π₁ OF HAMILTONIAN S¹ MANIFOLDS

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Abstract. Let \((M,\omega)\) be a connected, compact symplectic manifold equipped with a Hamiltonian \(S^1\) action. We prove that, as fundamental groups of topological spaces,
\[
\pi_1(M) = \pi_1(\text{minimum}) = \pi_1(\text{maximum}) = \pi_1(M_{\text{red}}),
\]
where \(M_{\text{red}}\) is the symplectic quotient at any value in the image of the moment map \(\phi\).

Let \((M,\omega)\) be a connected, compact symplectic manifold equipped with a circle action. If the action is Hamiltonian, then the moment map \(\phi : M \to \mathbb{R}\) is a perfect Bott-Morse function. Its critical sets are precisely the fixed point sets \(M^{S^1}\) of the \(S^1\) action, and \(M^{S^1}\) is a disjoint union of symplectic submanifolds. Each fixed point set has even index. By [1], \(\phi\) has a unique local minimum and a unique local maximum. We will use Morse theory to prove

**Theorem 0.1.** Let \((M^{2n},\omega)\) be a connected, compact symplectic manifold equipped with a Hamiltonian \(S^1\) action. Then as fundamental groups of topological spaces,
\[
\pi_1(M) = \pi_1(\text{minimum}) = \pi_1(\text{maximum}) = \pi_1(M_{\text{red}}),
\]
where \(M_{\text{red}}\) is the symplectic quotient at any value in the image of the moment map \(\phi\).

**Remark 0.2.** The theorem is not true for orbifold \(\pi_1\) of \(M_{\text{red}}\), as shown in the example below. (See [5] or [11] for the definition of orbifold \(\pi_1\).)

Let \(a \in \text{im}(\phi)\), and \(\phi^{-1}(a) = \{x \in M \mid \phi(x) = a\}\) be the level set. Define \(M_a = \phi^{-1}(a)/S^1\) to be the symplectic quotient.

Note that if \(a\) is a regular value of \(\phi\), and if the circle action on \(\phi^{-1}(a)\) is not free, then \(M_a\) is an orbifold, and we have an orbi-bundle:
\[
\begin{array}{ccc}
S^1 & \hookrightarrow & \phi^{-1}(a) \\
& \downarrow & \\
M_a
\end{array}
\] (0.1)

If \(a\) is a critical value of \(\phi\), then \(M_a\) is a stratified space ([10]).

Now, let \(S^1\) act on \((S^2 \times S^2, 2\rho \oplus \rho)\) (where \(\rho\) is the standard symplectic form on \(S^2\)) by \(\lambda(z_1, z_2) = (\lambda^2 z_1, \lambda z_2)\). Let 0 be the minimal value of the moment map. Then for \(a \in (1, 2)\), \(M_a\) is an orbifold which is homeomorphic to \(S^2\) and has two \(\mathbb{Z}_2\) singularities. The orbifold \(\pi_1\) of \(M_a\) is \(\mathbb{Z}_2\), but the \(\pi_1\) of \(M_a\) as a topological space is trivial.

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Let $a$ be a regular or a critical value of $\phi$. Define

$$M^a = \{x \in M \mid \phi(x) \leq a\}.$$  

By Morse theory, we have the following lemmas about how $M^a$ and $\phi^{-1}(a)$ change when $\phi$ doesn’t cross or crosses a critical level.

**Lemma 0.3** (Theorem 3.1 in [7]). Assume $[a, b] \subset \text{im(} \phi \text{)}$ is an interval consisting of regular values. Then $\phi^{-1}(a)$ is diffeomorphic to $\phi^{-1}(b)$.

**Lemma 0.4** (See [7] and [8]). Let $c \in (a, b]$ be the only critical value of $\phi$ in $[a, b]$, $F \subset \phi^{-1}(c)$ the fixed point set component, $D^-$ the negative disk bundle of $F$, and $S(D^-)$ its sphere bundle. Then $M^b$ is homotopy equivalent to $M^a \cup S(D^-) D^-$. 

