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ON A SUBSPACE PERTURBATION PROBLEM

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Abstract. We discuss the problem of perturbation of spectral subspaces for linear self-adjoint operators on a separable Hilbert space. Let A and V be bounded self-adjoint operators. Assume that the spectrum of A consists of two disjoint parts σ and Σ such that $d = \operatorname{dist}(\sigma, \Sigma) > 0$. We show that the norm of the difference of the spectral projections

$$\mathsf{E}_A(\sigma) \quad \text{and} \quad \mathsf{E}_{A+V}\big(\{\lambda \mid \operatorname{dist}(\lambda, \sigma) < d/2\}\big)$$

for A and $A+V$ is less than one whenever either (i) $\|V\| < \frac{2}{2+\pi} d$ or (ii) $\|V\| < \frac{1}{2} d$ and certain assumptions on the mutual disposition of the sets σ and Σ are satisfied.

1. Introduction

It is well known (see, e.g., [10]) that if A and V are bounded self-adjoint operators on a separable Hilbert space $\mathfrak{H}$, then (the perturbation) V does not close gaps of length greater than $2\|V\|$ in the spectrum of A. More precisely, if (a, b) is a finite interval and $(a, b) \subset \varrho(A)$, the resolvent set of A, then

$$(a + \|V\|, b - \|V\|) \subset \varrho(A + sV) \qquad \text{for all } s \in [-1, 1]$$

whenever $2\|V\| < b - a$. Hence, under the assumption that A has an isolated part σ of the spectrum separated from its remainder by gaps of length greater than or equal to $d > 0$, the spectrum of the operators $A + sV$, $s \in [-1, 1]$, will also have separated components, provided that the condition

$$\|V\| < \frac{d}{2} \tag{1.1}$$

holds.

Our main concern is to study the variation of the corresponding spectral subspace associated with the isolated part σ of the spectrum of A under perturbations satisfying (1.1).

For notational setup we assume the following hypothesis.

Hypothesis 1. *Assume that A and V are bounded self-adjoint operators on a separable Hilbert space $\mathfrak{H}$. Suppose that the spectrum of A has a part σ separated from the remainder of the spectrum Σ in the sense that*

$$\operatorname{spec}(A) = \sigma \cup \Sigma \quad \textit{and} \quad \operatorname{dist}(\sigma, \Sigma) = d > 0.$$

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Introduce the orthogonal projections $P = \mathsf{E}_A(\sigma)$ *and* $Q = \mathsf{E}_{A+V}(U_{d/2}(\sigma))$, *where* $U_\varepsilon(\sigma)$, $\varepsilon > 0$, *is the open* ε*-neighborhood of the set* σ. *Here* $\mathsf{E}_A(\Delta)$ *and* $\mathsf{E}_{A+V}(\Delta)$ *denote the spectral projections for operators* A *and* $A+V$, *respectively, corresponding to a Borel set* $\Delta \subset \mathbb{R}$.

In this note we address the following question: *Assuming Hypothesis 1, does condition* (1.1) *imply*

$$\|P - Q\| < 1?$$

We give a partially affirmative answer to this question. The precise statement reads as follows.

Theorem 1. *Assume Hypothesis 1 and suppose that either*

(i) $\|V\| < \frac{2}{2+\pi} d$

or

(ii) $\|V\| < \frac{1}{2} d$

and

$$\text{conv.hull}(\sigma) \cap \Sigma = \varnothing \quad \textit{or} \quad \text{conv.hull}(\Sigma) \cap \sigma = \varnothing. \tag{1.2}$$

Then

$$\|P - Q\| < 1.$$

Our strategy of the proof of Theorem 1 does not allow us to relax the condition

$$\|V\| < \frac{2}{2+\pi} d \tag{1.3}$$

and just assume the natural condition (1.1) with no additional hypotheses. It is an *open problem* whether Hypothesis 1 alone and the bounds

$$\frac{2}{2+\pi} \le \frac{\|V\|}{d} < \frac{1}{2} \tag{1.4}$$

on the perturbation V imply $\|P - Q\| < 1$.

