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FRACTALS AND DISTRIBUTIONS ON THE N-TORUS

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ABSTRACT. This paper establishes non-Cartesian product sets, called fractal carpets and fractal foam, as sets of uniqueness for a class of trigonometric series.

1. INTRODUCTION

Operating in $\mathbf{R}^N$, $N \geqq 2$, with $(x, y) = x_1y_1 + \cdots + x_Ny_N$, $|x| = (x, x)^{1/2}$, and $\alpha x + \beta y = (\alpha x_1 + \beta y_1, ..., \alpha x_N + \beta y_N)$, we call

$$T_N = \{x = (x_1, ..., x_N) : 0 \leqq x_j < 1,\ j = 1, ..., N\}$$

the N-torus. In the sequel, working mod 1 in each variable, we shall describe fractal sets on T_N, called carpets in two dimensions and fractal foam in three dimensions (see [M, p. 133]), which will be sets of uniqueness for a class of distributions on the N-torus. The results to be presented here constitute an extension of our previous work, [Sh1], but now, unlike before, we show that the sets of uniqueness can be non-Cartesian product sets. We illustrate this fact with two examples.

[Sh1] was motivated by the two-dimensional results in [Sh2] and analogous one-dimensional results to be found in [Z1], [Z2], [KS], and [Sa].

We employ the notion of distributions as defined in [BJS, p. 168]. In particular, a distribution S on T_N, also called a periodic distribution, is a real linear functional on $\mathcal{D}(T_N)$, the class of real functions in $\mathrm{C}^{(\infty)}(\mathbf{R}^N)$ which are periodic of period one in each variable, with the property that

(1.1) if $\phi_k \in \mathcal{D}(T_N)$ and $\|\phi_k\|_t \to 0$ as $k \to \infty$ for every $t > 0$, then $S(\phi_k) \to 0$,

where $\|\phi_k\|_t$ is defined in (1.5) below. We will denote the class of such S by $\mathcal{D}'(T_N)$.

For $S \in \mathcal{D}'(T_N)$, we set

$$S^\wedge(m) = S[\cos 2\pi(m, x)] - iS[\sin 2\pi(m, x)] \tag{1.2}$$

for every integral lattice point m and observe from [BJS, p. 168] that there exists a positive integer k such that

$$\sum_m |S^\wedge(m)| \, (|m| + 1)^{-k} < \infty.$$

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Also, we have from this last-named reference that for $\phi \in \mathcal{D}(T_N)$,

$$S(\phi) = \sum_m S^\wedge(m)\phi^\wedge(-m), \tag{1.3}$$

where

$$\phi^\wedge(m) = \int_{T_N} e^{-2\pi i(m,x)}\phi(x)\ dx. \tag{1.4}$$

Corresponding to [BJS, p. 166], we also have for t an integer

$$\|\phi\|_t^2 = \sum_m [1 + (2\pi)^2 |m|^2]^t \left|\phi^\wedge(m)\right|^2. \tag{1.5}$$

In this paper, we deal with a subclass of $\mathcal{D}'(T_N)$, namely the following:

$$\mathcal{B}(T_N) = \{S \in \mathcal{D}'(T_N) : \lim_{|m|\to\infty} |S^\wedge(m)| = 0\}. \tag{1.6}$$

Next, we observe that the set $E \subset T_N$ is closed in the torus topology if and only if the set E^* is a closed set in $\mathbf{R}^N$ where $E^* = \bigcup_m \{E + m\}$. If $\phi \in \mathcal{D}(T_N)$, we designate the set *supp* (ϕ), called the support of ϕ, by the following:

$$\textit{supp}\ (\phi) = \{x \in T_N :\ \phi(x) \neq 0\}^\sim \tag{1.7}$$

where $^\sim$ denotes the closure in the torus topology. Also, given $G \subset T_N$ open in the torus topology and $S \in \mathcal{D}'(T_N)$, we define $S = 0$ in G to mean

$$S(\phi) = 0 \quad \forall \phi \in \mathcal{D}(T_N) \text{ such that } \textit{supp}\ (\phi) \subset G. \tag{1.8}$$

A set $E \subset T_N$ closed in the torus topology will be called a $\mathcal{V}$*-set* if the following holds:

(i) $\exists\ \{p^k\}_{k=1}^\infty$, a sequence of integral lattice points, with $p^k = (p_1^k, ..., p_N^k)$ where p_j^k is a positive integer for $j = 1, ..., N$, and also

$$\lim_{k\to\infty} p_j^k = \infty \quad \text{for} \quad j = 1, ..., N; \tag{1.9}$$

(ii) $x \in E \Rightarrow (p_1^k x_1, ..., p_N^k x_N) \in E \ \textit{mod}\ 1$ in each variable $\forall k$.

