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A NOTE ON A THEOREM OF RAUBENHEIMER AND RODE

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Abstract. We prove the converse of Raubenheimer and Rode's Banach algebra version of the Perron-Frobenius Theorem.

Let $(A, ||\cdot||)$ be a complex Banach algebra with unit e and $||e|| = 1$. A set $W \subseteq A$ is called an algebra wedge in the case that W is closed, $W + W \subseteq W$, $\lambda W \subseteq W$ $(\lambda \geq 0)$, $W \cdot W \subseteq W$, and $e \in W$. As usual, by setting $a \leq b :\iff b - a \in W$ we obtain a reflexive and transitive relation on A, and call A ordered by W. In the sequel let $\sigma_A(a)$ and $r(a)$ denote the spectrum and the spectral radius of $a \in A$, respectively.

In [2] Raubenheimer and Rode proved the following version of the Perron-Frobenius Theorem.

Theorem 1. *Let A be ordered by an algebra wedge W such that the spectral radius is increasing on W. Then $r(a) \in \sigma_A(a)$ for all $a \in W$.*

The purpose of this note is to prove the converse of this result:

Theorem 2. *Let $a \in A$ be such that $r(a) \in \sigma_A(a)$. Then there exists an algebra wedge W such that the spectral radius is increasing on W, and $a \in W$.*

Proof. For $S \subseteq A$ let $\Gamma(S)$ denote the centralizer of S, that is,

$$\Gamma(S) = \{v \in A : vu = uv \ (u \in S)\},$$

and set $B = \Gamma(\Gamma(\{a\}))$. Then B is a closed commutative subalgebra of A, and $e, a \in B$. Moreover $\sigma_A(u) = \sigma_B(u)$ $(u \in B)$ (see for example [3], Theorem 11.22). In particular $r(u)$ does mean the same in A and B. Let Δ denote the set of all nontrivial multiplicative linear functionals on B.

Since B is commutative and $r(a) \in \sigma_B(a)$ there is a $\psi \in \Delta$ such that $\psi(a) = r(a)$. We define

$$W := \{u \in B : r(u) = \psi(u)\}.$$

Obviously $a \in W$, and $e \in W$ since $\psi(e) = 1$. To see that W is an algebra wedge first note that whenever $u, v \in B$ then, by Theorem 11.23 in [3],

$$(*) \qquad r(u + v) \leq r(u) + r(v), \quad r(uv) \leq r(u)r(v).$$

Moreover $r(u) = \max_{\varphi \in \Delta} |\varphi(u)|$ $(u \in B)$.

For $u, v \in W$ we obtain

$$r(uv) \leq r(u)r(v) = \psi(u)\psi(v) = \psi(uv) \leq r(uv),$$

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hence $uv \in W$, and analogously $u + v \in W$. Next, for each $u \in B$ and $\lambda \geq 0$ we have $r(\lambda u) = \lambda r(u)$, hence $\lambda W \subseteq W$ $(\lambda \geq 0)$.

To prove that W is closed, note that $(*)$ implies

$$|r(u) - r(v)| \leq r(u - v) \leq ||u - v|| \quad (u, v \in B).$$

If (u_n) is a convergent sequence in W with limit v, say, then

$$\lim_{n\to\infty} \psi(u_n) = \psi(v),$$

$$|\psi(u_n) - r(v)| = |r(u_n) - r(v)| \leq ||u_n - v|| \to 0 \quad (n \to \infty),$$

and therefore $r(v) = \psi(v)$. Altogether W is an algebra wedge.

Now let A be ordered by W. It remains to prove that r is increasing on W. Let $0 \leq u_1 \leq u_2$. Then

$$r(u_2) - r(u_1) = \psi(u_2 - u_1) = r(u_2 - u_1) \geq 0 \Longrightarrow r(u_1) \leq r(u_2).$$

□

Examples. Consider $A = \mathbb{C}^{2\times 2}$ and

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad r(a) = 1 \in \{-1, 1\} = \sigma_A(a).$$

In this case it is easy to check that

$$B = \left\{ u = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : \ \alpha, \beta \in \mathbb{C} \right\},$$

and that $\psi(u) = \alpha + \beta$ is in Δ with $\psi(a) = 1$. Since

$$\sigma_B\left(\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right) = \{\alpha + \beta, \alpha - \beta\}$$

we get

$$W = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : \ |\alpha - \beta| \leq \alpha + \beta \right\}.$$

The same considerations for

$$a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad r(a) = 0 \in \{0\} = \sigma_A(a)$$

lead to

$$B = \left\{ u = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \ \alpha, \beta \in \mathbb{C} \right\}, \quad \psi(u) = \alpha,$$

and

$$W = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \ \alpha \geq 0 \right\}.$$

In the first example W is an algebra cone, that is, in addition $W \cap (-W) = \{0\}$. In the second example W is not an algebra cone since $a \in W \cap (-W)$. This is not an accident, as the following results will show.

The radical of A, denoted by $\mathrm{rad}(A)$, is the intersection of all maximal ideals in A, and A is called semisimple in case $\mathrm{rad}(A) = \{0\}$. It follows from [3], Theorem 11.9, that if A is commutative, then

$$\mathrm{rad}(A) = \{u \in A : r(u) = 0\}.$$

Theorem 3. *Let $a \in A$, $r(a) \in \sigma(a)$, and let B, ψ and W be as in the proof of Theorem 2. Then* $\operatorname{rad}(B) = W \cap (-W)$. *In particular W is an algebra cone if and only if B is semisimple.*

Proof. 1. If $u \in \operatorname{rad}(B)$, then

$$\sigma(u) = \{\varphi(u) : \varphi \in \Delta\} = \{0\}.$$

Thus $\psi(u) = 0 = r(u)$, hence $u \in W$. We have $-u \in \operatorname{rad}(B)$, since $\operatorname{rad}(B)$ is an ideal in B. As above we conclude $-u \in W$. Therefore $u \in W \cap (-W)$.

2. Let $u \in W \cap (-W)$. Then

$$\psi(u) = r(u) = r(-u) = \psi(-u) = -\psi(u).$$

Hence $\psi(u) = r(u) = 0$, and therefore $u \in \operatorname{rad}(B)$. □

As usual, an algebra cone is called normal if there exists a constant $\alpha \geq 1$ such that $0 \leq u \leq v$ implies $||u|| \leq \alpha ||v||$.

If A is a C^*-algebra, then $a \in A$ is called normal if $aa^* = a^*a$. In this case we have $r(a) = ||a||$ (see for example [1], Theorem 58.3). As a consequence of Theorem 3 we get the following result.

Theorem 4. *Suppose A is a C^*-algebra, $a \in A$ is normal, and $r(a) \in \sigma(a)$. Let W be the algebra wedge from the proof of Theorem 2. Then:*

(1) *W is a normal algebra cone with constant $\alpha = 1$;*
(2) *each $u \in W$ is normal.*

Proof. Let B, ψ and W be as in the proof of Theorem 2. By the Fuglede-Putnam Theorem (see for example [1], Exercise 65.1) $\Gamma(\{a\})$ is a $*$-subalgebra of A, thus $B = \Gamma(\Gamma(\{a, a^*\}))$, and therefore each $u \in B$ is normal. This shows that (2) is valid, and that $\operatorname{rad}(B) = \{0\}$. Hence B is semisimple, and W is an algebra cone according to Theorem 3. Finally let $0 \leq u \leq v$. Since r is increasing on W we have

$$||u|| = r(u) \leq r(v) = ||v||.$$

□

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