**Lemma 0.5.** Under the same hypothesis of Lemma 0.4, $\phi^{-1}(a) \cup S(D^-) D^-$ has the homotopy type of $\phi^{-1}(c)$.

**Proof.** If $F$ is a point, then from the proof of Theorem 3.2 in [7], we can see that the region between $\phi^{-1}(a) \cup S(D^-) D^-$ and $\phi^{-1}(c)$ is homotopy equivalent to both $\phi^{-1}(a) \cup S(D^-) D^-$ and $\phi^{-1}(c)$. (See pp. 18 and 19 in [7].)

The same idea applies when $F$ is a submanifold. \hfill $\Box$

This lemma immediately implies the following

**Lemma 0.6.** Under the same hypothesis of Lemma 0.4, $M_c$ has the homotopy type of $M_a \cup S(D^-) D^- / S^1$. 

We will also need

**Lemma 0.7.** Assume $F$ is a critical set, $\phi(F) \in (a, b)$ and there are no other critical sets in $\phi^{-1}([a, b])$. If $\text{index}(F) = 2$, then there is an embedding $i$ from $F$ to $M_a$ such that $S(D^-)$ can be identified with the restriction of $\phi^{-1}(a)$ to $F$, i.e., we have the following bundle identification:

$$S^1 \hookrightarrow S(D^-) \twoheadrightarrow \phi^{-1}(a)
\downarrow \quad \downarrow
F \quad i \
M_a$$

(0.2)

**Proof.** Assume that the positive normal bundle $D^+$ of $F$ has complex rank $m$. We may assume $\phi(F) = 0$. By Lemma 0.3, we can assume $a = -\epsilon$ and $b = +\epsilon$ for $\epsilon$ small. By the equivariant symplectic embedding theorem ([8]), a tubular neighborhood of $F$ is equivariantly diffeomorphic to $P \times_G (\mathbb{C} \times \mathbb{C}^m)$, where $G = S^1 \times U(m)$ and $P$ is a principal $G$-bundle over $F$. The moment map can be written $\phi = -p_0|z_0|^2 + p_1 |z_1|^2 + \cdots + p_m |z_m|^2$, where $p_0, p_1, ..., p_m$ are positive integers. Then $\phi^{-1}(-\epsilon) = P \times_G (S^1 \times \mathbb{C}^m)$, $M_{-\epsilon} = P \times_G (S^1 \times \mathbb{C}^m) / S^1$, $F = P \times_G (S^1 \times 0) / S^1 \subset M_{-\epsilon}$, and $S(D^-) = P \times_G S^1$ is the restriction of $\phi^{-1}(-\epsilon)$ to $F$. \hfill $\Box$

We are now ready to prove the theorem.

**Proof.** Let us put the critical values of $\phi$ in the order

$$\text{minimal} = 0 < a_1 < a_2 < \cdots < a_k = \text{maximal}.$$ 

First, we prove $\pi_1(\text{minimum}) = \pi_1(M_{\text{red}})$. 

For $a \in (0, a_1)$, by the equivariant symplectic embedding theorem, $\phi^{-1}(a)$ is a sphere bundle over the minimum. Assume the fiber of this sphere bundle is $S^{2l+1}$ with $l \geq 0$. Then $M_a$ is diffeomorphic to a weighted $\mathbb{C}P^l$ bundle over the minimum.
(possibly an orbifold). The weighted $\mathbb{C}P^l$ is the symplectic reduction of $S^{2l+1}$ by the $S^1$ action with different weights. We can easily see that $S^{2l+1} \to$ weighted $\mathbb{C}P^l$ induces a surjection in $\pi_1$ since the inverse image of each point is connected. So the weighted $\mathbb{C}P^l$ is simply connected, hence $\pi_1(M_a) = \pi_1(\text{minimum})$.

Next, let $b \in (a_1, a_2)$, and let $F \subset \phi^{-1}(a_1)$ be the critical set. (If there are other critical sets on the same level, argue similarly for each connected component.)