A set $Z \subset T_N$ which is of N-dimensional Lebesgue measure zero and also closed in the torus topology will be called a set of uniqueness for the class $\mathcal{B}(T_N)$, defined in (1.6), if the following holds:

$$S \in \mathcal{B}(T_N) \ \text{ and } \ S = 0 \text{ in } T_N \setminus Z \quad \Rightarrow S \equiv 0. \tag{1.10}$$

We shall prove the following theorem.

Theorem. *Suppose $Z \subset T_N$ is a closed set in the torus topology and also of N-dimensional Lebesgue measure zero. Suppose furthermore that Z is a $\mathcal{V}$-set. Then Z is a set of uniqueness for the class $\mathcal{B}(T_N)$.*

2. Fundamental lemmas

If $\lambda \in \mathcal{D}(T_N)$ and $S \in \mathcal{D}'(T_N)$, then we define $\lambda S \in \mathcal{D}'(T_N)$ as follows:

$$\lambda S(\phi) = S(\lambda\phi) \quad \forall \phi \in \mathcal{D}(T_N). \tag{2.1}$$

It is easy to check that this definition is all right because it meets the condition set forth in (1.1).

Let m be a fixed integral lattice point in $\mathbf{R}^N$. Then if $\lambda \in \mathcal{D}(T_N)$, it follows from (1.4) that

$$\lambda e^{-2\pi i(m,x)} = \sum_p \lambda^\wedge(p) e^{-2\pi i(m-p,x)},$$

and consequently from (1.2), (1.3), and (2.1) that for $S \in \mathcal{D}'(T_N)$,

$$(\lambda S)^\wedge(m) = \sum_p \lambda^\wedge(p) S^\wedge(m-p). \tag{2.2}$$

The first lemma that we establish is the following:

Lemma 1. *Let $S \in D'(T_N)$, $\lambda \in D(T_N)$, and $G \subset T_N$ be open in the torus topology. Suppose $S = 0$ in G, and λ has its support in G. Then the distribution λS is such that $\lambda S \equiv 0$.*

To prove the lemma, let $\phi \in \mathcal{D}(T_N)$. Then $\lambda\phi$ has its support in G. Consequently, it follows from (2.1) and the hypothesis of the lemma that $\lambda S(\phi) = 0$, which establishes the lemma.

Lemma 2. *Let $Z \subset T_N$ be a set closed in the torus topology and of N-dimensional Lebesgue measure zero. Suppose that*

(1) *$S \in \mathcal{B}(T_N)$ and that $S = 0$ in $T_N \setminus Z$;*

(2) *$\exists$ $\{\lambda_k\}_{k=1}^\infty$ with the following properties:* (i) $\lambda_k \in \mathcal{D}(T_N)$ $\forall k$, (ii) *supp* $(\lambda_k) \subset T_N \setminus Z$ $\forall k$, (iii) $\exists$ $C > 0$ *such that* $\sum_m |\lambda_k^\wedge(m)| \leqq C$ $\forall k$, (iv) $\lim_{k\to\infty} |\lambda_k^\wedge(m)| = 0$ $\forall m \neq 0$, *and* (v) $\lim_{k\to\infty} \lambda_k^\wedge(0) = \alpha_0 \neq 0$.

Then $S \equiv 0$.

To prove the lemma, we observe from (ii) that for each k the support of λ_k is in the open set $T_N \setminus Z$. Furthermore, $S = 0$ in $T_N \setminus Z$. Hence, it follows from Lemma 1 that

$$\lambda_k S \equiv 0 \quad \forall k.$$

Consequently, we obtain from (2.2) that

$$0 = \sum_p \lambda_k^\wedge(p) S^\wedge(m-p)$$

for a fixed integral lattice point m. But then we have that

$$-\lambda_k^\wedge(0) S^\wedge(m) = \sum_{p\neq 0} \lambda_k^\wedge(p) S^\wedge(m-p). \tag{2.3}$$

