FRACTALS AND DISTRIBUTIONS ON THE N-TORUS

VICTOR L. SHAPIRO

(Communicated by Andreas Seeger)

Abstract. This paper establishes non-Cartesian product sets, called fractal carpets and fractal foam, as sets of uniqueness for a class of trigonometric series.

1. Introduction

Operating in \( \mathbb{R}^N \), \( N \geq 2 \), with \((x, y) = x_1 y_1 + \cdots + x_N y_N, |x| = (x, x)^{1/2} \), and \( \alpha x + \beta y = (\alpha x_1 + \beta y_1, \ldots, \alpha x_N + \beta y_N) \), we call

\[ T_N = \{ x = (x_1, \ldots, x_N) : 0 \leq x_j < 1, j = 1, \ldots, N \} \]

the \( N \)-torus. In the sequel, working mod 1 in each variable, we shall describe fractal sets on \( T_N \), called carpets in two dimensions and fractal foam in three dimensions (see [M, p. 133]), which will be sets of uniqueness for a class of distributions on the \( N \)-torus. The results to be presented here constitute an extension of our previous work, [Sh1], but now, unlike before, we show that the sets of uniqueness can be non-Cartesian product sets. We illustrate this fact with two examples.

[Sh1] was motivated by the two-dimensional results in [Sh2] and analogous one-dimensional results to be found in [Z1], [Z2], [KS], and [Sa].

We employ the notion of distributions as defined in [BJS, p. 168]. In particular, a distribution \( S \) on \( T_N \), also called a periodic distribution, is a real linear functional on \( \mathcal{D}(T_N) \), the class of real functions in \( \mathcal{C}^\infty(\mathbb{R}^N) \) which are periodic of period one in each variable, with the property that

\[ (1.1) \text{ if } \phi_k \in \mathcal{D}(T_N) \text{ and } \|\phi_k\|_t \to 0 \text{ as } k \to \infty \text{ for every } t > 0, \text{ then } S(\phi_k) \to 0, \]

where \( \|\phi_k\|_t \) is defined in (1.5) below. We will denote the class of such \( S \) by \( \mathcal{D}'(T_N) \).

For \( S \in \mathcal{D}'(T_N) \), we set

\[ S^\wedge(m) = S[\cos 2\pi(m, x)] - iS[\sin 2\pi(m, x)] \]

for every integral lattice point \( m \) and observe from [BJS, p. 168] that there exists a positive integer \( k \) such that

\[ \sum_m |S^\wedge(m)| (|m| + 1)^{-k} < \infty. \]

Received by the editors July 3, 2001 and, in revised form, May 25, 2002.

2000 Mathematics Subject Classification. Primary 42B35, 46F99; Secondary 42B05, 05A18.

Key words and phrases. Fractal, carpet, distribution, \( N \)-torus.
Also, we have from this last-named reference that for \( \phi \in \mathcal{D}(T_N) \),

\[
S(\phi) = \sum_{m} S^\wedge(m) \phi^\wedge(-m),
\]

where

\[
\phi^\wedge(m) = \int_{T_N} e^{-2\pi i (m,x)} \phi(x) \, dx.
\]

Corresponding to [BJS, p. 166], we also have for \( t \) an integer

\[
\|\phi\|^2_t = \sum_{m} [1 + (2\pi)^2 |m|^2]^t |\phi^\wedge(m)|^2.
\]

In this paper, we deal with a subclass of \( \mathcal{D}'(T_N) \); namely the following:

\[
B(T_N) = \{ S \in \mathcal{D}'(T_N) : \lim_{|m| \to \infty} |S^\wedge(m)| = 0 \}.
\]

Next, we observe that the set \( E \subset T_N \) is closed in the torus topology if and only if the set \( E^* \) is a closed set in \( \mathbb{R}^N \) where \( E^* = \bigcup_m \{ E + m \} \). If \( \phi \in \mathcal{D}(T_N) \), we designate the set \( \text{supp} (\phi) \), called the support of \( \phi \), by the following:

\[
\text{supp} (\phi) = \{ x \in T_N : \phi(x) \neq 0 \}^\sim
\]

where \( ^\sim \) denotes the closure in the torus topology. Also, given \( G \subset T_N \) open in the torus topology and \( S \in \mathcal{D}'(T_N) \), we define \( S = 0 \) in \( G \) to mean

\[
S(\phi) = 0 \quad \forall \phi \in \mathcal{D}(T_N) \text{ such that } \text{supp} (\phi) \subset G.
\]

