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THE RANK OF FINITELY GENERATED MODULES OVER GROUP ALGEBRAS

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ABSTRACT. We show the existence of a rank function on finitely generated modules over group algebras $K\Gamma$, where K is an arbitrary field and Γ is a finitely generated amenable group. This extends a result of W. Lück (1998).

1. INTRODUCTION

The goal of this paper is to construct a real-valued non-trivial rank function on finitely generated modules over a group algebra $K\Gamma$, where K is an arbitrary field and Γ is a finitely generated amenable group.

Theorem 1. *Let K be a field and let Γ be a finitely generated amenable group. Then there exists a function*

$$rk : Fin.Gen.Modules(K\Gamma) \to \mathbb{R}$$

such that

1. *If $M \simeq N$, then $rk(N) = rk(M)$.*
2. *If $0 \to M \to P \to N \to 0$ is an exact sequence, then $rk(M) + rk(N) = rk(P)$.*
3. *$rk(K\Gamma) = 1$.*

The existence of such rank function was proved by Lück [2] in the case when K is the field of complex numbers. In [2], for any finitely generated module M over $\mathbb{C}\Gamma$ the author associated an invariant subspace V_M of the Hilbert space $[l^2(\Gamma)]^n$. The rank of the module M was defined as the von Neumann dimension of V_M. In an earlier paper [1] we extended Lück's result for finite fields using the notion of topological entropy. In the general case, we will apply the following strategy. For any finitely generated module M over $K\Gamma$, we associate an invariant subspace V_M of the function field $\prod_{\gamma\in\Gamma} K^n$. Then we define an average dimension for such invariant spaces that behaves similarly to the von Neumann dimension. The rank of M will be the average dimension of the associated invariant subspace V_M. An observation of Lück suggests that Theorem 1 might serve as a ring theoretic characterisation of amenability. That is, such a rank function may not exist if Γ is a non-amenable group.

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2. Invariant subspaces

Let K be an arbitrary field equipped with the discrete topology and let Γ be a countable group. For any integer $n \geq 1$, one can consider $\prod_{\gamma\in\Gamma} K^n$, the space of vector-valued functions with the product topology. The space $\prod_{\gamma\in\Gamma} K^n$ is metrizable and $\{f_m\}_{m=1}^\infty \to f$ if for any $\gamma \in \Gamma$, $f_m(\gamma) = f(\gamma)$ for sufficiently large m depending on γ. The natural right Γ-action on $\prod_{\gamma\in\Gamma} K^n$ is defined by

$$(f \cdot \gamma)(\delta) = f(\delta\gamma^{-1})\,.$$

Note that we also have a left Γ-action on $\prod_{\gamma\in\Gamma} K^n$ commuting with the right Γ-action:

$$(\gamma \cdot f)(\delta) = f(\gamma^{-1}\delta)\,.$$

Both the left and right Γ-actions are continuous and K-linear. Denote by $K\Gamma$ the group algebra of Γ over K. The left Γ-action defines a representation of $K\Gamma$ in the homomorphism ring of $\prod_{\gamma\in\Gamma} K^n$. A subspace $V \subseteq \prod_{\gamma\in\Gamma} K^n$ is called *invariant* if it is closed, K-linear and invariant under the right Γ-action.

3. Γ-equivariant maps

Let us consider $Mat_{n\times m}(K\Gamma)$ the space of finitely supported $Mat_{n\times m}(K)$-valued functions on Γ. If $M = \sum_{\gamma\in\Gamma} M(\gamma)\gamma \in Mat_{n\times m}(K\Gamma)$ and $f \in \prod_{\gamma\in\Gamma} K^m$, then the matrix multiplication $M : \prod_{\gamma\in\Gamma} K^m \to \prod_{\gamma\in\Gamma} K^n$ is defined by

$$Mf(\delta) = \sum_{\gamma\in\Gamma} M(\gamma) f(\gamma^{-1}\delta) \in K^n\,.$$

Note that $M : \prod_{\gamma\in\Gamma} K^m \to \prod_{\gamma\in\Gamma} K^n$ is continuous, K-linear and Γ-equivariant, that is, it commutes with the right Γ-action.