Since m is a fixed integral lattice point, given $\varepsilon > 0$, we have from (1.6) that there exists $R_o > 1$ such that $|S^\wedge(m-p)| < \varepsilon$ for $|p| \geqq R_o$. From (iii) in the lemma and from (2.3), we then infer that

$$|\lambda_k^\wedge(0) S^\wedge(m)| < \sum_{1\leqq |p| < R_o} |\lambda_k^\wedge(p) S^\wedge(m-p)| + \varepsilon C.$$

Passing to the limit as $\varepsilon \to 0$ on both sides of this last inequality and simultaneously making use of (iv) and (v) of the lemma gives us that

$$|\alpha_0 S^\wedge(m)| \leqq \varepsilon C.$$

But ε is an arbitrary positive number and $\alpha_0 \neq 0$; so we conclude that $S^\wedge(m) = 0$ for every lattice point m. It then follows from (1.3) that $S(\phi) = 0$ $\forall$ $\phi \in \mathcal{D}(T_N)$. Therefore, $S \equiv 0$, and the proof of the lemma is complete.

3. Proof of the Theorem

To prove the Theorem, given that $Z \subset T_N$ is a closed set in the torus topology which is of N-dimensional Lebesgue measure zero and which is also a $\mathcal{V}$-*set*, we shall show the existence of a sequence $\{\lambda_k\}_{k=1}^{\infty}$ which meets the conditions 2(i)-2(v) in the hypothesis of Lemma 2. The Theorem will then follow from definition (1.10) and Lemma 2.

It is clear from the assumptions concerning Z that there is an $x_0 \in T_N$ and open ball with center x_0 and radius $r_0 > 0$, which we call $B(x_0, r_0)$, such that $Z \cap B(x_0, r_0) = \emptyset$. Hence, there is

$$0 < a_j^1 < a_j^2 < a_j^3 < b_j^3 < b_j^2 < b_j^1 < 1, \;\; j = 1, ..., N, \tag{3.1}$$

with

$$Q^l = [a_1^l, b_1^l] \times \cdots \times [a_N^l, b_N^l], \quad l = 1, 2, 3, \tag{3.2}$$

such that $Z \cap Q^1 = \emptyset$. Since $Q^3 \subset Q^2$ are rectangular parallelopipeds with the conditions in (3.1) holding, it is well known that there is a function $\lambda(x) \in \mathcal{D}(T_N)$ with the follow in g properties:

(i) $\lambda(x) = 1$ for $x \in Q^3$;
(ii) $\lambda(x) = 0$ for $x \in T_N \diagdown Q^2$, i.e., *supp* $(\lambda) \subset Q^2$;
(iii) $\lambda(x) \geqq 0$ for $x \in T_N$.

To obtain the sequence $\{\lambda_k\}_{k=1}^{\infty}$ alluded to in the first paragraph above, we let $\{p^k\}_{k=1}^{\infty}$be the sequence of integral lattice points introduced in (1.9)(i), (ii) used in the definition of Z being a $\mathcal{V}$-*set*, and we define

$$\lambda_k(x) = \lambda(p_1^k x_1, ..., p_N^k x_N) \quad \forall k \text{ and } \forall x \in \mathbf{R}^N. \tag{3.3}$$

Since p_j^k is a positive integer for $j = 1, ..., N$, and since $\lambda \in \mathcal{D}(T_N)$, it follows from (3.3) that $\lambda_k \in \mathcal{D}(T_N)$. Hence, (2)(i) in Lemma 2 for our sequence $\{\lambda_k\}$ is established.

To establish (2)(ii) in Lemma 2, we observe that both Q^1 and Z are compact subsets of T_N in the torus topology and $Z \cap Q^1 = \emptyset$. Consequently, $\exists\, \varepsilon_0 > 0$ such that

$$B(x, \varepsilon_0) \cap Q^{1*} = \emptyset \;\; \text{for } x \in Z^* \tag{3.4}$$

where $Q^{1*} = \bigcup_m \{Q^1 + m\}$ and $Z^* = \bigcup_m \{Z + m\}$. Now for k fixed, let

$$\left|p^k\right|^2 = \sum_{j=1}^{N} \left|p_j^k\right|^2.$$

Then it follows from (3.4) with $\varepsilon_1 = \varepsilon_0 / \left|p^k\right|$ that

$$x + (p_1^k y_1, ..., p_N^k y_N) \notin Q^{1*} \quad \text{for } |y| < \varepsilon_1 \text{ and } \; x \in Z^*.$$