A set \( E \subset T_N \) closed in the torus topology will be called a \( V \)-set if the following holds:

(i) \( \exists \{ p^k \}_{k=1}^\infty \), a sequence of integral lattice points, with \( p^k = (p^k_1, \ldots, p^k_N) \) where \( p^k_j \) is a positive integer for \( j = 1, \ldots, N \), and also

\[
\lim_{k \to \infty} p^k_j = \infty \quad \text{for } j = 1, \ldots, N;
\]

(ii) \( x \in E \Rightarrow (p^k_1 x_1, \ldots, p^k_N x_N) \in E \mod 1 \) in each variable \( \forall k \).

A set \( Z \subset T_N \) which is of \( N \)-dimensional Lebesgue measure zero and also closed in the torus topology will be called a set of uniqueness for the class \( \mathcal{B}(T_N) \), defined in (1.6), if the following holds:

\[
S(\phi) = 0 \quad \forall \phi \in \mathcal{D}(T_N) \quad \text{and } S = 0 \text{ in } T_N \setminus Z \Rightarrow S \equiv 0.
\]

We shall prove the following theorem.

**Theorem.** Suppose \( Z \subset T_N \) is a closed set in the torus topology and also of \( N \)-dimensional Lebesgue measure zero. Suppose furthermore that \( Z \) is a \( V \)-set. Then \( Z \) is a set of uniqueness for the class \( \mathcal{B}(T_N) \).

### 2. Fundamental lemmas

If \( \lambda \in \mathcal{D}(T_N) \) and \( S \in \mathcal{D}'(T_N) \), then we define \( \lambda S \in \mathcal{D}'(T_N) \) as follows:

\[
\lambda S(\phi) = S(\lambda \phi) \quad \forall \phi \in \mathcal{D}(T_N).
\]

It is easy to check that this definition is all right because it meets the condition set forth in (1.1).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let $m$ be a fixed integral lattice point in $\mathbb{R}^N$. Then if $\lambda \in \mathcal{D}(T_N)$, it follows from (1.4) that
\[
\lambda e^{-2\pi i (m, x)} = \sum_{\mathbf{p}} \lambda^{\wedge}(\mathbf{p}) e^{-2\pi i (m - \mathbf{p}, x)},
\]
and consequently from (1.2), (1.3), and (2.1) that for $S \in \mathcal{D}'(T_N)$,
\[
(\lambda S)^\wedge(m) = \sum_{\mathbf{p}} \lambda^{\wedge}(\mathbf{p}) S^{\wedge}(m - \mathbf{p}).
\]

The first lemma that we establish is the following:

**Lemma 1.** Let $S \in \mathcal{D}'(T_N)$, $\lambda \in \mathcal{D}(T_N)$, and $G \subset T_N$ be open in the torus topology. Suppose $S = 0$ in $G$, and $\lambda$ has its support in $G$. Then the distribution $\lambda S$ is such that $\lambda S = 0$.

To prove the lemma, let $\phi \in \mathcal{D}(T_N)$. Then $\lambda \phi$ has its support in $G$. Consequently, it follows from (2.1) and the hypothesis of the lemma that $\lambda S(\phi) = 0$, which establishes the lemma.

**Lemma 2.** Let $Z \subset T_N$ be a set closed in the torus topology and of $N$-dimensional Lebesgue measure zero. Suppose that

1. $S \in \mathcal{B}(T_N)$ and that $S = 0$ in $T_N \setminus Z$;
2. $\{\lambda_k\}_{k=1}^\infty$ with the following properties: (i) $\lambda_k \in \mathcal{D}(T_N)$ $\forall k$, (ii) $\text{supp} \ (\lambda_k) \subset T_N \setminus Z$ $\forall k$, (iii) $\exists C > 0$ such that $\sum_{m} |\lambda_k^{\wedge}(m)| \leq C$ $\forall k$, (iv) $\lim_{k \to \infty} |\lambda_k^{\wedge}(m)| = 0$ $\forall m \neq 0$, and (v) $\lim_{k \to \infty} \lambda_k^{\wedge}(0) = a_0 \neq 0$.