For compact perturbations V satisfying inequality (1.1) we can however state that the pair (P, Q) of the orthogonal projections is a Fredholm pair with zero index. Recall that the pair (P, Q) of orthogonal projections is called Fredholm if the operator QP viewed as a map from $\operatorname{Ran} P$ to $\operatorname{Ran} Q$ is a Fredholm operator [3]. The index of this operator is called the index of the pair (P, Q).

Theorem 2. *Assume Hypothesis 1 and suppose that* V *is a compact operator satisfying* (1.1). *Then the pair* (P, Q) *is Fredholm with zero index. In particular, the subspaces* $\operatorname{Ker}(PQ^\perp - I)$ *and* $\operatorname{Ker}(P^\perp Q - I)$ *are finite-dimensional and*

$$\dim \operatorname{Ker}(PQ^\perp - I) = \dim \operatorname{Ker}(P^\perp Q - I).$$

In the "overcritical" case $\|V\| > d/2$, the perturbed operator $A + V$ may not have separated parts of the spectrum at all. In this case we give an example where the spectral measure of the perturbed operator $A+V$ is "concentrated" on the unit sphere in the space of bounded operators $\mathcal{B}(\mathfrak{H})$ centered at the point $P = \mathsf{E}_A(\sigma)$, with the norm of the perturbation being arbitrarily close to $d/2$. That is, given $d > 0$, for any $\varepsilon > 0$ one can find a self-adjoint operator A satisfying Hypothesis 1 and a self-adjoint perturbation V with $\|V\| = d/2 + \varepsilon$ such that

$$\|\mathsf{E}_A(\sigma) - \mathsf{E}_{A+V}(\Delta)\| = 1$$

for any Borel set $\Delta \subset \mathbb{R}$.

2. Proof of Theorem 1

Our proof of Theorem 1 is based on the following sharp result (see [9] and references cited therein) taken from geometric perturbation theory initiated by C. Davis [6] and developed further in [4], [5], [7], [8], [10].

Proposition 2.1. *Let A and B be bounded self-adjoint operators and δ and Δ two Borel sets on the real axis $\mathbb{R}$. Then*

$$\operatorname{dist}(\delta,\Delta)\|\mathsf{E}_A(\delta)\mathsf{E}_B(\Delta)\| \le \frac{\pi}{2}\|A-B\|.$$

If, in addition, the convex hull of the set δ does not intersect the set Δ, or the convex hull of the set Δ does not intersect the set δ, then one has the stronger result

$$\operatorname{dist}(\delta,\Delta)\|\mathsf{E}_A(\delta)\mathsf{E}_B(\Delta)\| \le \|A-B\|.$$

We split the proof of Theorem 1 into the following two lemmas.

Lemma 2.2. *Assume Hypothesis 1. Assume, in addition, that* (1.3) *holds. Then*

$$\|P-Q\|<1.$$

Proof. Clearly $\operatorname{spec}(A+V)\subset\overline{U_{\|V\|}(\sigma\cup\Sigma)}$, where the bar denotes the (usual) closure in $\mathbb{R}$, and then

$$Q^{\perp}=\mathsf{E}_{A+V}\big(\overline{U_{\|V\|}(\Sigma)}\big).$$

By the first claim of Proposition 2.1,

$$\|PQ^{\perp}\|\le\frac{\pi}{2}\frac{\|V\|}{\operatorname{dist}(\sigma,U_{\|V\|}(\Sigma))}. \tag{2.1}$$