By (1.9)(ii), $x = (x_1, ..., x_N) \in Z \Longrightarrow (p_1^k x_1, ..., p_N^k x_N) \in Z^*$. Therefore if $x \in Z$ and $|y| < \varepsilon_1$,

$$(p_1^k x_1, ..., p_N^k x_N) + (p_1^k y_1, ..., p_N^k y_N) \notin Q^{1*}. \tag{3.5}$$

From (3.1) and (3.2), we see that $Q^{2*} \subset Q^{1*}$ and from (ii) in the properties of $\lambda(x)$ that *supp* $(\lambda) \subset Q^2$. Consequently, it follows from (3.3) and (3.5) that if $x \in Z$ and $|y| < \varepsilon_1$, $\lambda_k(x + y) = 0$. Hence, $Z \cap$ *supp* $(\lambda_k) = \emptyset$, and (2)(ii) of Lemma 2 is established.

To establish (2)(iii) in Lemma 2, we observe that $\lambda(x) \in \mathcal{D}(T_N)$. Consequently,

$$\lambda(x) = \sum_m \lambda\hat{}(m) e^{2\pi i(m,x)} \tag{3.6}$$

where

$$\sum_m |\lambda\hat{}(m)| = C < \infty. \tag{3.7}$$

It follows therefore from (3.3) that

$$\lambda_k(x) = \lambda\hat{}(0) + \sum_{m \neq 0} \lambda\hat{}(m) e^{2\pi i(p_1^k m_1 x_1 + \cdots + p_N^k m_N x_N)} \tag{3.8}$$

for $x \in T_N$. Hence,

$$\sum_m |\lambda_k^\wedge(m)| = \sum_m |\lambda\hat{}(m)| = C < \infty,$$

which fact establishes (2)(iii) in Lemma 2.

To establish (2)(iv) in Lemma 2, let m_0 be an arbitrary but fixed integral lattice with $m_0 \neq 0$. It follows from (1.9) that $\exists\ k_0 > 0$ such that for $k > k_0$

$$\left(\sum_{j=1}^N \left| p_j^k m_j \right|^2 \right)^{1/2} \geqq \min(p_1^k, ..., p_N^k) \geqq |m_0| + 1$$

for all $m \neq 0$. Consequently, it follows from (3.8) that for $k > k_0$,

$$\lambda_k\hat{}(m_0) = 0,$$

and (2)(iv) in Lemma 2 is established.

Next, we use (3.8) once again and obtain that

$$\lambda_k\hat{}(0) = \lambda\hat{}(0) \quad \forall k.$$

From the defining properties (i), (ii), (iii) of $\lambda(x)$ stated above, we see that

$$0 < \int_{T_N} \lambda(x) = \lambda\hat{}(0).$$

We conclude from these last two facts that indeed $\lim_{k\to\infty} \lambda_k\hat{}(0) = \alpha_0 \neq 0$. Hence (2)(v) in Lemma 2 is established, and the proof of the theorem is complete.

4. Examples of sets of uniqueness

In order to show that a set $Z \subset T_N$ is a set of uniqueness for the class $\mathcal{B}(T_N)$, according to the theorem, we need only show that (i) it is closed in the torus sense, (ii) it is of N-dimensional Lebesgue measure zero, and (iii) it is a $\mathcal{V}$-*set.* We shall do this for two different examples; the first will take place in dimension $N = 3$ and the second in dimension $N = 2$. Each example will constitute a non-Cartesian product set. It will also be clear that both examples hold for $N \geqq 3$, but the notation in the higher dimensional cases is considerably more cumbersome. Also, example 2 covers example 1 in dimension $N = 2$.

For $N = 3$, the set we will deal with is alluded to in Mandelbrot's book as triadic fractal foam [M, p. 133], and we will refer to it as TFF. We will define TFF in $\bar{T}_3$ where

$$\bar{T}_3 = \{x = (x_1, x_2, x_3) : 0 \leqq x_j \leqq 1,\ j = 1, 2, 3\}.$$

Our set of uniqueness Z will then be

$$Z = TFF \cap T_3. \tag{4.1}$$

To define TFF, subdivide $\bar{T}_3$ into 27 closed congruent cubes by cutting $\bar{T}_3$ with planes parallel to the three axes, i.e., $x_j = 1/3,\ 2/3$ for $j = 1, 2, 3$. Each cube has a distinguished point within it, namely $x^{j_1,1}$ which is the point with smallest Euclidean norm in each cube. Each $x^{j_1,1}$ corresponds to a unique triple