Proposition 3.1. *Let $V \subseteq \prod_{\gamma\in\Gamma} K^m$, $W \subseteq \prod_{\gamma\in\Gamma} K^n$ be invariant subspaces and let $T : V \to W$ be a continuous K-linear Γ-equivariant map. Then there exists $M_T \in Mat_{n\times m}(K\Gamma)$ such that for any $v \in V$, $M_T v = Tv$.*

Proof. Let e_Γ be the unit element of Γ. Note that if T_1 and T_2 are Γ-equivariant maps and $T_1 v(e_\Gamma)$ and $T_2 v(e_\Gamma)$ for all $v \in V$, then $T_1 v = T_2 v$ for all $v \in V$. Since T is continuous, there exists a finite subset $A \subseteq \Gamma$ such that $Tv(e_\Gamma)$ depends only on the values of v at the elements of A. Since

$$Hom_K(\bigoplus_{\gamma\in A} K^m, K^n) = \bigoplus_{\gamma\in A} Hom_K(K^m, K^n)\,,$$

it is easy to see that there exist elements $\{M_\gamma\}_{\gamma\in A} \subseteq Mat_{n\times m}(K)$ such that $Tv(e_\Gamma) = \sum_{\gamma\in A} M_\gamma \cdot v(\gamma)$, for any $v \in V$. Let $M_T(\gamma) = M_{\gamma^{-1}}$. Then $Tv(e_\Gamma) = M_T v(e_\Gamma)$, for any $v \in V$.

4. Duality

In this section we study the baby-version of Pontrjagin's duality. All proofs are straightforward and are left to the reader. Let $\sum_{\gamma\in\Gamma} K^n$ be the space of finitely supported vector-valued functions on Γ equipped with the discrete topology. Then one has the pairing

$$\langle , \rangle : \prod_{\gamma\in\Gamma} K^n \times \sum_{\gamma\in\Gamma} K^n \to K$$

defined by $\langle v, f\rangle = \sum_{\gamma\in\Gamma}(v(\gamma), f(\gamma))$, where $(,)$ is the usual scalar product on K^n. Denote by $(\sum_{\gamma\in\Gamma} K^n)^*$ the space of all K-linear maps $T : \sum_{\gamma\in\Gamma} K^n \to K$. If $U \subseteq \sum_{\gamma\in\Gamma} K^n$ is a K-linear subspace, then its *orthogonal set* $U^\perp$ is defined as

$$U^\perp = \{v \in \prod_{\gamma\in\Gamma} K^n : \langle v, f\rangle = 0,\ \text{for any } f \in U\}\,.$$

Note that $U^\perp$ is a closed K-linear subspace in $\prod_{\gamma\in\Gamma} K^n$. Similarly, if $W \subseteq \prod_{\gamma\in\Gamma} K^n$, then

$$W^\perp = \{f \in \sum_{\gamma\in\Gamma} K^n : \langle v, f\rangle = 0,\ \text{for any } v \in W\}\,.$$

Lemma 4.1. *Let $W \subseteq \prod_{\gamma\in\Gamma} K^n$ be a K-linear subspace. Then $(W^\perp)^\perp = \overline{W}$, the closure of W in $\prod_{\gamma\in\Gamma} K^n$.*

Proof. It is enough to prove that if $w \notin \overline{W}$, then there exists $f \in W^\perp$ such that $\langle w, f\rangle \neq 0$. Since $w \notin \overline{W}$, there exists a finite subset $A \subseteq \Gamma$ such that $w\,|_A \notin W\,|_A$. Therefore there exists an element $f \in \sum_{\gamma\in\Gamma} K^n$ supported on A that is orthogonal to W and has a non-zero scalar product with w. □