The distance between the set σ and the $\|V\|$-neighborhood of the set Σ can be estimated from below as follows:

$$\operatorname{dist}(\sigma,U_{\|V\|}(\Sigma))\ge d-\|V\|>0.$$

Then (2.1) implies the inequality

$$\|PQ^{\perp}\|\le\frac{\pi}{2}\frac{\|V\|}{d-\|V\|}.$$

Hence, from inequality (1.3) it follows that

$$\|PQ^{\perp}\|\le\frac{\pi}{2}\frac{\|V\|}{d-\|V\|}<1. \tag{2.2}$$

Interchanging the roles of σ and Σ one obtains the analogous inequality

$$\|P^{\perp}Q\|<1. \tag{2.3}$$

Since

$$\|P-Q\|=\max\{\|PQ^{\perp}\|,\|P^{\perp}Q\|\} \tag{2.4}$$

(see, e.g., [2, Ch. III, Section 39]), inequalities (2.2) and (2.3) prove the assertion. □

Under additional assumptions on mutual disposition of the parts σ and Σ of the spectrum of A one can relax the condition (1.3) on the norm of perturbation and replace it by the natural condition (1.1).

Lemma 2.3. *Assume Hypothesis 1 and suppose that condition* (1.1) *holds.*

(i) *If either* $\sigma \cap \mathrm{conv.hull}(\Sigma) = \varnothing$ *or* $\mathrm{conv.hull}(\sigma) \cap \Sigma = \varnothing$*, then*

$$\|P - Q\| < 1. \tag{2.5}$$

(ii) *If in addition the sets* σ *and* Σ *are subordinated, that is,*

$$\mathrm{conv.hull}(\sigma) \cap \mathrm{conv.hull}(\Sigma) = \varnothing,$$

then the following sharp estimate holds:

$$\|P - Q\| < \frac{\sqrt{2}}{2}. \tag{2.6}$$

Proof. (i) The proof follows that of Lemma 2.2. Applying the second assertion of Proposition 2.1 instead of inequality (2.1), one derives the estimates

$$\|PQ^\perp\| \le \frac{\|V\|}{\mathrm{dist}(\sigma, U_{\|V\|}(\Sigma))} \le \frac{\|V\|}{d - \|V\|} < 1, \tag{2.7}$$

under hypothesis (1.4), and then the inequality $\|P^\perp Q\| < 1$, proving assertion (2.5) using (2.4).

(ii) First assume that V is off-diagonal, that is,

$$\mathsf{E}_A(\sigma) V \mathsf{E}_A(\sigma) = \mathsf{E}_A(\sigma)^\perp V \mathsf{E}_A(\sigma)^\perp = 0.$$

Then the inequality $\|P - Q\| < \frac{\sqrt{2}}{2}$ follows from the $\tan 2\Theta$-Theorem proven first by C. Davis (see, e.g., [8])

$$\|P - Q\| \le \sin\left(\frac{1}{2}\arctan\frac{2\|V\|}{d}\right) < \frac{\sqrt{2}}{2}.$$

A related result can be found in [1].

The general case can be reduced to the off-diagonal one by the following trick. Assume that V is not necessarily off-diagonal. Decomposing the perturbation V into the diagonal V_{diag} and off-diagonal V_{off} parts with respect to the orthogonal decomposition $\mathfrak{H} = \mathrm{Ran}\,\mathsf{E}_A(\sigma) \oplus \mathrm{Ran}\,\mathsf{E}_{\mathbf{A}}(\sigma)^\perp$ associated with the range of the projection $\mathsf{E}_A(\sigma)$

$$V = V_{\mathrm{diag}} + V_{\mathrm{off}},$$

one concludes that

$$\mathsf{E}_{A+V_{\mathrm{diag}}}(U_{d/2}(\sigma)) = \mathsf{E}_A(\sigma).$$

Moreover, the distance between the spectrum of the part of $A+V_{\mathrm{diag}}$ associated with the invariant subspace $\mathrm{Ran}\,\mathsf{E}_{A+V_{\mathrm{diag}}}(U_{d/2}(\sigma))$ and the remainder of the spectrum of $A + V_{\mathrm{diag}}$ does not exceed $d - 2\|V_{\mathrm{diag}}\| > 0$. Using the $\tan 2\Theta$-Theorem then yields