$$x^{j_1,1} \longleftrightarrow (\varepsilon_1, \delta_1, \zeta_1) \tag{4.2}$$

with $x^{j_1,1} = (\varepsilon_1/3, \delta_1/3, \zeta_1/3)$ where $\varepsilon_1, \delta_1, \zeta_1$ run through the numbers $0, 1, 2$ with one caveat: we do not allow the triple with $\varepsilon_1 = \delta_1 = \zeta_1 = 1$ since we are going to remove the open cube corresponding to this point. We shall define an ordering on different triples of the nature $(\varepsilon_1, \delta_1, \zeta_1) \neq (\varepsilon_1', \delta_1', \zeta_1')$ as follows:

$$(\varepsilon_1, \delta_1, \zeta_1) \prec (\varepsilon_1', \delta_1', \zeta_1') \text{ means} \tag{4.3}$$

(i) $\varepsilon_1 < \varepsilon_1'$ or (ii) $\varepsilon_1 = \varepsilon_1'$ and $\delta_1 < \delta_1'$ or (iii) $\varepsilon_1 = \varepsilon_1'$ and $\delta_1 = \delta_1'$ and $\zeta_1 < \zeta_1'$. This also imposes an ordering on $\{x^{j_1,1}\}$ via (4.2).

Now we have 26 triples, and we count them out according to this $\prec$-ordering, giving us $\{x^{j_1,1}\}_{j_1=1}^{26}$. Thus $x^{1,1} = (0,0,0)$, $x^{2,1} = (0,0,1/3)$, $x^{3,1} = (0,0,2/3)$, $x^{4,1} = (0,1/3,0), \ldots, x^{26,1} = (2/3,2/3,2/3)$. The closed cube which has $x^{j_1,1}$ as its distinguished point, we label $I^{j_1,1}$. We then define $I^1 \subset \bar{T}_3$ to be the closed set

$$I^1 = \bigcup_{j_1=1}^{26} I^{j_1,1}.$$

In each of the 26 cubes, which have sides of length $1/3$, we now perform the same operation as above, obtaining $(26)^2$ cubes, which now have sides of length $(1/3)^2$. Each of these last-mentioned cubes has a distinguished point

$$x^{j_2,2} = x^{j_1,1} + (\varepsilon_2/3^2, \delta_2/3^2, \zeta_2/3^2)$$

where $\varepsilon_2, \delta_2, \zeta_2$ run through the numbers $0, 1, 2$, and we do not allow the triple with $\varepsilon_2 = \delta_2 = \zeta_2 = 1$. These triples have an ordering imposed on them by (4.3) which in turn gives an ordering on $\{x^{j_2,2}\}$ defined as follows:

$x^{j_2,2} \prec x^{j_2',2}$ means

$$\text{(i) } x^{j_1,1} \prec x^{j_1',1} \text{ or (ii) } x^{j_1,1} = x^{j_1',1} \text{ and } (\varepsilon_2, \delta_2, \zeta_2) \prec (\varepsilon_2', \delta_2', \zeta_2'). \tag{4.4}$$

We then count out the $(26)^2$ points according to this ordering and obtain $\{x^{j_2,2}\}_{j_2=1}^{(26)^2}$. The closed cube containing $x^{j_2,2}$ as its distinguished point we call $I^{j_2,2}$. We then define $I^2 \subset I^1 \subset \bar{T}_3$ to be the closed set

$$I^2 = \bigcup_{j_2=1}^{(26)^2} I^{j_2,2}. \tag{4.5}$$

In each of the $(26)^2$ cubes which have sides of length $(1/3)^2$, we now perform the same operation as before obtaining $(26)^3$ cubes with each having sides of length $(1/3)^3$. We get distinguished points in each of these cubes and put an ordering on them similar to the procedure in (4.4) to obtain $\{x^{j_3,3}\}_{j_3=1}^{(26)^3}$. Next, in a procedure similar to (4.5), we get the closed set I^3 with $I^3 \subset I^2 \subset I^1 \subset \bar{T}_3$.