Then $S \equiv 0$.

To prove the lemma, we observe from (ii) that for each $k$ the support of $\lambda_k$ is in the open set $T_N \setminus Z$. Furthermore, $S = 0$ in $T_N \setminus Z$. Hence, it follows from Lemma 1 that
\[
\lambda_k S \equiv 0 \quad \forall k.
\]
Consequently, we obtain from (2.2) that
\[
0 = \sum_{\mathbf{p}} \lambda_k^{\wedge}(\mathbf{p}) S^{\wedge}(m - \mathbf{p})
\]
for a fixed integral lattice point $m$. But then we have that
\[
(2.3) \quad -\lambda_k^{\wedge}(0) S^{\wedge}(m) = \sum_{\mathbf{p} 
eq \mathbf{0}} \lambda_k^{\wedge}(\mathbf{p}) S^{\wedge}(m - \mathbf{p}).
\]
Since $m$ is a fixed integral lattice point, given $\varepsilon > 0$, we have from (1.6) that there exists $R_0 > 1$ such that $|S^{\wedge}(m - \mathbf{p})| < \varepsilon$ for $|\mathbf{p}| \geq R_0$. From (iii) in the lemma and from (2.3), we then infer that
\[
|\lambda_k^{\wedge}(0) S^{\wedge}(m)| < \sum_{1 \leq |\mathbf{p}| < R_0} |\lambda_k^{\wedge}(\mathbf{p}) S^{\wedge}(m - \mathbf{p})| + \varepsilon C.
\]
Passing to the limit as $\varepsilon \to 0$ on both sides of this last inequality and simultaneously making use of (iv) and (v) of the lemma gives us that
\[
|a_0 S^{\wedge}(m)| \leq \varepsilon C.
\]
But $\varepsilon$ is an arbitrary positive number and $a_0 \neq 0$; so we conclude that $S^{\wedge}(m) = 0$ for every lattice point $m$. It then follows from (1.3) that $S(\phi) = 0 \quad \forall \phi \in \mathcal{D}(T_N)$. Therefore, $S \equiv 0$, and the proof of the lemma is complete.
3. PROOF OF THE THEOREM

To prove the Theorem, given that \( Z \subset T_N \) is a closed set in the torus topology which is of \( N \)-dimensional Lebesgue measure zero and which is also a \( \mathcal{V} \)-set, we shall show the existence of a sequence \( \{ \lambda_k \}_{k=1}^\infty \) which meets the conditions (2)(i)-2(v) in the hypothesis of Lemma 2. The Theorem will then follow from definition (1.10) and Lemma 2.

It is clear from the assumptions concerning \( Z \) that there is an \( x_0 \in T_N \) and open ball with center \( x_0 \) and radius \( r_0 > 0 \), which we call \( B(x_0, r_0) \), such that \( Z \cap B(x_0, r_0) = \emptyset \). Hence, there is

\[
0 < a_j^1 < b_j^1 < a_j^2 < b_j^2 < b_j^3 < 1, \ j = 1, \ldots, N,
\]

with

\[
Q^l = [a_{1}^{l}, b_{1}^{l}] \times \cdots \times [a_{N}^{l}, b_{N}^{l}], \ l = 1, 2, 3,
\]

such that \( Z \cap Q^1 = \emptyset \). Since \( Q^2 \subset Q^3 \) are rectangular parallelepipeds with the conditions in (3.1) holding, it is well known that there is a function \( \lambda(x) \in \mathcal{D}(T_N) \) with the follow in \( g \) properties:

(i) \( \lambda(x) = 1 \) for \( x \in Q^3 \);
(ii) \( \lambda(x) = 0 \) for \( x \in T_N \setminus Q^2 \), i.e., \( \text{supp} (\lambda) \subset Q^2 \);
(iii) \( \lambda(x) \preceq 1 \) for \( x \in T_N \).