$$\begin{aligned}\|P - Q\| &\le \sin\left(\frac{1}{2}\arctan\frac{2\|V_{\mathrm{off}}\|}{d - 2\|V_{\mathrm{diag}}\|}\right)\\ &\le \sin\left(\frac{1}{2}\arctan\frac{2\|V\|}{d - 2\|V\|}\right) < \frac{\sqrt{2}}{2},\end{aligned}$$

completing the proof. □

The sharpness of estimate (2.6) is shown by the following example.

Example 2.4. Let $\mathfrak{H} = \mathbb{C}^2$. For an arbitrary $\varepsilon \in (0, 3/4)$ consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1/2-\varepsilon & \sqrt{\varepsilon}/2 \\ \sqrt{\varepsilon}/2 & -1/2+\varepsilon \end{pmatrix}.$$

Let $\sigma = \{0\}$ and $\Sigma = \{1\}$. Obviously, $\operatorname{dist}(\sigma, \Sigma) = 1$. Since

$$\|V\| = \frac{1}{2}\sqrt{1-3\varepsilon+4\varepsilon^2} < \frac{1}{2},$$

the perturbation V satisfies the hypotheses of Lemma 2.3. Simple calculations yield

$$\begin{aligned} Q &= \mathsf{E}_{A+V}\big(U_{1/2}(\sigma)\big) = \mathsf{E}_{A+V}\big((-1/2, 1/2)\big) \\ &= \frac{1}{1+(2\sqrt{\varepsilon}+\sqrt{1+4\varepsilon})^2}\begin{pmatrix} (2\sqrt{\varepsilon}+\sqrt{1+4\varepsilon})^2 & -2\sqrt{\varepsilon}-\sqrt{1+4\varepsilon} \\ -2\sqrt{\varepsilon}-\sqrt{1+4\varepsilon} & 1 \end{pmatrix}, \end{aligned}$$

and hence,

$$\|P-Q\| = \big[1+(2\sqrt{\varepsilon}+\sqrt{1+4\varepsilon})^2\big]^{-1/2} < \frac{\sqrt{2}}{2}.$$

Taking ε sufficiently small, the norm $\|P-Q\|$ can be made arbitrarily close to $\sqrt{2}/2$.

3. Proof of Theorem 2

Lemma 3.1. *Assume Hypothesis 1 and suppose, in addition, that V is a compact operator satisfying condition* (1.1)*. Then there is a unitary W such that $Q = WPW^*$ and $W - I$ is compact.*

Proof. Fix $\varepsilon > 0$ such that $(1+\varepsilon)\|V\| < d/2$ and introduce the family of spectral projections

$$\mathcal{P}(s) = \mathsf{E}_{A+sV}(U_{d/2}(\sigma)), \quad s \in (-\varepsilon, 1+\varepsilon).$$

Clearly, $\mathcal{P}(0) = P$ and $\mathcal{P}(1) = Q$. From the analytical perturbation theory (see [10]) one concludes that the operator-valued function $\mathcal{P}(s)$ is real-analytic on $(-\varepsilon, 1+\varepsilon)$. Moreover (see [10, Section II.4.2]),

$$\mathcal{P}(s) = X(s)\mathcal{P}(0)X(s)^*, \quad s \in [0,1],$$

where $X(s)$ is the unique unitary solution to the initial value problem

$$\begin{aligned} X'(s) &= H(s)X(s), \quad s \in [0,1], \\ X(0) &= I, \end{aligned}$$

with $H(s) = \mathcal{P}'(s)\mathcal{P}(s) - \mathcal{P}(s)\mathcal{P}'(s)$.