Continuing in this manner, we get the decreasing sequence of closed sets $\{I^n\}_{n=1}^{\infty}$ with $I^{n+1} \subset I^n \subset \bar{T}_3$ where each I^n consists of $(26)^n$ cubes each with sides of length $(1/3)^n$. The set TFF is then defined to be

$$TFF = \bigcap_{n=1}^{\infty} I^n. \tag{4.6}$$

With Z defined by (4.1) where TFF is defined by (4.6), we see that Z is closed in the torus sense because every point in the boundary of $\bar{T}_3$ is contained in TFF. Also since the Lebesgue measure of each I^n in (4.6) is $(26/27)^n$, we see that TFF is of N-dimensional Lebesgue measure zero; hence by (4.1) the same can be said of Z. Consequently, we conclude from the conditions in the hypothesis of the theorem to show that Z is a set of uniqueness for the class $\mathcal{B}(T_3)$; it only remains to show that Z is a $\mathcal{V}$*-set* according to the definition given in (1.9). We claim

$$x \in Z \Rightarrow (3^k x_1, 3^k x_2, 3^k x_3) \in Z \bmod 1 \text{ in each variable} \tag{4.7}$$

for k a positive integer where $x = (x_1, x_2, x_3)$. Once (4.7) is established, it then follows from (1.9) that Z is indeed a $\mathcal{V}$*-set.* To show that (4.7) holds, it is clearly sufficient to show that it holds in the special case when $k = 1$, i.e.,

$$x \in Z \Rightarrow (3x_1, 3x_2, 3x_3) \in Z \quad \bmod 1 \text{ in each variable.} \tag{4.8}$$

It follows from the definition of TFF in (4.6) that given $x_o \in TFF$, $\exists\ \{x_o^{j_n,n}\}_{n=1}^{\infty}$ where each $x_o^{j_n,n}$ is a distinguished point of one of the $(26)^n$ cubes in I^n of sides $(1/3)^n$ such that

$$\left\| x_o^{j_n,n} - x_o \right\| \to 0 \text{ as } n \to \infty.$$

Consequently, to show that (4.8) holds it is sufficient to show that it holds when x is a distinguished point $x^{j_n,n}$.

If $x = x^{j_1,1}$, then it follows from the enumeration of the 26 such points given below (4.3) that the conclusion in (4.8) holds. Hence from the above discussion Z will be a $\mathcal{V}$*-set*, if we show the following:

Given $x^{j_n,n} = (x_1^{j_n,n}, x_2^{j_n,n}, x_3^{j_n,n})$ a distinguished point in an $I^{j_n,n}$, then

$$(3x_1^{j_n,n}, 3x_2^{j_n,n}, 3x_3^{j_n,n}) = x^{j_{n-1},n-1} \bmod 1 \text{ in each variable} \tag{4.9}$$

for $n \geqq 2$ where $x^{j_{n-1},n-1}$ is a distinguished point in an $I^{j_{n-1},n-1}$.

It is clear from the representation of $x^{j_2,2}$ given above (4.4) that

$$x^{j_2,2} = \left(\frac{\varepsilon_1}{3} + \frac{\varepsilon_2}{3^2}, \frac{\delta_1}{3} + \frac{\delta_2}{3^2}, \frac{\zeta_1}{3} + \frac{\zeta_2}{3^2} \right)$$

where $\varepsilon_i, \delta_i, \zeta_i$ run through the numbers $0, 1, 2$, and we do not allow $\varepsilon_i = \delta_i = \zeta_i = 1$ for $i = 1, 2$. Exactly similar reasoning shows that

$$x^{j_n,n} = \left(\sum_{i=1}^{n} \frac{\varepsilon_i}{3^i}, \sum_{i=1}^{n} \frac{\delta_i}{3^i}, \sum_{i=1}^{n} \frac{\zeta_i}{3^i} \right) \tag{4.10}$$

where now $\varepsilon_i = \delta_i = \zeta_i = 1$ is not allowed for $i = 1, ..., n$. From (4.10), we see that

$$(3x_1^{j_n,n}, 3x_2^{j_n,n}, 3x_3^{j_n,n}) = \left(\varepsilon_1 + \sum_{i=1}^{n-1} \frac{\varepsilon_{i+1}}{3^i}, \delta_1 + \sum_{i=1}^{n-1} \frac{\delta_{i+1}}{3^i}, \zeta_1 + \sum_{i=1}^{n-1} \frac{\zeta_{i+1}}{3^i} \right).$$

But ε_1, δ_1, and ζ_1 are each non-negative integers, and we conclude from this last equality that (4.9) does indeed hold. Hence Z defined by (4.1) is a $\mathcal{V}$*-set*, and our example is complete.