To obtain the sequence \( \{ \lambda_k \}_{k=1}^\infty \) alluded to in the first paragraph above, we let \( \{ p^k \}_{k=1}^\infty \) be the sequence of integral lattice points introduced in (1.9)(i), (ii) used in the definition of \( Z \) being a \( \mathcal{V} \)-set, and we define

\[
\lambda_k(x) = \lambda(p^k_{1}x_1, \ldots, p^k_{N}x_N) \ \forall k \text{ and } \forall x \in \mathbb{R}^N.
\]

Since \( p^k_{j} \) is a positive integer for \( j = 1, \ldots, N \), and since \( \lambda \in \mathcal{D}(T_N) \), it follows from (3.3) that \( \lambda_k \in \mathcal{D}(T_N) \). Hence, (2)(i) in Lemma 2 for our sequence \( \{ \lambda_k \} \) is established.

To establish (2)(ii) in Lemma 2, we observe that both \( Q^1 \) and \( Z \) are compact subsets of \( T_N \) in the torus topology and \( Z \cap Q^1 = \emptyset \). Consequently, \( \exists \varepsilon_0 > 0 \) such that

\[
B(x, \varepsilon_0) \cap Q^{1*} = \emptyset \quad \text{for } x \in Z^* \tag{3.4}
\]

where \( Q^{1*} = \bigcup_m \{ Q^1 + m \} \) and \( Z^* = \bigcup_m \{ Z + m \} \). Now for \( k \) fixed, let

\[
|p^k| = \sum_{j=1}^{N} |p^k_{j}|.
\]

Then it follows from (3.4) with \( \varepsilon_1 = \varepsilon_0 / |p^k| \) that

\[
x + (p^k_{1}y_1, \ldots, p^k_{N}y_N) \notin Q^{1*} \quad \text{for } |y| < \varepsilon_1 \text{ and } x \in Z^*.
\]

By (1.9)(ii), \( x = (x_1, \ldots, x_N) \in Z \implies (p^k_{1}x_1, \ldots, p^k_{N}x_N) \in Z^* \). Therefore if \( x \in Z \) and \( |y| < \varepsilon_1 \),

\[
(p^k_{1}x_1, \ldots, p^k_{N}x_N) + (p^k_{1}y_1, \ldots, p^k_{N}y_N) \notin Q^{1*} \tag{3.5}
\]

From (3.1) and (3.2), we see that \( Q^{2*} \subset Q^{1*} \) and from (ii) in the properties of \( \lambda(x) \) that \( \text{supp} (\lambda) \subset Q^2 \). Consequently, it follows from (3.3) and (3.5) that if \( x \in Z \) and \( |y| < \varepsilon_1 \), \( \lambda_k(x + y) = 0 \). Hence, \( Z \cap \text{supp} (\lambda_k) = \emptyset \), and (2)(ii) of Lemma 2 is established.
To establish (2)(iii) in Lemma 2, we observe that
\[ (3.6) \quad \lambda(x) = \sum_m \lambda^\wedge(m) e^{2\pi i (m, x)} \]
where
\[ (3.7) \quad \sum_m |\lambda^\wedge(m)| = C < \infty. \]
It follows therefore from (3.3) that
\[ (3.8) \quad \lambda_k(x) = \lambda^\wedge(0) + \sum_{m \neq 0} \lambda^\wedge(m) e^{2\pi i (p^k_1 x_1 + \cdots + p^k_N x_N)} \]
for \( x \in T_N \). Hence,
\[ \sum_m |\lambda_k^\wedge(m)| = \sum_m |\lambda^\wedge(m)| = C < \infty, \]
which fact establishes (2)(iii) in Lemma 2.

To establish (2)(iv) in Lemma 2, let \( m_0 \) be an arbitrary but fixed integral lattice with \( m_0 \neq 0 \). It follows from (1.9) that \( \exists k_0 > 0 \) such that for \( k > k_0 \)
\[ \left( \sum_{j=1}^N |p^k_j m_j|^2 \right)^{1/2} \geq \min(p^k_1, \ldots, p^k_N) \geq |m_0| + 1 \]
for all \( m \neq 0 \). Consequently, it follows from (3.8) that for \( k > k_0 \),
\[ \lambda_k^\wedge(m_0) = 0, \]
and (2)(iv) in Lemma 2 is established.

