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CIRCLE MAPS HAVING AN INFINITE ω-LIMIT SET WHICH CONTAINS A PERIODIC ORBIT HAVE POSITIVE TOPOLOGICAL ENTROPY

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ABSTRACT. Let f be a continuous map from the circle to itself. The main result of this paper is that the topological entropy of f is positive if and only if f has an infinite ω-limit set which contains a periodic orbit.

1. Introduction

Let f be a continuous map from a continuum X to itself. We denote the n-fold composition f^n of f with itself by $f \circ \ldots \circ f$ and f^0 the identity map. Let x be a point of X. We define the orbit $\mathrm{Orb}(x, f)$ of x by $\{f^n(x) | n \geq 0\}$, and we define the *ω-limit set* of x to be the set $\omega(x, f) = \{y \in X|$ for each neighborhood V of y and each positive integer n, $V \cap \mathrm{Orb}(f^n(x), f)$ is nonempty$\}$. It is known that $\omega(x, f)$ is nonempty and strongly invariant, i.e. $f(\omega(x, f)) = \omega(x, f)$. See [BC, p.72] for details. If $\omega(x, f)$ is finite, by [BC, Lemma IV4, p.72], $\omega(x, f)$ is a periodic orbit of some point.

Let z be a periodic point of f. The *unstable set* of z is defined to be the set $W(z, f) = \{x \in X|$ for any neighborhood V of z, $x \in f^k(V)$ for some $k > 0\}$. A point y is *homoclinic for f* if there exists a point $z \neq y$ such that $f^n(z) = z$ for some $n > 0$, $y \in W(z, f^n)$ and $f^{kn}(y) = z$ for some $k > 0$. This definition of homoclinic points first appeared in [B]. A point $x \in X$ is a *nonwandering point for f* if for any open set U containing x there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.

The following theorem is well known.

Theorem 1.1. *Let f be a continuous map from a compact interval I to itself. The following statements are equivalent:*

(*a*) *f has positive topological entropy,*
(*b*) *f^n is strictly turbulent for some positive integer n,*
(*c*) *f has a nonwandering homoclinic point, and*
(*d*) *for some $c \in I$, $\omega(c, f)$ properly contains a periodic orbit.*

See [BC, p.25] for strictly turbulent, [BC, Section VIII] for topological entropy and [BC, p.124, p.153 and p.218] or [SKSF, Theorem 4.19, p.112] for Theorem 1.1.

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We can consider the following theorem, analogous to the corresponding conditions in Theorem 1.1.

Theorem 1.2. *Let f be a continuous map from the circle S^1 to itself. The following statements are equivalent:*

(a) *f has positive topological entropy,*
(b) *f^n is strictly turbulent for some positive integer n,*
(c) *f has a nonwandering homoclinic point, and*
(d) *for some $c \in S^1$, $\omega(c, f)$ properly contains a periodic orbit.*

See [BC, p.229] for strictly turbulent of circle maps. Although the aim of this paper is to prove Theorem 1.2, it is known that (a), (b) and (c) are equivalent (see [BC, p.229] or [BCMN, Theorem B+, p.529] for details) and that conditions (a), (b) and (c) imply condition (d) (see [BC, p.230] for details). Therefore, in this paper, we prove that condition (d) implies condition (a). This is the answer of the question in [BC, p.230].

2. Definitions

Notation 2.1. Let Y be a subspace of a space X, and let $\text{int}\, Y$ and $\text{Cl}Y$ denote the interior and the closure of Y in a space X, respectively.

Definition 2.2. Let f be a continuous map from a space X to itself. A point $x \in X$ is a *fixed point for f* if $f(x) = x$. A point $x \in X$ is a *periodic point of period $n \geq 1$ for f* if $f^n(x) = x$. We denote the sets of fixed points, periodic points and nonwandering points for f by $\mathrm{F}(f)$, $\mathrm{P}(f)$ and $\Omega(f)$, respectively.