The weak topology of $(\sum_{\gamma\in\Gamma} K^n)^*$ can be defined the usual way via base sets. Let $T \in (\sum_{\gamma\in\Gamma} K^n)^*$ and let $f_1, f_2, \dots, f_r \in \sum_{\gamma\in\Gamma} K^n$. Then,

$$N_{T,f_1,f_2,\dots,f_r} = \{S \in (\sum_{\gamma\in\Gamma} K^n)^* : Tf_i = Sf_i,\ \text{for all } i,\ 1 \le i \le r\}\,.$$

Proposition 4.1. *There exists a continuous isomorphism Ψ between $\prod_{\gamma\in\Gamma} K^n$ and $(\sum_{\gamma\in\Gamma} K^n)^*$, defined by $\Psi(v) = \langle v, .\rangle$. Similarly, if $A \subseteq \sum_{\gamma\in\Gamma} K^n$ is a K-linear subspace, then $\Psi_A(v) = \langle v, .\rangle$ defines a continuous isomorphism between $A^\perp$ and $(\sum_{\gamma\in\Gamma} K^n/A)^*$.* □

Let $A, B \subseteq \sum_{\gamma\in\Gamma} K^n$ be K-linear subspaces and $T : \sum_{\gamma\in\Gamma} K^n/A \to \sum_{\gamma\in\Gamma} K^n/B$ a K-linear transformation. Then the adjoint of T is the well-defined continuous map $T^* : B^\perp \to A^\perp$ such that $\langle T^*v, f\rangle = \langle v, Tf\rangle$, for any $v \in B^\perp$ and $f \in \sum_{\gamma\in\Gamma} K^n/A$. Let us suppose that $A \subseteq \sum_{\gamma\in\Gamma} K^n$ is invariant under the right Γ-action. Then $A^\perp$ is an invariant subspace. Also, if T is Γ-equivariant, then T^* is Γ-equivariant as well. Indeed, the adjoint of the right translation by $\gamma \in \Gamma$ is just the right translation by γ^{-1}. Observe that $\sum_{\gamma\in\Gamma} K^n$ can be identified with $(K\Gamma)^n$ as a right $K\Gamma$-module. The subspaces $A \in \sum_{\gamma\in\Gamma} K^n$ which are invariant under the right Γ-action can be identified with the $K\Gamma$-submodules. Hence the spaces $\sum_{\gamma\in\Gamma} K^n/A$ can be identified with finitely generated right $K\Gamma$-modules. We can formulate the previous observations in the following proposition.

Proposition 4.2. *The duality functor $M \to M^*$ associates an invariant subspace to a finitely generated $K\Gamma$-module. The adjoint functor $T \to T^*$ associates a Γ-equivariant continuous map between invariant subspaces to any module homomorphism.* □

Later we will see that if M and N are isomorphic modules, then M^* and N^* are isomorpic as well; also if T is an isomorphism, then T^* is an isomorphism as well.

5. Weak exactness

Proposition 5.1. *Let* $0 \to M \xrightarrow{i} P \xrightarrow{\pi} N \to 0$ *be an exact sequence of finitely generated right* $K\Gamma$*-modules. Then* $0 \to N^* \xrightarrow{\pi^*} P^* \xrightarrow{i^*} M^* \to 0$ *is weak exact, that is,*

(1) π^* *is injective.*
(2) $M^* = \overline{Ran\, i^*}$
(3) $Ker\, i^* = \overline{Ran\, \pi^*}$.