Let Γ be a Jordan counterclockwise oriented contour encircling $U_{d/2}(\sigma)$ in a way such that no point of $U_{d/2}(\Sigma)$ lies within Γ. Then

$$\mathcal{P}(s) = -\frac{1}{2\pi i}\int_\Gamma (A+sV-z)^{-1}dz, \quad s \in [0,1],$$

and hence,

$$\mathcal{P}'(s) = \frac{1}{2\pi i}\int_\Gamma (A+sV-z)^{-1}V(A+sV-z)^{-1}dz, \quad s \in [0,1].$$

By the hypothesis V is compact, and hence, $\mathcal{P}'(s)$, $s \in [0,1]$, is also compact, which implies that $H(s)$ is a compact operator for $s \in [0,1]$.

Applying the successive approximation method

$$X_n(s) = I + \int_0^s H(t)X_{n-1}(t)dt, \quad X_0(s) = I,$$

yields that $X_n(s)$ converges to $X(s)$, $s \in [0,1]$, in the norm topology and $X_n(s) - I$ is compact for all $n \in \mathbb{N}$. Thus, $X(s) - I$ is a compact operator for all $s \in [0,1]$. Taking $W = X(1)$ yields $Q = WPW^*$, completing the proof. □

Lemma 3.1 implies that the operator PWP viewed as a map from $\operatorname{Ran} P$ to $\operatorname{Ran} P$ is Fredholm with zero index. By Theorem 5.2 of [3] it follows that the pair (P,Q) is Fredholm and $\operatorname{index}(P,Q) = \operatorname{index}(PW|_{\operatorname{Ran} P}) = 0$, proving Theorem 2.

4. Overcritical perturbations

If the perturbation V closes a gap between the separated parts σ and Σ of the spectrum of the unperturbed operator A, then, necessarily, we are dealing with the case $\|V\| \geq d/2$. In this case one encounters a new phenomenon: It may happen that any invariant subspace of the operator $A+V$ contains a nontrivial element orthogonal to $\operatorname{Ran} P = \operatorname{Ran} \mathsf{E}_A(\sigma)$.

To illustrate this phenomenon we need the following abstract result.

Lemma 4.1. *Let A and V be bounded self-adjoint operators and $\sigma \neq \varnothing$ be a finite set consisting of isolated eigenvalues of A of finite multiplicity. Assume that the spectrum of the operator $A+V$ has no pure point component. Then for the orthogonal projection Q onto an arbitrary invariant subspace of the operator $A+V$, the subspace $\operatorname{Ker}(P^\perp Q - I)$, where $P = \mathsf{E}_A(\sigma)$, is infinite-dimensional. In particular,*

$$\|P - Q\| = 1. \tag{4.1}$$

Proof. Since $A+V$ has no eigenvalues, $\operatorname{Ran} Q$ is an infinite-dimensional subspace. By hypothesis, $\operatorname{Ran} P$ is a finite-dimensional subspace. Thus, there exists an orthonormal system $\{f_n\}_{n\in\mathbb{N}}$ in $\operatorname{Ran} Q$ such that f_n is orthogonal to $\operatorname{Ran} P$ for any $n \in \mathbb{N}$ and hence $P^\perp Q f_n = f_n$, $n \in \mathbb{N}$, proving $\dim\big(\operatorname{Ker}(P^\perp Q - I)\big) = \infty$. Now equality (4.1) follows from representation (2.4). □

The next lemma shows that an isolated *eigenvalue* of the unperturbed operator A separated from the remainder of the spectrum of A by a gap of length 1 may "dissolve" in the essential spectrum of the perturbed operator $A+V$ turning into a "*resonance*", with the norm of the perturbation being larger but arbitrarily close to 1/2.