Definition 2.3. Let us denote a subspace $\{z||z| = 1\}$ of the complex plane, i.e., the circle, by S^1. Let x and y be two distinct points of S^1. We denote the closed arc from x counterclockwise to y by $[x, y]$, and we denote $(x, y) = [x, y] \setminus \{x, y\}$, $(x, y] = [x, y] \setminus \{x\}$ and $[x, y) = [x, y] \setminus \{y\}$.

Let π be the canonical projection from the real line onto S^1 defined by $\pi(t) = e^{2\pi i t}$, $\widetilde{x}$ and $\widetilde{y}$ two points of the real line such that $\widetilde{y} \in (\widetilde{x}, \widetilde{x}+1)$, $\pi(\widetilde{x}) = x$ and $\pi(\widetilde{y}) = y$. We see that $\pi|[\widetilde{x}, \widetilde{y}] : [\widetilde{x}, \widetilde{y}] \to [x, y]$ is a homeomorphism. Every continuous map f from the circle S^1 to itself has countable many lifts, i.e., continuous maps $\widetilde{f}$ from the real line to itself satisfying $f \circ \pi = \pi \circ \widetilde{f}$.

Definition 2.4. Let p be a fixed point of a continuous map f from S^1 to itself. If V is a neighborhood of p in $[p, -p)$ (in $(-p, p]$, respectively), we say V is an *R-neighborhood* of p (*L-neighborhood* of p, respectively). Let $S = R, L$. The *S-sided unstable manifold of p* is defined by

$$W(p, f, S) = \{x \in S^1 | \text{ for any } S\text{-neighborhood } V \text{ of } p,\ x \in f^k(V) \text{ for some } k > 0\}.$$

Let $\widetilde{p}$ be a fixed point of a continuous map $\widetilde{f}$ from the real line to itself. If $\widetilde{V}$ is a neighborhood of $\widetilde{p}$ in $[\widetilde{p}, \widetilde{p}+1)$ (in $(\widetilde{p}-1, \widetilde{p}]$, respectively), we say $\widetilde{V}$ is an *R-neighborhood* of $\widetilde{p}$ (*L-neighborhood* of $\widetilde{p}$, respectively). The *S-sided unstable manifold of $\widetilde{p}$* is defined by $W(\widetilde{p}, \widetilde{f}, S) = \{\widetilde{x}|$ for any S-neighborhood $\widetilde{V}$ of $\widetilde{p}$, $\widetilde{x} \in \widetilde{f}^k(\widetilde{V})$ for some $k > 0\}$.

We see that $W(p, f) = W(p, f, R) \cup W(p, f, L)$ and that $W(\widetilde{p}, \widetilde{f}) = W(\widetilde{p}, \widetilde{f}, R) \cup W(\widetilde{p}, \widetilde{f}, L)$.

3. Elementary lemmas

By [BC, Proposition II 2, p.48], we have the following lemma.

Lemma 3.1. *Let $\widetilde{f}$ be a continuous map from the real line to itself and $\widetilde{z} \in \mathrm{F}(\widetilde{f})$.*

1. *If $W(\widetilde{z}, \widetilde{f}, L) \cap (\widetilde{z}, \infty) \neq \emptyset$, then $W(\widetilde{z}, \widetilde{f}, R) \subset W(\widetilde{z}, \widetilde{f}, L)$.*
2. *If $W(\widetilde{z}, \widetilde{f}, L) \cap (-\infty, \widetilde{z}) = \emptyset$, then $W(\widetilde{z}, \widetilde{f}, L) = \{\widetilde{z}\}$ or $W(\widetilde{z}, \widetilde{f}, R)$.*
3. *If $W(\widetilde{z}, \widetilde{f}, R) \cap (-\infty, \widetilde{z}) \neq \emptyset$, then $W(\widetilde{z}, \widetilde{f}, L) \subset W(\widetilde{z}, \widetilde{f}, R)$.*
4. *If $W(\widetilde{z}, \widetilde{f}, R) \cap (\widetilde{z}, \infty) = \emptyset$, then $W(\widetilde{z}, \widetilde{f}, L) = \{\widetilde{z}\}$ or $W(\widetilde{z}, \widetilde{f}, R)$.*

By [BC, Proposition II 1 and 3] and [BCMN, Lemma 1], we have the following lemma.