Proof. (1) If $\pi^*(f) = 0$, then $\langle \pi^*(f), p\rangle = 0$ for any $p \in P$. That is, $\langle f, \pi(p)\rangle = 0$. Hence $\langle f, n\rangle = 0$ for any $n \in N$. Therefore $f = 0$. (2) Let $\langle i^*(g), m\rangle = 0$, for any $g \in P^*$. Then $\langle g, i(m)\rangle = 0$, thus m must be 0. Therefore $i^*(g)$ is dense in M^*, by Lemma 4.1. (3) $\langle i^* \circ \pi^*(f), m\rangle = \langle f, \pi \circ i(m)\rangle = 0$. This shows that $Ran\, \pi^* \subseteq Ker\, i^*$. Since $Ker\, i^*$ is closed, it means that $\overline{Ran\, \pi^*} \subseteq Ker\, i^*$. Now suppose that $p \perp Ran\, \pi^*$. By Lemma 4.1, it is enough to see that $p \perp Ker\, i^*$ as well. However, $0 = \langle p, \pi^*(f)\rangle = \langle \pi(p), f\rangle$, that is, $\pi(p) = 0$. Hence by the exactness, $p = i(m)$. Now $\langle i(m), g\rangle = 0$ if $g \in Ker\, i^*$. □

6. Amenability

Let us recall the important notion of quasi-tilings of amenable groups following the paper of Ornstein & Weiss [3]. Let Γ be a finitely generated group with symmetric generating set $\{g_1, g_2, \dots, g_k\}$. The generators determine a word-metric d on Γ in such a way that the right multiplications by the elements of Γ are isometries. The group Γ is called *amenable* if it has an exhaustion

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \bigcup_{k=1}^{\infty} \mathcal{F}_k = \Gamma$$

by finite subsets such that for any fixed $r \in \mathbb{N}$,

$$\lim_{k\to\infty} \frac{|\partial_r \mathcal{F}_k|}{|\mathcal{F}_k|} = 0\,,$$

where $\partial_r \mathcal{F}_k = \{\gamma \in \Gamma :\ d(\gamma, \mathcal{F}_k) \leq r \text{ and } d(\gamma, \Gamma - \mathcal{F}_k) \leq r\}$. Such exhaustions are called Følner-exhaustions. The finite subsets $A_1, A_2, \dots, A_k \subseteq \Gamma$ are called ϵ-disjoint if there are subsets $B_1, B_2, \dots, B_k \subseteq \Gamma$ such that

(1) $B_i \subseteq A_i$ for any $i = 1, 2, \dots, k$.
(2) $\frac{|B_i|}{|A_i|} > 1 - \epsilon$.
(3) $|B_i \cap B_j| = \emptyset$ if $i \neq j$.

Also, we say that the finite subsets $A_1, A_2, \dots, A_k \subseteq \Gamma$ δ-covers the set $A \subseteq \Gamma$, if

$$\frac{|A \cap (\bigcup_{i=1}^k A_i)|}{|A|} \geq \delta\,.$$

Finally, the finite subsets $A_1, A_2, \dots, A_k \subseteq \Gamma$ ϵ-quasitile the set $A \subseteq \Gamma$, if there are finite sets $C_1, C_2, \dots, C_k \subseteq \Gamma$ such that

(1) $A_i C_i \subseteq A$, for any $i = 1, 2, \dots, k$.
(2) $A_i C_i \cap A_j C_j = \emptyset$ if $i \neq j$.
(3) $\{A_i c : c \in C_i\}$ form an ϵ-disjoint family.
(4) $\{A_i C_i,\ i = 1, 2, \dots, k\}$ form a $(1-\epsilon)$-cover of A.

The sets C_i are called the tiling centers. The following result is due to Ornstein & Weiss [3]:

Proposition 6.1. *Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \dots$ be Følner-exhaustions. Let $\epsilon \in (0, \frac{1}{4})$ and $N > 0$. Then there exist integers $N \leq n_1 < n_2 < \cdots < n_l$ and $M > 0$ such that $\{\mathcal{F}_{n_1}, \mathcal{F}_{n_2}, \dots, \mathcal{F}_{n_l}\}$ ϵ-quasitile $\mathcal{G}_m$ for any $m \geq M$.*

7. Average dimension

Let $B \subseteq \Gamma$ be a finite subset and let $V \subseteq \prod_{\gamma\in\Gamma} K^n$ be an invariant subset. Denote by V_B the space of vectors v supported on B such that there exists $w \in V$, $v\mid_B = w\mid_B$. The *average dimension*, $\dim^A_\Gamma(V)$, is defined as

$$\dim^A_\Gamma(V) = \limsup_{k\to\infty} \frac{\dim V_{\mathcal{F}_k}}{|\mathcal{F}_k|}\,.$$

We have the following proposition.