Lemma 4.2. *Let $\varepsilon > 0$. Let A and V be 2×2 operator matrices in $\mathfrak{H} = L^2(0,1) \oplus \mathbb{C}$,*

$$A = \begin{pmatrix} M & 0 \\ 0 & -I_{\mathbb{C}} \end{pmatrix} \quad \textit{and} \quad V = \begin{pmatrix} -\left(\frac{1}{2}+\varepsilon\right) I_{L^2(0,1)} & \sqrt{\varepsilon} v \\ \sqrt{\varepsilon} v^* & (\frac{1}{2}+\varepsilon) I_{\mathbb{C}} \end{pmatrix}$$

with respect to the decomposition $\mathfrak{H} = L^2(0,1) \oplus \mathbb{C}$. Here M denotes the multiplication operator in $L^2(0,1)$,

$$(Mf)(\mu) = \mu f(\mu), \quad 0 < \mu < 1, \quad f \in L^2(0,1),$$

and $v \in \mathcal{B}\big(\mathbb{C}, L^2(0,1)\big)$

$$(vg)(\mu) = w(\mu) g, \quad \mu \in (0,1), \quad g \in \mathbb{C}, \qquad w(\mu) = \sqrt{\mu(1-\mu)}.$$

If $\varepsilon < 2/5$, then the operator $A+V$ has no eigenvalues.

Proof. Assume to the contrary that $\lambda \in \mathbb{R}$ is an eigenvalue of the perturbed operator $A+V$, that is,

$$(\mu - 1/2 - \varepsilon)f(\mu) + \sqrt{\varepsilon}w(\mu)g = \lambda f(\mu) \quad \text{a.e. } \mu \in (0,1)$$

and

$$\sqrt{\varepsilon}\int_0^1 d\mu f(\mu)w(\mu) + (-1/2+\varepsilon)g = \lambda g$$

for some $f \in L^2(0,1)$ and $g \in \mathbb{C}$. In particular,

$$f(\mu) = \sqrt{\varepsilon}\frac{w(\mu)}{\lambda - (\mu - \frac{1}{2} - \varepsilon)}g,$$

and hence $f \notin L^2(0,1)$ whenever $\lambda \in [-1/2-\varepsilon, 1/2-\varepsilon]$ (unless $f=0$ and $g=0$). Thus, the interval $[-1/2-\varepsilon, 1/2-\varepsilon]$ does not intersect the point spectrum of $A+V$. Moreover, $\lambda \in (-\infty, -1/2-\varepsilon) \cup (1/2-\varepsilon, \infty)$ is an eigenvalue of $A+V$ if and only if

$$\lambda + \frac{1}{2} - \varepsilon + \varepsilon\int_0^1 d\mu \frac{\mu(1-\mu)}{\mu - \frac{1}{2} - \varepsilon - \lambda} = 0. \tag{4.2}$$

Elementary analysis of the graph of the function on the left-hand side of (4.2) then yields that under the condition $0 < \varepsilon < 2/5$ there is no solution of equation (4.2) in $(-\infty, -1/2-\varepsilon) \cup (1/2-\varepsilon, \infty)$. Thus, the point spectrum of $A+V$ is empty. □

Remark 4.3. We note that $\operatorname{spec}(A) = \{-1\} \cup [0,1]$ and hence $\operatorname{spec}(A)$ has two components separated by a gap of length one, and the norm of the perturbation V may be arbitrarily close to $1/2$ (from above):

$$\|V\| = \sqrt{\left(\frac{1}{2}+\varepsilon\right)^2 + \frac{1}{6}\varepsilon} = \frac{1}{2} + \frac{7}{6}\varepsilon + \mathcal{O}(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0. \tag{4.3}$$

Using scaling arguments, Remark 4.3 combined with the result of Lemma 4.1 shows that given $d > 0$, for any $\varepsilon > 0$ one can find a self-adjoint operator A satisfying Hypothesis 1 and a self-adjoint perturbation V with $\|V\| = d/2 + \varepsilon$ such that

$$\|\mathsf{E}_A(\sigma) - Q\| = 1$$

for the orthogonal projection Q onto an arbitrary invariant subspace of the operator $A+V$.

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