Lemma 3.2. *Let X be either a compact interval or the circle or the real line, f a continuous map from X to itself, $z \in \mathrm{F}(f)$ and $S = R, L$. Then $W(z, f, S)$ and $W(z, f)$ are connected, $f(W(z, f, S)) = W(z, f, S)$ and $f(W(z, f)) = W(z, f)$.*

Lemma 3.3. *Let f be a continuous map from the circle S^1 to itself, $z \in \mathrm{F}(f)$ and $S = R, L$. Also, let $\widetilde{f}$ be the lift of f with $\widetilde{z} \in \mathrm{F}(\widetilde{f})$ satisfying $\pi(\widetilde{z}) = z$. Then $\pi(W(\widetilde{z}, \widetilde{f}, S)) = W(z, f, S)$ and $\pi(W(\widetilde{z}, \widetilde{f})) = W(z, f)$.*

Proof. We give the proof for the first assertion. Let $\widetilde{x} \in W(\widetilde{z}, \widetilde{f}, S)$ and U a small S-neighborhood of z. There exists a small S-neighborhood $\widetilde{V}$ of $\widetilde{z}$ with $\pi(\widetilde{V}) \subset U$. Since $\widetilde{x} \in W(\widetilde{z}, \widetilde{f}, S)$, we have a positive integer n such that $\widetilde{x} \in \widetilde{f}^n(\widetilde{V})$, thus, $\pi(\widetilde{x}) \in \pi(\widetilde{f}^n(\widetilde{V})) = f^n(\pi(\widetilde{V})) \subset f^n(U)$. We conclude that $\pi(\widetilde{x}) \in W(z, f, S)$ and that $\pi(W(\widetilde{z}, \widetilde{f}, S)) \subset W(z, f, S)$.

Let $x \in W(z, f, S) \setminus \{z\}$ and let $\{\widetilde{U_n}\}_{n=1}^{\infty}$ be a sequence of small connected S-neighborhoods of $\widetilde{z}$ with $\bigcap_{n=1}^{\infty} \widetilde{U_n} = \{\widetilde{z}\}$. Since $\pi(\widetilde{U_n})$ is a small S-neighborhood of z for each n, there exists a positive integer k_n such that $x \in f^{k_n}(\pi(\widetilde{U_n}))$. Since $x \in \pi(\widetilde{f}^{k_n}(\widetilde{U_n}))$, we see that $\pi^{-1}(x) \cap \widetilde{f}^{k_n}(\widetilde{U_n}) \neq \emptyset$. Thus there exist $\widetilde{x} \in \pi^{-1}(x) \cap (\widetilde{z}-1, \widetilde{z}+1)$ and a sequence $\ell_1, \ell_2, \dots$ of positive integers such that $\widetilde{x} \in \widetilde{f}^{k_{\ell_n}}(\widetilde{U_{\ell_n}})$ for each n. This shows that $\widetilde{x} \in W(\widetilde{z}, \widetilde{f}, S)$ and that $\pi(W(\widetilde{z}, \widetilde{f}, S)) \supset W(z, f, S)$. □

Corollary 3.4. *Let f be a continuous map from the circle S^1 to itself with lift $\widetilde{f}$. If $\widetilde{y}$ is a homoclinic point for $\widetilde{f}$, then $y = \pi(\widetilde{y})$ is a homoclinic point for f.*