Proposition 7.1. *The average dimension of an invariant subspace does not depend on the particular choice of the Følner-exhaustion and in fact,*

$$\dim^A_\Gamma(V) = \lim_{k\to\infty} \frac{\dim V_{\mathcal{F}_k}}{|\mathcal{F}_k|}\,.$$

Proof. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \dots$ be Følner-exhaustions. It is enough to prove that

$$\limsup_{k\to\infty} \frac{\dim V_{\mathcal{F}_k}}{|\mathcal{F}_k|} \geq \limsup_{k\to\infty} \frac{\dim V_{\mathcal{G}_k}}{|\mathcal{G}_k|}\,.$$

First of all note that by sorting out a suitable subsequence we may suppose that

$$\dim^A_\Gamma(V) = \lim_{k\to\infty} \frac{\dim V_{\mathcal{F}_k}}{|\mathcal{F}_k|}\,.$$

It is enough to see that for any $\delta > 0$, there exists $Q > 0$ such that if $q > Q$, then

$$\frac{\dim V_{\mathcal{G}_q}}{|\mathcal{G}_q|} \leq (1+\delta)\dim^A_\Gamma(V) + \delta\,. \tag{1}$$

Pick an integer N such that if $k > N$, then

$$\frac{\dim V_{\mathcal{F}_k}}{|\mathcal{F}_k|} \leq \dim^A_\Gamma(V) + \epsilon\,,$$

where the exact value of ϵ will be determined at the end of the proof. By Proposition 6.1, we have integers $N \leq n_1 < n_2 \cdots < n_l$ and $M > 0$ such that $\{\mathcal{F}_{n_1}, \mathcal{F}_{n_2}, \dots, \mathcal{F}_{n_l}\}$ ϵ-quasitile $\mathcal{G}_m$ for any $m \geq M$. Denote by $S_1, S_2, \dots, S_{c_m}$ the translated copies of Følner-sets used in the ϵ-quasitiling of $\mathcal{G}_m$ and denote by $T_1, T_2, \dots, T_{c_m}$ the ϵ-disjoint parts of the S_i's as in the definition of ϵ-disjoint sets. Then by the invariance of V, we have the following estimate for any $1 \leq i \leq c_m$:

$$\dim V_{T_i} \leq \dim V_{S_i} \leq |S_i|(\dim^A_\Gamma(V) + \epsilon)\,.$$

Hence, $\dim V_{T_i} \leq (1-\epsilon)^{-1}|T_i|(\dim^A_\Gamma(V) + \epsilon)$. Also, $|\mathcal{G}_m| \geq \sum_{i=1}^{c_m} |T_i|$. By a straightforward pigeon-hole argument,

$$\dim V_{\mathcal{G}_m} \leq \sum_{i=1}^{c_m} \dim V_{T_i} + \epsilon^2 n |\mathcal{G}_m|\,.$$

Therefore,

$$\dim V_{\mathcal{G}_m} \leq (1-\epsilon)^{-1}(\dim_\Gamma^A(V)+\epsilon)|\mathcal{G}_m| + \epsilon^2 n|\mathcal{G}_m|\,,$$

that is,

$$\frac{\dim V_{\mathcal{G}_m}}{|\mathcal{G}_m|} \leq (1-\epsilon)^{-1}(\dim_\Gamma^A(V)+\epsilon) + \epsilon^2 n\,.$$

Choosing ϵ carefully, we can assert that (1) holds. □

8. Extended configurations

In this section we use a technique that proved to be quite useful in the case of calculating topological entropies [4], [1]. Let $V \subseteq \prod_{\gamma\in\Gamma} K^n$ be an invariant subspace. An extended configuration V^E is defined the following way. For any finite set $B \subseteq \Gamma$, let $V_B^E \subseteq \prod_{\gamma\in\Gamma} K^n$ be a finite-dimensional K-linear vectorspace supported on B that satisfies the following four conditions:

