that $S(D^-)$ can be identified with the restriction of $\phi^{-1}(a)$ to F, i.e., we have the following bundle identification:*

$$\begin{array}{ccccc} S^1 & \hookrightarrow & S(D^-) & \rightarrow & \phi^{-1}(a) \\ & & \downarrow & & \downarrow \\ & & F & \xrightarrow{i} & M_a \end{array} \tag{0.2}$$

Proof. Assume that the positive normal bundle D^+ of F has complex rank m. We may assume $\phi(F) = 0$. By Lemma 0.3, we can assume $a = -\epsilon$ and $b = +\epsilon$ for ϵ small. By the equivariant symplectic embedding theorem ([8]), a tubular neighborhood of F is equivariantly diffeomorphic to $P \times_G (\mathbb{C} \times \mathbb{C}^m)$, where $G = S^1 \times U(m)$ and P is a principal G-bundle over F. The moment map can be written $\phi = -p_0|z_0|^2 + p_1|z_1|^2 + \cdots + p_m|z_m|^2$, where $p_0, p_1, ..., p_m$ are positive integers. Then $\phi^{-1}(-\epsilon) = P \times_G (S^1 \times \mathbb{C}^m)$, $M_{-\epsilon} = P \times_G (S^1 \times \mathbb{C}^m)/S^1$. $F = P \times_G (S^1 \times 0)/S^1 \subset M_{-\epsilon}$, and $S(D^-) = P \times_G S^1$ is the restriction of $\phi^{-1}(-\epsilon)$ to F. □

We are now ready to prove the theorem.

Proof. Let us put the critical values of ϕ in the order

$$\text{minimal} = 0 < a_1 < a_2 < \cdots < a_k = \text{maximal}.$$

First, we prove $\pi_1(\text{minimum}) = \pi_1(M_{red})$.

For $a \in (0, a_1)$, by the equivariant symplectic embedding theorem, $\phi^{-1}(a)$ is a sphere bundle over the minimum. Assume the fiber of this sphere bundle is S^{2l+1} with $l \geq 0$. Then M_a is diffeomorphic to a weighted $\mathbb{C}P^l$ bundle over the minimum (possibly an orbifold). The weighted $\mathbb{C}P^l$ is the symplectic reduction of S^{2l+1} by the S^1 action with different weights. We can easily see that $S^{2l+1} \to$ weighted $\mathbb{C}P^l$ induces a surjection in π_1 since the inverse image of each point is connected. So the weighted $\mathbb{C}P^l$ is simply connected, hence $\pi_1(M_a) = \pi_1(\text{minimum})$.

Next, let $b \in (a_1, a_2)$, and let $F \subset \phi^{-1}(a_1)$ be the critical set. (If there are other critical sets on the same level, argue similarly for each connected component.)

By Lemma 0.6 and the Van-Kampen theorem, we have

$$\pi_1(M_{a_1}) = \pi_1(M_a) *_{\pi_1(S(D^-)/S^1)} \pi_1(D^-/S^1) = \pi_1(M_a),$$

since $S(D^-)/S^1$ is a weighted projectivized bundle over F, and D^-/S^1 is homotopy equivalent to F, so $\pi_1(S(D^-)/S^1)$ is isomorphic to $\pi_1(D^-/S^1)$.

Similarly, using $-\phi$, we can obtain $\pi_1(M_b) = \pi_1(M_{a_1})$.

By induction on the critical values, and by repeating the argument each time ϕ crosses a critical level, we see that if $a' \in (a_{k-1}, a_k)$, then $\pi_1(M_{a'}) = \pi_1(\text{minimum})$. Similarly to the proof of $\pi_1(M_a) = \pi_1(\text{minimum})$ when $a \in (0, a_1)$, we have $\pi_1(M_{a'}) = \pi_1(\text{maximum})$.

Therefore we have proved that $\pi_1(M_{red}) = \pi_1(\text{minimum}) = \pi_1(\text{maximum})$.

Next, we prove $\pi_1(M) = \pi_1(\text{minimum})$.

Consider M^a, for $a \in (0, a_1)$. Since M^a is a complex disk bundle over the minimum, $\pi_1(M^a) = \pi_1(\text{minimum}) = \pi_1(M_a)$.

Consider $b \in (a_1, a_2)$, and let $F \subset \phi^{-1}(a_1)$ be the critical set.

First assume index$(F) = 2$. By Lemma 0.4, and the Van-Kampen theorem,

$$\pi_1(M^b) = \pi_1(M^a) *_{\pi_1(S(D^-))} \pi_1(D^-) = \pi_1(M^a) *_{\pi_1(S(D^-))} \pi_1(F).$$

Consider the fibration

$$\begin{array}{ccc} S^1 & \hookrightarrow & S(D^-) \\ & & \downarrow \\ & & F \end{array} \tag{0.3}$$

and its homotopy exact sequence

$$\cdots \to \pi_1(S^1) \xrightarrow{j} \pi_1(S(D^-)) \xrightarrow{f} \pi_1(F) \to 0. \tag{0.4}$$

The map f is surjective. By Lemma 0.7, the image of $\ker(f) = \text{im}(j)$ in $\pi_1(M_a)$ is 0. By induction, $\pi_1(M^a) = \pi_1(M_a)$. So the image of $\ker(f)$ in $\pi_1(M^a)$ is 0. Hence, $\pi_1(M^b) = \pi_1(M^a) = \pi_1(\text{minimum})$.

If index$(F) > 2$, then the corresponding map $\pi_1(S(D^-)) \to \pi_1(F)$ is an isomorphism. So we also have $\pi_1(M^b) = \pi_1(M^a)$.

By induction, we see that $\pi_1(M) = \pi_1(\text{minimum})$. □

Remark 0.8. The proof that $\pi_1(\text{minimum}) = \pi_1(M_{red})$ can be achieved by using known results about how the reduced space changes after ϕ crosses a critical level. (See [4], for instance, or [6] where the action is semi-free.) After the first induction step, when ϕ crosses a critical set F, if index$(F) = 2$, then M_a is homeomorphic to M_{a_1}; if index$(F) > 2$, then M_{a_1} can be obtained from M_a by a blow-up followed by a blow-down. M_b and M_{a_1} are similarly related. Then we modify the proof of D. McDuff's ([9]) result

Proposition 0.9. *If $\tilde{X}$ is the blow-up of X along a submanifold N, then $\pi_1(\tilde{X}) = \pi_1(X)$.*

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