Lemma 3.5. *Let f be a continuous map from the circle S^1 to itself with lift $\widetilde{f}$ and $\widetilde{z} \in \mathrm{F}(\widetilde{f})$, $S, S' \in \{R, L\}$, and $\widetilde{y}, \widetilde{z}'$ two points of the real line satisfying that $\widetilde{y} \in W(\widetilde{z}, \widetilde{f}, S)$, $\pi(\widetilde{z}) = \pi(\widetilde{z}') \neq \pi(\widetilde{y})$ and $\widetilde{f}^n(\widetilde{y}) = \widetilde{z}'$ for some $n \geq 1$. If $\widetilde{f}^n(\widetilde{U})$ contains an S-neighborhood of $\widetilde{z}'$ for each S'-neighborhood $\widetilde{U}$ of $\widetilde{y}$, then $y = \pi(\widetilde{y})$ is a nonwandering homoclinic point for f.*

Proof. We notice that $z = \pi(\widetilde{z}) \in \mathrm{F}(f)$ and that $f^n(y) = \pi(\widetilde{f}^n(\widetilde{y})) = \pi(\widetilde{z}') = z$. It follows from Lemma 3.3 that $y \in \pi(W(\widetilde{z}, \widetilde{f}, S)) = W(z, f, S) \subset W(z, f)$. It suffices to show that $y \in \Omega(f)$. Let U be a small S'-neighborhood of y and $\widetilde{U}$ an S'-neighborhood of $\widetilde{y}$ with $\pi(\widetilde{U}) = U$. From the assumption, $\widetilde{f}^n(\widetilde{U})$ contains some small S-neighborhood $\widetilde{V}$ of $\widetilde{z}'$. Since $\pi(\widetilde{z}) = \pi(\widetilde{z}')$, there exists the integer δ such that $\widetilde{z}' = \widetilde{z} + \delta$. Set $\widetilde{V} - \delta = \{\widetilde{x} - \delta | \widetilde{x} \in \widetilde{V}\}$. We notice that $\widetilde{V} - \delta$ is an S-neighborhood of $\widetilde{z}$ and that $\pi(\widetilde{V} - \delta) = \pi(\widetilde{V})$ is an S-neighborhood of z. Also, since $\widetilde{y} \in W(\widetilde{z}, \widetilde{f}, S)$, we have a positive integer m such that $\widetilde{y} \in \widetilde{f}^m(\widetilde{V} - \delta)$. This shows that $y \in \pi(\widetilde{f}^m(\widetilde{V}-\delta)) = f^m(\pi(\widetilde{V}-\delta)) = f^m(\pi(\widetilde{V})) \subset f^m(\pi(\widetilde{f}^n(\widetilde{U}))) = f^{m+n}(\pi(\widetilde{U})) = f^{m+n}(U)$, thus, $y \in \Omega(f)$. □

4. A proof of Theorem 1.2

Lemma 4.1. *Let f be a continuous map from the circle S^1 to itself. If there exists a point $c \in S^1$ such that $\omega(c,f)$ is infinite containing some fixed point, then f has positive topological entropy.*

Proof. Set $c_m = f^m(c)$ for each m. Choose $z \in \omega(c,f) \cap \mathrm{F}(f)$. We have an increasing sequence $\{n_k\}$ of positive integers, $S = R, L$ and an S-neighborhood U_z of z such that $c_{n_k} \in U_z \setminus \{z\}$ for all k and that $\lim_{k\to\infty} c_{n_k} = z$. Since $\omega(c,f)$ is infinite, we see that $W(z,f,S) \neq \{z\}$.

We show that $\mathrm{Orb}(c,f) \cap W(z,f,S) \neq \emptyset$. We suppose that $\mathrm{Orb}(c,f) \cap W(z,f,S) = \emptyset$. Since $W(z,f,S)$ is connected by Lemma 3.2, there exist a point x of $\omega(c,f)$ and a compact interval A such that $\omega(c,f) \subset A$, $A \setminus \mathrm{int}\, A = \{z,x\}$ and $W(z,f,S) \subset S^1 \setminus \mathrm{int}\, A$. Since $\omega(c,f)$ is infinite, by the definition of $W(z,f,S)$, there exists a point $x' \in \mathrm{Cl}\, W(z,f,S) \cap \mathrm{int}\, A$. This is a contradiction.