(1) $V_B \subseteq V_B^E$.
(2) $V_{B\gamma}^E = V_B^E \cdot \gamma$, for any $\gamma \in \Gamma$.
(3) If $B \subseteq C$ and $f \in V_C^E$, then there is a $g \in V_B^E$ such that $f\,|_B = g\,|_B$.
(4) If $v \in \prod_{\gamma\in\Gamma} K^n$ and for any finite set $B \subseteq \Gamma$ there exists $f \in V_B^E$ such that $f\,|_B = v\,|_B$, then $f \in V$.

We call such a system of functions an extended configuration of V. The average dimension of V^E is defined as

$$\dim_\Gamma^A(V^E) = \limsup_{k\to\infty} \frac{\dim V_{\mathcal{F}_k}^E}{|\mathcal{F}_k|}\,.$$

For any $r > 0$ and a finite set $A \subseteq \Gamma$ let

$$B_r(A) = \{\gamma \in \Gamma : d(\gamma, A) \leq r\}\,.$$

The following proposition is a vectorspace version of the well-known König lemma.

Proposition 8.1. *Let V^E be an extended configuration of an invariant vectorspace $V \subseteq \prod_{\gamma\in\Gamma} K^n$ and let $A \subseteq \Gamma$ be a finite set. Denote by V_A^r the space of K^n-valued functions f supported on A such that there exists $v \in V_{B_r(A)}^E$ so that $v\,|_A = f\,|_A$. Then if r is sufficiently large, depending on A, then $V_A^r = V_A$.*

Proof. Let $C \subseteq \Gamma$ be a finite subset. Call a vector $v \in V_C^E$ extendable if for any finite set $C \subseteq D \subseteq \Gamma$ there exists $w \in V_D^E$ such that $w\,|_C = v\,|_C$. Denote the space of extendable vectors by V_C^∞. Obviously, $V_C \subseteq V_C^\infty$. For $v \in V_C^E$, the height of v, $h(v)$ is defined as follows:

$$h(v) = \sup_{r\geq 0}\{r : \text{there exists } w \in V_{B_r(C)}^E \text{ such that } w\,|_C = v\,|_C\}\,.$$

Note that $h(v) = \infty$ means that v is extendable.

Lemma 8.1. *If $v \in V_C^\infty$, then there exists $w \in V_{B_1(C)}^\infty$ such that $v\,|_C = w\,|_C$.*

Proof. Denote by H_v the affine space of vectors in $V_{B_1(C)}^E$ such that if $w \in H_v$, then $w\,|_C = v\,|_C$. Since v is extendable, the height function is unbounded on H_v. Let $H_v^k = \{w \in H_v \;:\; h(w) \geq k\}$. Then H_v^k are affine spaces as well and

$$H_v \supseteq H_v^1 \supseteq H_v^2 \supseteq \dots\,.$$

Hence for certain k, $H^k_v = H^{k+1}_v = \dots$, therefore $H^k_v \subseteq V^\infty_{B_1(C)}$. This proves our lemma. □

Lemma 8.2. $V_C = V_C^\infty$.

Proof. Let $v \in V_C^\infty$. Then by the previous lemma, there exist $v_k \in V^\infty_{B_k(C)}$ such that $v_k \mid_{B_k(C)} = v_{k+1} \mid_{B_k(C)}$. Hence the pointwise limit $l \in \prod_{\gamma\in\Gamma} K^n$ of $\{v_k\}_{k=1}^\infty$ exists. Then $l \mid_{B_k(C)} = v_k$ and $l \mid_C = v \mid_C$. Hence by condition (4), $l \in V$ and $v \in V_C$. □

Now we turn back to the proof of our proposition. Note that $V_C^E \supseteq V_C^1 \supseteq V_C^2 \supseteq \dots$ is a decreasing sequence of subspaces containing V_C. Hence for some r, $V_C^r = V_C^{r+1} = \dots$. Thus $V_C^r = V_C^\infty$. Hence by our previous lemma $V_C^r = V_C$. □