Next, we use (3.8) once again and obtain that
\[ \lambda_k^\wedge(0) = \lambda^\wedge(0) \quad \forall k. \]
From the defining properties (i), (ii), (iii) of \( \lambda(x) \) stated above, we see that
\[ 0 < \int_{T_N} \lambda(x) = \lambda^\wedge(0). \]
We conclude from these last two facts that indeed \( \lim_{k \to \infty} \lambda_k^\wedge(0) = \alpha_0 \neq 0 \). Hence (2)(v) in Lemma 2 is established, and the proof of the theorem is complete.

4. Examples of sets of uniqueness

In order to show that a set \( Z \subset T_N \) is a set of uniqueness for the class \( \mathcal{B}(T_N) \), according to the theorem, we need only show that (i) it is closed in the torus sense, (ii) it is of \( N \)-dimensional Lebesgue measure zero, and (iii) it is a \( V \)-set. We shall do this for two different examples: the first will take place in dimension \( N = 3 \) and the second in dimension \( N = 2 \). Each example will constitute a non-Cartesian product set. It will also be clear that both examples hold for \( N \geq 3 \), but the notation in the higher dimensional cases is considerably more cumbersome. Also, example 2 covers example 1 in dimension \( N = 2 \).

For \( N = 3 \), the set we will deal with is alluded to in Mandelbrot’s book as triadic fractal foam [M, p. 133], and we will refer to it as \( TFF \). We will define \( TFF \) in \( \bar{T}_3 \) where
\[ \bar{T}_3 = \{ x = (x_1, x_2, x_3) : 0 \leq x_j \leq 1, j = 1, 2, 3 \}. \]
Our set of uniqueness $Z$ will then be
\begin{equation}
Z = TFF \cap T_3.
\end{equation}

To define $TFF$, subdivide $T_3$ into 27 closed congruent cubes by cutting $T_3$ with planes parallel to the three axes, i.e., $x_j = 1/3$, $2/3$ for $j = 1, 2, 3$. Each cube has a distinguished point within it, namely $x^{j_1, 1}$ which is the point with smallest Euclidean norm in each cube. Each $x^{j_1, 1}$ corresponds to a unique triple
\begin{equation}
x^{j_1, 1} \longleftrightarrow (\varepsilon_1, \delta_1, \zeta_1)
\end{equation}
with $x^{j_1, 1} = (\varepsilon_1/3, \delta_1/3, \zeta_1/3)$ where $\varepsilon_1, \delta_1, \zeta_1$ run through the numbers 0, 1, 2 with one caveat: we do not allow the triple with $\varepsilon_1 = \delta_1 = \zeta_1 = 1$ since we are going to remove the open cube corresponding to this point. We shall define an ordering on different triples of the nature $(\varepsilon_1, \delta_1, \zeta_1) \neq (\varepsilon'_1, \delta'_1, \zeta'_1)$ as follows:
\begin{equation}
(\varepsilon_1, \delta_1, \zeta_1) \prec (\varepsilon'_1, \delta'_1, \zeta'_1)
\end{equation}

(i) $\varepsilon_1 < \varepsilon'_1$ or (ii) $\varepsilon_1 = \varepsilon'_1$ and $\delta_1 < \delta'_1$ or (iii) $\varepsilon_1 = \varepsilon'_1$ and $\delta_1 = \delta'_1$ and $\zeta_1 < \zeta'_1$.

This also imposes an ordering on $\{x^{j_1, 1}\}$ via (4.2).

Now we have 26 triples, and we count them out according to this $\prec$-ordering, giving us $\{x^{j_1, 1}\}_{j_1=1}^{26}$. Thus $x^{1, 1} = (0, 0, 0)$, $x^{2, 1} = (0, 0, 1/3)$, $x^{3, 1} = (0, 0, 2/3)$, $x^{4, 1} = (0, 1/3, 0)$, ..., $x^{26, 1} = (2/3, 2/3, 2/3)$. The closed cube which has $x^{j_1, 1}$ as its distinguished point, we label $I^{j_1, 1}$. We then define $I^1 \subset T_3$ to be the closed set
\begin{equation}
I^1 = \bigcup_{j_1=1}^{26} I^{j_1, 1}.
\end{equation}