Since $\mathrm{Orb}(c,f) \cap W(z,f,S) \neq \emptyset$, Lemma 3.2 implies that $\omega(c,f) \subset \mathrm{Cl}W(z,f,S)$. We suppose that $\mathrm{Cl}W(z,f,S) \neq S^1$. By Lemma 3.2, we see that $\mathrm{Cl}W(z,f,S)$ is a compact interval. Since $\mathrm{Orb}(c,f) \cap W(z,f,S) \neq \emptyset$, we have $c_m \in W(z,f,S)$ for some m. Since $\omega(c_m,f) = \omega(c,f)$ and $f(\mathrm{Cl}\, W(z,f,S)) = \mathrm{Cl}\, W(z,f,S)$, we have $\omega(c,f) = \omega(c_m, f|\,\mathrm{Cl}\, W(z,f,S))$, where $f|\,\mathrm{Cl}\, W(z,f,S) : \mathrm{Cl}\, W(z,f,S) \to \mathrm{Cl}\, W(z,f,S)$ is the restriction of f. From Theorem 1.1, $f|\,\mathrm{Cl}\, W(z,f,S)$ has positive topological entropy. We conclude from [BC, Proposition VIII 5, p.193] that f has positive topological entropy. We may assume that $\mathrm{Cl}W(z,f,S) = S^1$.

Since (a) and (c) in Theorem 1.2 are equivalent, we are going to show that f has a nonwandering homoclinic point.

Let $\widetilde{f}$ be the lift of f with $\widetilde{z} \in \mathrm{F}(\widetilde{f})$ satisfying $\pi(\widetilde{z}) = z$. We suppose that $\mathrm{Cl}W(\widetilde{z},\widetilde{f},S)$ is compact, i.e., bounded. Since $\pi(\mathrm{Cl}W(\widetilde{z},\widetilde{f},S)) = \mathrm{Cl}W(z,f,S) = S^1$ by Lemma 3.3, there exists a point $\widetilde{c} \in \mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S)$ satisfying $\pi(\widetilde{c}) = c$. Set $\widetilde{c_m} = \widetilde{f}^m(\widetilde{c})$ for each m. Let $\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S) : \mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S) \to \mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S)$ be the restriction of $\widetilde{f}$. Since $\pi(\widetilde{c_m}) = c_m$ for each m, we have $\pi(\omega(\widetilde{c},\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S))) \subset \omega(c,f)$. Let $x \in \omega(c,f)$. We have a sequence $n_1, n_2, \dots$ of positive integers such that $\lim_{k\to\infty} c_{n_k} = x$. Since $\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S)$ is compact, there exist a subsequence $n_{i_1}, n_{i_2}, \dots$ and $\widetilde{x} \in \mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S)$ such that $\lim_{k\to\infty} \widetilde{c_{n_{i_k}}} = \widetilde{x}$. Since $\widetilde{x} \in \omega(\widetilde{c},\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S))$ and $\pi(\widetilde{x}) = x$, we see that $\pi(\omega(\widetilde{c},\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S))) \supset \omega(c,f)$ and conclude that $\pi(\omega(\widetilde{c},\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S))) = \omega(c,f)$. This shows that $\omega(\widetilde{c},\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S))$ properly contains a periodic orbit. By Theorem 1.1, $\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S)$ has a homoclinic point $\widetilde{y} \in \Omega(\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S))$. Since $\pi(\Omega(\widetilde{f}|\,\mathrm{Cl}\, W(\widetilde{z},\widetilde{f},S))) \subset \pi(\Omega(\widetilde{f})) \subset \Omega(f)$, by Corollary 3.4, f has a homoclinic point $\pi(\widetilde{y}) \in \Omega(f)$. We may assume that $\mathrm{Cl}W(\widetilde{z},\widetilde{f},S)$ is unbounded.

