A NOTE ON A THEOREM OF RAUBENHEIMER AND RODE

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Abstract. We prove the converse of Raubenheimer and Rode’s Banach algebra version of the Perron-Frobenius Theorem.

Let \((A, ||\cdot||)\) be a complex Banach algebra with unit \(e\) and \(||e|| = 1\). A set \(W \subseteq A\) is called an algebra wedge in the case that \(W\) is closed, \(W + W \subseteq W\), \(\lambda W \subseteq W\) \((\lambda \geq 0)\), \(W \cdot W \subseteq W\), and \(e \in W\). As usual, by setting \(a \leq b : \iff b - a \in W\) we obtain a reflexive and transitive relation on \(A\), and call \(A\) ordered by \(W\). In the sequel let \(\sigma_A(a)\) and \(r(a)\) denote the spectrum and the spectral radius of \(a \in A\), respectively.

In \([2]\) Raubenheimer and Rode proved the following version of the Perron-Frobenius Theorem.

Theorem 1. Let \(A\) be ordered by an algebra wedge \(W\) such that the spectral radius is increasing on \(W\). Then \(r(a) \in \sigma_A(a)\) for all \(a \in W\).

The purpose of this note is to prove the converse of this result:

Theorem 2. Let \(a \in A\) be such that \(r(a) \in \sigma_A(a)\). Then there exists an algebra wedge \(W\) such that the spectral radius is increasing on \(W\), and \(a \in W\).

Proof. For \(S \subseteq A\) let \(\Gamma(S)\) denote the centralizer of \(S\), that is,

\[\Gamma(S) = \{v \in A : vu = uv \ (u \in S)\},\]

and set \(B = \Gamma(\Gamma(\{a\}))\). Then \(B\) is a closed commutative subalgebra of \(A\), and \(e, a \in B\). Moreover \(\sigma_A(u) = \sigma_B(u)\) \((u \in B)\) (see for example \([3]\), Theorem 11.22). In particular \(r(u)\) does mean the same in \(A\) and \(B\). Let \(\Delta\) denote the set of all nontrivial multiplicative linear functionals on \(B\).

Since \(B\) is commutative and \(r(a) \in \sigma_B(a)\) there is a \(\psi \in \Delta\) such that \(\psi(a) = r(a)\). We define

\[W := \{u \in B : r(u) = \psi(u)\},\]

Obviously \(a \in W\), and \(e \in W\) since \(\psi(e) = 1\). To see that \(W\) is an algebra wedge first note that whenever \(u, v \in B\) then, by Theorem 11.23 in \([3]\),

\[r(u + v) \leq r(u) + r(v), \quad r(uv) \leq r(u)r(v).\]

Moreover \(r(u) = \max_{\psi \in \Delta} |\psi(u)|\) \((u \in B)\).

For \(u, v \in W\) we obtain

\[r(uv) \leq r(u)r(v) = \psi(u)\psi(v) = \psi(uv) \leq r(uv),\]

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hence $uv \in W$, and analogously $u + v \in W$. Next, for each $u \in B$ and $\lambda \geq 0$ we have $r(\lambda u) = \lambda r(u)$, hence $\lambda W \subseteq W$ ($\lambda \geq 0$).

To prove that $W$ is closed, note that (1.1) implies $||r(u) - r(v)|| \leq ||u - v||$. Hence $W$ is closed.

If $(u_n)$ is a convergent sequence in $W$ with limit $v$, say, then

$$\lim_{n \to \infty} \psi(u_n) = \psi(v),$$

$$||\psi(u_n) - r(v)|| = ||r(u_n) - r(v)|| \leq ||u_n - v|| \to 0 \quad (n \to \infty),$$

and therefore $r(v) = \psi(v)$. Altogether $W$ is an algebra wedge.