In each of the 26 cubes, which have sides of length 1/3, we now perform the same operation as above, obtaining $(26)^2$ cubes, which now have sides of length $(1/3)^2$. Each of these last-mentioned cubes has a distinguished point
\[x^{j_2, 2} = x^{j_1, 1} + (\varepsilon_2/3^2, \delta_2/3^2, \zeta_2/3^2)\]
where $\varepsilon_2, \delta_2, \zeta_2$ run through the numbers 0, 1, 2, and we do not allow the triple with $\varepsilon_2 = \delta_2 = \zeta_2 = 1$. These triples have an ordering imposed on them by (4.3) which in turn gives an ordering on $\{x^{j_2, 2}\}$ defined as follows:
\begin{equation}
x^{j_2, 2} \prec x^{j'_2, 2} \text{ means (i) } x^{j_1, 1} \prec x^{j'_1, 1} \text{ or (ii) } x^{j_1, 1} = x^{j'_1, 1} \text{ and } (\varepsilon_2, \delta_2, \zeta_2) \prec (\varepsilon'_2, \delta'_2, \zeta'_2).
\end{equation}

We then count out the $(26)^2$ points according to this ordering and obtain $\{x^{j_2, 2}\}_{j_2=1}^{26}$. The closed cube containing $x^{j_2, 2}$ as its distinguished point we call $I^{j_2, 2}$. We then define $I^2 \subset I^1 \subset T_3$ to be the closed set
\begin{equation}
I^2 = \bigcup_{j_2=1}^{26} I^{j_2, 2}.
\end{equation}

In each of the $(26)^2$ cubes which have sides of length $(1/3)^2$, we now perform the same operation as before obtaining $(26)^3$ cubes with each having sides of length $(1/3)^3$. We get distinguished points in each of these cubes and put an ordering on them similar to the procedure in (4.4) to obtain $\{x^{j_3, 3}\}_{j_3=1}^{26}$. Next, in a procedure similar to (4.5), we get the closed set $I^3$ with $I^3 \subset I^2 \subset I^1 \subset T_3$. 

Continuing in this manner, we get the decreasing sequence of closed sets \( \{ I^n \}_{n=1}^\infty \) with \( I^{n+1} \subset I^n \subset \mathbb{T}_3 \) where each \( I^n \) consists of \((26)^n\) cubes each with sides of length \((1/3)^n\). The set \( TFF \) is then defined to be

\[
TFF = \bigcap_{n=1}^\infty I^n.
\]

With \( Z \) defined by (4.1) where \( TFF \) is defined by (4.6), we see that \( Z \) is closed in the torus sense because every point in the boundary of \( \mathbb{T}_3 \) is contained in \( TFF \). Also since the Lebesgue measure of each \( I^n \) in (4.6) is \((26/27)^n\), we see that \( TFF \) is of \( N \)-dimensional Lebesgue measure zero; hence by (4.1) the same can be said of \( Z \).

Consequently, we conclude from the conditions in the hypothesis of the theorem to show that \( Z \) is a set of uniqueness for the class \( \mathcal{B}(T_3) \); it only remains to show that \( Z \) is a \( \mathcal{V} \)-set according to the definition given in (1.9). We claim

\[
x \in Z \Rightarrow (3^kx_1, 3^kx_2, 3^kx_3) \in Z \mod 1 \text{ in each variable}
\]

for \( k \) a positive integer where \( x = (x_1, x_2, x_3) \). Once (4.7) is established, it then follows from (1.9) that \( Z \) is indeed a \( \mathcal{V} \)-set. To show that (4.7) holds, it is clearly sufficient to show that it holds in the special case when \( k = 1 \), i.e.,

\[
x \in Z \Rightarrow (3x_1, 3x_2, 3x_3) \in Z \mod 1 \text{ in each variable.}
\]

It follows from the definition of \( TFF \) in (4.6) that given \( x_o \in TFF \), \( \exists \{x^{j,n}_o\}_{n=1}^\infty \) where each \( x^{j,n}_o \) is a distinguished point of one of the \((26)^n\) cubes in \( I^n \) of sides \((1/3)^n\) such that

\[
\|x^{j,n}_o - x_o\| \to 0 \text{ as } n \to \infty.
\]

Consequently, to show that (4.8) holds it is sufficient to show that it holds when \( x \) is a distinguished point \( x^{j,n}_o \).