Proposition 8.2. *Let V^E be an extended configuration of the invariant subspace V, then*

$$\dim^A_\Gamma(V) = \dim^A_\Gamma(V^E)\ .$$

Proof. We need to show that for any $\delta > 0$, there exists $Q > 0$ such that if $q \geq Q$, then

$$\frac{\dim V^E_{\mathcal{F}_q}}{|\mathcal{F}_q|} \leq (1+\delta)\dim^A_\Gamma(V) + \delta\,. \tag{2}$$

Let ϵ be a real number; its exact value will be chosen at the end of the proof. By Proposition 6.1, we have integers $N \leq n_1 < n_2 < \dots < n_l$ and $M > 0$ such that $\{\mathcal{F}_{n_1}, \mathcal{F}_{n_2}, \dots, \mathcal{F}_{n_l}\}$ ϵ-quasitile $\mathcal{G}_m$ for any $m \geq M$, and

$$\frac{\dim V_{\mathcal{F}_{n_i}}}{|\mathcal{F}_{n_i}|} \leq \dim^A_\Gamma(V) + \epsilon\,, \quad \text{for any } 1 \leq i \leq k\,.$$

By our previous lemma, there exists $r > 0$ such that for any $1 \leq i \leq k$,

$$V^r_{\mathcal{F}_{n_i}} = V_{\mathcal{F}_{n_i}}\,.$$

Let $P_1, P_2, \dots, P_{c_s}$ be those translated copies in the ϵ-quasitiling of $\mathcal{F}_s$ such that $B_r(P_i) \subseteq \mathcal{F}_s$. Let $R_i \subseteq P_i$ denote the disjoint parts. Let N_s be the cardinality of those points in $\mathcal{F}_s$ which are not covered by $\bigcup_{i=1}^{c_s} R_i$. Then $N_s \leq \epsilon^2|\mathcal{F}_s| + |\partial_{r+D+1}\mathcal{F}_s|$, where $D = \max_{1\leq i\leq k} diam\,(\mathcal{F}_{n_i})$. Then by the third condition in the definition of extended configurations,

$$\dim V^E_{\mathcal{F}_s} \leq \sum_{i=1}^{c_s} \dim V_{R_i} + n(\epsilon^2|\mathcal{F}_s| + |\partial_{r+D+1}\mathcal{F}_s|)\,.$$

Hence,

$$\dim V^E_{\mathcal{F}_s} \leq |\mathcal{F}_s|(1-\epsilon)^{-1}(\dim^A_\Gamma(V) + \epsilon) + n(\epsilon^2|\mathcal{F}_s| + |\partial_{r+D+1}\mathcal{F}_s|)\,.$$

Therefore, for large s,

$$\frac{\dim V^E_{\mathcal{F}_s}}{|\mathcal{F}_s|} \leq (1-\epsilon)^{-1}(\dim^A_\Gamma(V) + \epsilon) + 2n\epsilon^2\ .$$

Picking the right ϵ one can assert that (2) holds. □

9. Additivity

Proposition 9.1. *Let $V \subseteq \prod_{\gamma\in\Gamma} K^n$, $W \subseteq \prod_{\gamma\in\Gamma} K^m$ be invariant subspaces. Let $T : V \to W$ be a continuous Γ-equivariant map. Then*

$$\dim_\Gamma^A(V) = \dim_\Gamma^A(Ker\,T) + \dim_\Gamma^A(\overline{Ran\,T})\,.$$