Our next example will take place in dimension $N = 2$. We will call it a generalized carpet and refer to it as GC_{pq} where p and q are both positive integers strictly greater than 2. The set GC_{pq} will be a subset of $\bar{T}_2$ where

$$\bar{T}_2 = \{x = (x_1, x_2) : 0 \leqq x_j \leqq 1,\ j = 1, 2\}.$$

In particular, when $p = q = 3$, GC_{pq} will be the set referred to in the literature as the Sierpinski carpet [M, p. 144].

To define GC_{pq}, subdivide $\bar{T}_2$ into pq closed congruent rectangles by cutting $\bar{T}_2$ with lines parallel to the two axes, i.e., $x_1 = 1/p, 2/p, ..., (p-1)/p$, $x_2 = 1/q, 2/q, ..., (q-1)/q$. Each rectangle has a distinguished point within it, namely $x^{j_1,1}$ which is the point with smallest Euclidean norm in each rectangle. Each $x^{j_1,1}$ corresponds to a unique double

$$x^{j_1,1} \longleftrightarrow (\varepsilon_1, \delta_1) \tag{4.11}$$

with $x^{j_1,1} = (\varepsilon_1/p, \delta_1/q)$ where ε_1 and δ_1 run through the numbers $0, 1, ...,$ $p-1$ and $0, 1, ...,\ q-1$, respectively. There is a caveat however; the doubles with $\varepsilon_1 = 1, ..., p-2$, and simultaneously $\delta_1 = 1, ..., q-2$, are not allowed, for the rectangles corresponding to these points will be removed, i.e., the middle $(p-2)(q-2)$ rectangles will be deleted. An ordering on different doubles of the nature $(\varepsilon_1, \delta_1) \neq (\varepsilon_1', \delta_1')$ is then defined as follows:

$$(\varepsilon_1, \delta_1) \prec (\varepsilon_1', \delta_1') \text{ means} \tag{4.12}$$

(i) $\varepsilon_1 < \varepsilon_1'$ or (ii) $\varepsilon_1 = \varepsilon_1'$ and $\delta_1 < \delta_1'$. This also imposes an ordering on $\{\ x^{j_1,1}\}_{j_1=1}^{\gamma}$ via (4.11) where γ is the integer

$$\gamma = pq - (p-2)(q-2).$$

In particular, we see that $x^{1,1} = (0, 0)$, $x^{2,1} = (0, 1/q)$, $x^{3,1} = (0, 2/q), ...,\ x^{\gamma,1} = ((p-1)/p, (q-1)/q)$. We also observe that $x^{q,1} = (0, (q-1)/q)$, $x^{q+1,1} = (1/p, 0)$, and $x^{q+2,1} = (1/p, (q-1)/q)$. The closed rectangle which has $x^{j_1,1}$ as its distinguished point we label $I^{j_1,1}$. We then define $I^1 \subset \bar{T}_2$ to be the closed set

$$I^1 = \bigcup_{j_1=1}^{\gamma} I^{j_1,1}.$$

In each of the γ closed rectangles, which have sides of length $1/p$ and $1/q$, we now perform the same operation as above, obtaining γ^2 closed rectangles, which now have sides of length $(1/p)^2$ and $(1/q)^2$. Each of these last-mentioned rectangles has a distinguished point within it, namely $x^{j_2,2}$, where

$$x^{j_2,2} = x^{j_1,1} + (\varepsilon_2/p^2, \delta_2/q^2),$$

and where ε_2 and δ_2 run through the numbers $0, 1, ..., p-1$ and $0, 1, ..., q-1$, respectively. Also, we do not allow the doubles with $\varepsilon_2 = 1, ..., p-2$, and simultaneously $\delta_2 = 1, ..., q-2$. The doubles $(\varepsilon_2, \delta_2) \neq (\varepsilon_2', \delta_2')$ have an ordering imposed upon them by (4.12), which, in turn, imposes an ordering on the distinguished points given by $x^{j_2,2} \prec x^{j_2',2}$ akin to the ordering given in (4.4). We then count out the γ^2 points according to this ordering and obtain $\{x^{j_2,2}\}_{j_2=1}^{\gamma^2}$. The closed rectangle of sides $(1/p)^2$ and $(1/q)^2$ containing $x^{j_2,2}$ as its distinguished point we call $I^{j_2,2}$.