If \( x = x^{j+1,n}_o \), then it follows from the enumeration of the \( 26 \) such points given below (4.3) that the conclusion in (4.8) holds. Hence from the above discussion \( Z \) will be a \( \mathcal{V} \)-set, if we show the following:

Given \( x^{j,n}_o = (x^{j,n}_1, x^{j,n}_2, x^{j,n}_3) \) a distinguished point in an \( I^{j,n} \), then

\[
(3x^{j,n}_1, 3x^{j,n}_2, 3x^{j,n}_3) = x^{j,n-1,n-1}_o \mod 1 \text{ in each variable}
\]

for \( n \geq 2 \) where \( x^{j,n-1,n-1}_o \) is a distinguished point in an \( I^{j,n-1,n-1} \).

It is clear from the representation of \( x^{j+2,n} \) given above (4.4) that

\[
x^{j+2,n} = \left( \frac{\varepsilon_1}{3} + \frac{\varepsilon_2}{3^2} \right) + \left( \frac{\delta_1}{3} \right) + \left( \frac{\zeta_1}{3} \right)
\]

where \( \varepsilon_i, \delta_i, \zeta_i \) run through the numbers 0, 1, 2, and we do not allow \( \varepsilon_i = \delta_i = \zeta_i = 1 \) for \( i = 1, 2 \).

Exactly similar reasoning shows that

\[
x^{j,n} = \left( \sum_{i=1}^n \frac{\varepsilon_i}{3^i} \right) + \left( \sum_{i=1}^n \frac{\delta_i}{3^i} \right) + \left( \sum_{i=1}^n \frac{\zeta_i}{3^i} \right)
\]

where now \( \varepsilon_i = \delta_i = \zeta_i = 1 \) is not allowed for \( i = 1, ..., n \). From (4.10), we see that

\[
(3x^{j,n}_1, 3x^{j,n}_2, 3x^{j,n}_3) = \left( \varepsilon_1 + \sum_{i=1}^{n-1} \frac{\varepsilon_{i+1}}{3^i}, \delta_1 + \sum_{i=1}^{n-1} \frac{\delta_{i+1}}{3^i}, \zeta_1 + \sum_{i=1}^{n-1} \frac{\zeta_{i+1}}{3^i} \right).
\]

But \( \varepsilon_i, \delta_i, \) and \( \zeta_i \) are each non-negative integers, and we conclude from this last equality that (4.9) does indeed hold. Hence \( Z \) defined by (4.1) is a \( \mathcal{V} \)-set, and our example is complete.
Our next example will take place in dimension $N = 2$. We will call it a generalized carpet and refer to it as $GC_{pq}$ where $p$ and $q$ are both positive integers strictly greater than 2. The set $GC_{pq}$ will be a subset of $T_2$ where

$$T_2 = \{x = (x_1, x_2) : 0 \leq x_j \leq 1, \ j = 1, 2\}.$$ 

In particular, when $p = q = 3$, $GC_{pq}$ will be the set referred to in the literature as the Sierpinski carpet [M, p. 144].

To define $GC_{pq}$, subdivide $T_2$ into $pq$ closed congruent rectangles by cutting $T_2$ with lines parallel to the two axes, i.e., $x_1 = 1/p, 2/p, ..., (p - 1)/p, x_2 = 1/q, 2/q, ..., (q - 1)/q$. Each rectangle has a distinguished point within it, namely $x^{j_1, 1}$ which is the point with smallest Euclidean norm in each rectangle. Each $x^{j_1, 1}$ corresponds to a unique double

$$x^{j_1, 1} \rightarrow (\varepsilon_1, \delta_1)$$

with $x^{j_1, 1} = (\varepsilon_1/p, \delta_1/q)$ where $\varepsilon_1$ and $\delta_1$ run through the numbers $0, 1, ..., p - 1$ and $0, 1, ..., q - 1$, respectively. There is a caveat however; the doubles with $\varepsilon_1 = 1, ..., p - 2$, and simultaneously $\delta_1 = 1, ..., q - 2$, are not allowed, for the rectangles corresponding to these points will be removed, i.e., the middle $(p - 2)(q - 2)$ rectangles will be deleted. An ordering on different doubles of the nature $(\varepsilon_1, \delta_1) \neq (\varepsilon'_1, \delta'_1)$ is then defined as follows:

$$(\varepsilon_1, \delta_1) \prec (\varepsilon'_1, \delta'_1)$$

(i) $\varepsilon_1 < \varepsilon'_1$ or (ii) $\varepsilon_1 = \varepsilon'_1$ and $\delta_1 < \delta'_1$. This also imposes an ordering on $\{x^{j_1, 1}\}^{\gamma}_{j_1=1}$ via (4.11) where $\gamma$ is the integer $\gamma = pq - (p - 2)(q - 2)$.