Proof. First note that both $Ker\,T$ and $\overline{Ran\,T}$ are invariant subspaces. By Proposition 3.1 it is given by matrix multiplication. Therefore T has finite width, that is, there exists $L > 0$ such that for any $\gamma \in \Gamma$, $Tf(\gamma) = Tg(\gamma)$, whenever $f\,|_{B_L(\gamma)} = g\,|_{B_L(\gamma)}$. For any finite subset $A \subseteq \Gamma$, let

$$N_A^E = \{v \in \prod_{\gamma\in\Gamma} K^n : \text{supp } v \subseteq A \text{ and there exists } w \in V_{B_L(A)} \text{ such that } w\,|_A = v\,|_A \text{ and } Tv\,|_A = 0\}\,.$$

Then N_A^E is an extended configuration of $Ker\,T$. Also let

$$M_A^E = \{v \in \prod_{\gamma\in\Gamma} K^n : \text{supp } v \subseteq A \text{ and there exists } w \in V_{B_L(A)} \text{ such that } Tw\,|_A = v\,|_A\}\,.$$

Then M_A^E is an extended configuration of $\overline{Ran\,T}$. Let $R_A : V_{B_L(A)} \to \prod_{\gamma\in A} K^m$ be the restriction of T onto A. Then $Ran\,(R_A) \cong M_A^E$. Obviously, for any Følner-set $\mathcal{F}_s$, $\dim N_{\mathcal{F}_s}^E \leq \dim Ker\,R_{\mathcal{F}_s}$. However, any element of $Ker\,R_{\mathcal{F}_s}$ is determined by its restriction onto $\mathcal{F}_s \cup \partial_r \mathcal{F}_s$. Hence,

$$\dim Ker\,R_{\mathcal{F}_s} \leq \dim N_{\mathcal{F}_s}^E + n|\partial_r \mathcal{F}_s|\,. \tag{3}$$

Now,

$$\dim Ker\,R_{\mathcal{F}_s} + \dim Ran\,R_{\mathcal{F}_s} = \dim V_{B_L(\mathcal{F}_s)}. \tag{4}$$

Observe that

$$\lim_{s\to\infty} \frac{\dim V_{B_L(\mathcal{F}_s)}}{|\mathcal{F}_s|} = \dim_\Gamma^A(V)\,.$$

Also, by Proposition 8.2 and (3),

$$\lim_{s\to\infty} \frac{\dim Ker\,R_{\mathcal{F}_s}}{|\mathcal{F}_s|} = \dim_\Gamma^A(Ker\,T)$$

and

$$\lim_{s\to\infty} \frac{\dim Ran\,R_{\mathcal{F}_s}}{|\mathcal{F}_s|} = \dim_\Gamma^A(\overline{Ran\,T})\,.$$

Hence by (4) our proposition holds. □

Proposition 9.2. *Let $V \subseteq \prod_{\gamma\in\Gamma} K^n$, $W \subseteq \prod_{\gamma\in\Gamma} K^m$ and $Z \subseteq \prod_{\gamma\in\Gamma} K^s$ be invariant subspaces. Also let $T : V \to W$ and $S : W \to Z$ be continuous Γ-equivariant maps such that*

$$0 \to V \xrightarrow{T} W \xrightarrow{S} Z \to 0$$

is weak-exact. Then,

$$\dim_\Gamma^A(V) + \dim_\Gamma^A(Z) = \dim_\Gamma^A(W)\,.$$

Proof. By our previous proposition, $\dim_\Gamma^A(Ker\,S) = \dim_\Gamma^A(\overline{Ran\,T}) = \dim_\Gamma^A(V)$ and $\dim_\Gamma^A(Ker\,S) + \dim_\Gamma^A(Z) = \dim_\Gamma^A(W)$, which imply our proposition. □

10. The end of the proof of Theorem 1

Let M be a finitely generated module over $K\Gamma$. Define $rk(M)$ as $\dim_\Gamma^A(M^*)$. Then by Proposition 9.2, $rk(M) = rk(N)$ if $M \cong N$, and $rk([K\Gamma]) = 1$. Now, let $0 \to M \to P \to N \to 0$ be an exact sequence of finitely generated modules. Then by Proposition 9.2 and Proposition 5.1, $rk(P) = rk(N) + rk(M)$. This finishes the proof of our theorem. □

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