We then define $I^2 \subset I^1 \subset \bar{T}_2$ to be the closed set

$$I^2 = \bigcup_{j_2=1}^{\gamma^2} I^{j_2,2}.$$

Continuing in this manner, we get the decreasing sequence of closed sets $\{I^n\}_{n=1}^{\infty}$ with $I^{n+1} \subset I^n \subset \bar{T}_2$ where each I^n consists of γ^n rectangles each with sides of length $(1/p)^n$ and $(1/q)^n$. The set GC_{pq} is then defined to be

$$GC_{pq} = \bigcap_{n=1}^{\infty} I^n. \tag{4.13}$$

Next, we define Z to be the set

$$Z = GC_{pq} \cap T_2, \tag{4.14}$$

and observe that Z is closed in the torus sense because every point in the boundary of $\bar{T}_2$ is contained in GC_{pq}. Also, since the Lebesgue measure of each I^n is $(\gamma/pq)^n$ where $\gamma = pq-(p-2)(q-2)$, we see from (4.13) and (4.14) that the 2-dimensional Lebesgue measure of Z is zero. Hence according to the conditions in the hypothesis of the theorem, to show that Z is a set of uniqueness for the class $\mathcal{B}(T_2)$, it only remains to show that Z is a $\mathcal{V}$-*set*, i.e., that the conditions set forth in (1.9) hold.

We claim

$$x \in Z \Rightarrow (p^k x_1, q^k x_2) \in Z \quad \text{mod } 1 \text{ in each variable} \tag{4.15}$$

for k a positive integer and with $x = (x_1, x_2)$. Once (4.15) is established, then it follows from (1.9) that Z is indeed a $\mathcal{V}$-*set*. To show that (4.15) holds, it is clearly sufficient to show that it holds in the special case when $k = 1$, i.e.,

$$x \in Z \Rightarrow (px_1, qx_2) \in Z \quad \text{mod } 1 \text{ in each variable.} \tag{4.16}$$

Using the same argument that we used after (4.8), we see that to show that Z is a $\mathcal{V}$-*set*, we need only show that (4.16) holds for the special case when $x = x^{j_n,n}$, a distinguished point in one of the closed rectangles $I^{j_n,n}$ with sides (1/p)^n and (1/q)^n.

If $x = x^{j_1,1}$, then it follows from the enumeration of such points below (4.12) that (4.16) does indeed hold. Hence to show that Z is a $\mathcal{V}$-*set*, it only remains to establish the fact that (4.16) holds when $x = x^{j_n,n}$ for $n \geqq 2$. This will be accomplished if we show that the following fact holds:

Given $x^{j_n,n} = (x_1^{j_n,n}, x_2^{j_n,n})$ a distinguished point in an $I^{j_n,n}$, then

$$(px_1^{j_n,n}, qx_2^{j_n,n}) = x^{j_{n-1},n-1} \text{ mod 1 in each variable} \tag{4.17}$$

for $n \geqq 2$ where $x^{j_{n-1},n-1}$ is a distinguished point in an $I^{j_{n-1},n-1}$.

It is clear from the representation of $x^{j_2,2}$ given above that

$$x^{j_2,2} = \left(\frac{\varepsilon_1}{p} + \frac{\varepsilon_2}{p^2}, \frac{\delta_1}{q} + \frac{\delta_2}{q^2}\right)$$

where ε_i and δ_i run through the numbers $0, ..., p-1$, and $0, ..., q-1$, respectively, and we do not allow $\varepsilon_i = 1, ..., p-2$ and simultaneously $\delta_i = 1, ..., q-2$ for $i = 1, 2$. Exactly similar reasoning shows that

$$x^{j_n,n} = \left(\sum_{i=1}^{n} \frac{\varepsilon_i}{p^i}, \sum_{i=1}^{n} \frac{\delta_i}{q^i}\right) \tag{4.18}$$

where ε_i and δ_i are exactly as before, now with $i = 1, ..., n$. From (4.18), we see that

$$(px_1^{j_n,n}, qx_2^{j_n,n}) = \left(\varepsilon_1 + \sum_{i=1}^{n-1} \frac{\varepsilon_{i+1}}{p^i}, \delta_1 + \sum_{i=1}^{n-1} \frac{\delta_{i+1}}{q^i}\right).$$

But ε_i and $\delta_i$are each non-negative integers, and we conclude from this last equality and (4.18) that (4.17) does indeed hold. Hence, Z defined by (4.14) is a $\mathcal{V}$-*set*, and our example is complete.

In closing, we point out that the H^J-sets, defined in [Sh2] for dimension $N = 2$ and in [AW] for dimensions $N \geqq 3$, can also be shown to be sets of uniqueness for the class $\mathcal{B}(T_N)$ with respect to distributions on the N-torus.

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