We suppose that $W(\widetilde{z},\widetilde{f},S)$ contains some small open connected S-neighborhood $\widetilde{U}$ of $\widetilde{z}$. Since $\mathrm{Cl}W(\widetilde{z},\widetilde{f},S)$ is unbounded containing $\widetilde{z}$, we have $\delta = 1, -1$ such that $\widetilde{z} + \delta \in W(\widetilde{z},\widetilde{f},S)$. We suppose that $S = L$. Since $\widetilde{z} + \delta \in W(\widetilde{z},\widetilde{f},L)$, there exists a positive integer n such that $\widetilde{z} + \delta \in \widetilde{f}^n(\widetilde{U})$. Set $\widetilde{y} = \max\{\widetilde{y}' \in \widetilde{U} \mid \widetilde{f}^n(\widetilde{y}') = \widetilde{z} + \delta\}$.

If $\delta = 1$, we see that $\widetilde{f}^n(\widetilde{V})$ is an L-neighborhood of $\widetilde{z}+1$ for each small R-neighborhood $\widetilde{V}$ of $\widetilde{y}$. It follows from Lemma 3.5 that $y = \pi(\widetilde{y})$ is a nonwandering homoclinic point.

Next we suppose that $\delta = -1$ and $\widetilde{z}+1 \notin W(\widetilde{z}, \widetilde{f}, L)$. Since $(-\infty, \widetilde{z}-1) \subset W(\widetilde{z}, \widetilde{f}, L)$, there exist a point $\widetilde{x} \in (\widetilde{z}-1, \widetilde{z})$ and a positive integer m such that $\widetilde{f}^m(\widetilde{x}) < \widetilde{z}-1$. Thus, we have $\widetilde{y}' = \min(\widetilde{x}, \widetilde{z}) \cap \widetilde{f}^{-m}(\widetilde{z}-1)$. By the definition of $\widetilde{y}'$, we see that $\widetilde{f}^m(\widetilde{V}')$ is an L-neighborhood of $\widetilde{z}-1$ for each small L-neighborhood $\widetilde{V}'$ of $\widetilde{y}'$. It follows from Lemma 3.5 that $y' = \pi(\widetilde{y}')$ is a nonwandering homoclinic point.

We can prove $S = R$ by an argument similar to that for $S = L$. Thus, we may assume that $W(\widetilde{z}, \widetilde{f}, S)$ contains no S-neighborhood of $\widetilde{z}$.

Without loss of generality, we may assume that $S = L$. We see from Lemma 3.1(2) that $W(\widetilde{z}, \widetilde{f}, R) = W(\widetilde{z}, \widetilde{f}, L) = [\widetilde{z}, \infty)$. Let $\widetilde{U}$ be a small connected R-neighborhood of $\widetilde{z}$. Since $\widetilde{z}+1 \in W(\widetilde{z}, \widetilde{f}, R)$, there exists a positive integer n such that $\widetilde{z}+1 \in \widetilde{f}^n(\widetilde{U})$. Set $\widetilde{y} = \min\{\widetilde{y}' \in \widetilde{U} | \widetilde{f}^n(\widetilde{y}') = \widetilde{z}+1\}$. As above, we can show that $y = \pi(\widetilde{y})$ is a nonwandering homoclinic point. □

Theorem 4.2. *Condition* (*d*) *in Theorem* 1.2 *implies condition* (*a*) *in Theorem* 1.2.

Proof. Let f be a continuous map from the circle S^1 to itself. Let $c \in S^1$ such that $\omega(c, f)$ properly contains a periodic point of period n. By [BC, Lemma IV 4, p.72], $\omega(c, f)$ is infinite, thus, $\omega(f^j(c), f^n)$ is also infinite for all j with $0 \le j < n$ by [BC, p.70]. Since at least one of these ω-limit sets contains a fixed point of f^n by [BC, p.70], we see from Lemma 4.1 that f^n has positive topological entropy. We conclude from [BC, Proposition VIII 2, p.191] that f has positive topological entropy. □

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