Now let $A$ be ordered by $W$. It remains to prove that $r$ is increasing on $W$. Let $0 \leq u_1 \leq u_2$. Then

$$r(u_2) - r(u_1) = \psi(u_2 - u_1) = r(u_2 - u_1) \geq 0 \Longrightarrow r(u_1) \leq r(u_2).$$

\hspace{4cm} \square

**Examples.** Consider $A = \mathbb{C}^{2 \times 2}$ and

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad r(a) = 1 \in \{-1, 1\} = \sigma_A(a).$$

In this case it is easy to check that

$$B = \left\{ u = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\},$$

and that $\psi(u) = \alpha + \beta$ is in $\Delta$ with $\psi(a) = 1$. Since

$$\sigma_B \left( \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right) = \{\alpha + \beta, \alpha - \beta\}$$

we get

$$W = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : |\alpha - \beta| \leq \alpha + \beta \right\}.$$

The same considerations for

$$a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad r(a) = 0 \in \{0\} = \sigma_A(a)$$

lead to

$$B = \left\{ u = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}, \quad \psi(u) = \alpha,$$

and

$$W = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha \geq 0 \right\}.$$

In the first example $W$ is an algebra cone, that is, in addition $W \cap (-W) = \{0\}$. In the second example $W$ is not an algebra cone since $a \in W \cap (-W)$. This is not an accident, as the following results will show.

The radical of $A$, denoted by $\text{rad}(A)$, is the intersection of all maximal ideals in $A$, and $A$ is called semisimple in case $\text{rad}(A) = \{0\}$. It follows from [3], Theorem 11.9, that if $A$ is commutative, then

$$\text{rad}(A) = \{ u \in A : r(u) = 0 \}.$$
Theorem 3. Let $a \in A$, $r(a) \in \sigma(a)$, and let $B$, $\psi$ and $W$ be as in the proof of Theorem 2. Then $\text{rad}(B) = W \cap (-W)$. In particular $W$ is an algebra cone if and only if $B$ is semisimple.

Proof. 1. If $u \in \text{rad}(B)$, then
\[
\sigma(u) = \{ \varphi(u) : \varphi \in \Delta \} = \{0\}.
\]
Thus $\psi(u) = 0 = r(u)$, hence $u \in W$. We have $-u \in \text{rad}(B)$, since $\text{rad}(B)$ is an ideal in $B$. As above we conclude $-u \in W$. Therefore $u \in W \cap (-W)$.

2. Let $u \in W \cap (-W)$. Then
\[
\psi(u) = r(u) = r(-u) = \psi(-u) = -\psi(u).
\]
Hence $\psi(u) = r(u) = 0$, and therefore $u \in \text{rad}(B)$.

As usual, an algebra cone is called normal if there exists a constant $\alpha \geq 1$ such that $0 \leq u \leq v$ implies $||u|| \leq \alpha ||v||$.

If $A$ is a $C^*$-algebra, then $a \in A$ is called normal if $aa^* = a^*a$. In this case we have $r(a) = ||a||$ (see for example [1], Theorem 58.3). As a consequence of Theorem 3 we get the following result.

Theorem 4. Suppose $A$ is a $C^*$-algebra, $a \in A$ is normal, and $r(a) \in \sigma(a)$. Let $W$ be the algebra wedge from the proof of Theorem 2. Then:

(1) $W$ is a normal algebra cone with constant $\alpha = 1$;

(2) each $u \in W$ is normal.

Proof. Let $B$, $\psi$ and $W$ be as in the proof of Theorem 2. By the Fuglede-Putnam Theorem (see for example [1], Exercise 65.1) $\Gamma(\{a\})$ is a $*$-subalgebra of $A$, thus $B = \Gamma(\Gamma(\{a, a^*\}))$, and therefore each $u \in B$ is normal. This shows that (2) is valid, and that $\text{rad}(B) = \{0\}$. Hence $B$ is semisimple, and $W$ is an algebra cone according to Theorem 3. Finally let $0 \leq u \leq v$. Since $r$ is increasing on $W$ we have
\[
||u|| = r(u) \leq r(v) = ||v||.
\]

\[
\square
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REFERENCES