By Lemma 0.9 and the Van-Kampen theorem, we have

$$\pi_1(M_{a_1}) = \pi_1(M_a) *_{\pi_1(S(D^-)/S^1)} \pi_1(D^-/S^1) = \pi_1(M_a),$$

since $S(D^-)/S^1$ is a weighted projectivized bundle over $F$, and $D^-/S^1$ is homotopy equivalent to $F$, so $\pi_1(S(D^-)/S^1)$ is isomorphic to $\pi_1(D^-/S^1)$.

Similarly, using $-\phi$, we can obtain $\pi_1(M_b) = \pi_1(M_{a_1})$.

By induction on the critical values, and by repeating the argument each time $\phi$ crosses a critical level, we see that if $a' \in (a_{k-1}, a_k)$, then $\pi_1(M_{a'}) = \pi_1(\text{minimum})$.

Similarly to the proof of $\pi_1(M_a) = \pi_1(\text{minimum})$ when $a \in (0, a_1)$, we have $\pi_1(M_{a'}) = \pi_1(\text{maximum})$.

Therefore we have proved that $\pi_1(M_{\text{red}}) = \pi_1(\text{minimum}) = \pi_1(\text{maximum})$.

Next, we prove $\pi_1(M) = \pi_1(\text{minimum})$.

Consider $M^a$, for $a \in (0, a_1)$. Since $M^a$ is a complex disk bundle over the minimum, $\pi_1(M^a) = \pi_1(\text{minimum}) = \pi_1(M_a)$.

Consider $b \in (a_1, a_2)$, and let $F \subset \phi^{-1}(a_1)$ be the critical set.

First assume $\text{index}(F) = 2$. By Lemma 0.4 and the Van-Kampen theorem,

$$\pi_1(M^b) = \pi_1(M^a) *_{\pi_1(S(D^-)/S^1)} \pi_1(D^-) = \pi_1(M^a) *_{\pi_1(S(D^-)/S^1)} \pi_1(F).$$

Consider the fibration

$$S^1 \hookrightarrow S(D^-) \xrightarrow{j} F$$

and its homotopy exact sequence

$$(0.4) \quad \cdots \to \pi_1(S^1) \xrightarrow{j} \pi_1(S(D^-)) \xrightarrow{j} \pi_1(F) \to 0.$$ 

The map $f$ is surjective. By Lemma 0.7, the image of $\ker(f) = \text{im}(j)$ in $\pi_1(M_a)$ is $0$. By induction, $\pi_1(M^a) = \pi_1(M_a)$. So the image of $\ker(f)$ in $\pi_1(M^a)$ is $0$. Hence, $\pi_1(M^b) = \pi_1(M^a) = \pi_1(\text{minimum})$.

If $\text{index}(F) > 2$, then the corresponding map $\pi_1(S(D^-)) \to \pi_1(F)$ is an isomorphism. So we also have $\pi_1(M^b) = \pi_1(M^a)$.

By induction, we see that $\pi_1(M) = \pi_1(\text{minimum})$. 

\textbf{Remark} 0.8. The proof that $\pi_1(\text{minimum}) = \pi_1(M_{\text{red}})$ can be achieved by using known results about how the reduced space changes after $\phi$ crosses a critical level. (See [4], for instance, or [6] where the action is semi-free.) After the first induction step, when $\phi$ crosses a critical set $F$, if $\text{index}(F) = 2$, then $M_a$ is homeomorphic to $M_{a_1}$; if $\text{index}(F) > 2$, then $M_a$ can be obtained from $M_a$ by a blow-up followed by a blow-down. $M_b$ and $M_{a_1}$ are similarly related. Then we modify the proof of D. McDuff’s (9) result.

\textbf{Proposition 0.9.} If $\tilde{X}$ is the blow-up of $X$ along a submanifold $N$, then $\pi_1(\tilde{X}) = \pi_1(X)$. 

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REFERENCES


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