In particular, we see that $x^{3, 1} = (0, 0), x^{2, 1} = (0, 1/q), x^{3, 1} = (0, 2/q), ..., x^{\gamma, 1} = ((p - 1)/p, (q - 1)/q)$. We also observe that $x^{0, 1} = (0, (q - 1)/q), x^{0, 1} = (1/p, 0)$, and $x^{q + 2, 1} = (1/p, (q - 1)/q)$. The closed rectangle which has $x^{3, 1}$ as its distinguished point we label $I^{3, 1}$. We then define $I^1 \subset T_2$ to be the closed set

$$I^1 = \bigcup_{j_1=1}^{\gamma} I^{j_1, 1}.$$

In each of the $\gamma$ closed rectangles, which have sides of length $1/p$ and $1/q$, we now perform the same operation as above, obtaining $\gamma^2$ closed rectangles, which now have sides of length $(1/p)^2$ and $(1/q)^2$. Each of these last-mentioned rectangles has a distinguished point within it, namely $x^{j_2, 2}$, where

$$x^{j_2, 2} = x^{j_1, 1} + (\varepsilon_2/p^2, \delta_2/q^2),$$

and where $\varepsilon_2$ and $\delta_2$ run through the numbers $0, 1, ..., p - 1$ and $0, 1, ..., q - 1$, respectively. Also, we do not allow the doubles with $\varepsilon_2 = 1, ..., p - 2$, and simultaneously $\delta_2 = 1, ..., q - 2$. The doubles $(\varepsilon_2, \delta_2) \neq (\varepsilon'_2, \delta'_2)$ have an ordering imposed upon them by (4.12), which, in turn, imposes an ordering on the distinguished points given by $x^{j_2, 2} < x^{j'_2, 2}$ akin to the ordering given in (4.4). We then count out the $\gamma^2$ points according to this ordering and obtain $\{x^{j_2, 2}\}^{\gamma^2}_{j_2=1}$. The closed rectangle of sides $(1/p)^2$ and $(1/q)^2$ containing $x^{j_2, 2}$ as its distinguished point we call $I^{j_2, 2}$.
We then define $I^2 \subset I^1 \subset \bar{T}_2$ to be the closed set
\[ I^2 = \bigcup_{j_2=1}^{\gamma^2} I^{j_2,2}. \]

Continuing in this manner, we get the decreasing sequence of closed sets \( \{I^n\}_{n=1}^{\infty} \) with \( I^{n+1} \subset I^n \subset \bar{T}_2 \) where each \( I^n \) consists of \( \gamma^n \) rectangles each with sides of length \( (1/p)^n \) and \( (1/q)^n \). The set \( GC_{pq} \) is then defined to be
\[ (4.13) \quad GC_{pq} = \bigcap_{n=1}^{\infty} I^n. \]

Next, we define \( Z \) to be the set
\[ (4.14) \quad Z = GC_{pq} \cap T_2, \]
and observe that \( Z \) is closed in the torus sense because every point in the boundary of \( \bar{T}_2 \) is contained in \( GC_{pq} \). Also, since the Lebesgue measure of each \( I^n \) is \( (\gamma/pq)^n \) where \( \gamma = pq - (p-2)(q-2) \), we see from (4.13) and (4.14) that the 2-dimensional Lebesgue measure of \( Z \) is zero. Hence according to the conditions in the hypothesis of the theorem, to show that \( Z \) is a set of uniqueness for the class \( B(T_2) \), it only remains to show that \( Z \) is a \( \mathcal{V} \)-set, i.e., that the conditions set forth in (1.9) hold.

We claim
\[ (4.15) \quad x \in Z \Rightarrow (p^{j_1}x_1, q^{j_2}x_2) \in Z \mod 1 \text{ in each variable} \]
for \( k \) a positive integer and with \( x = (x_1, x_2) \). Once (4.15) is established, then it follows from (1.9) that \( Z \) is indeed a \( \mathcal{V} \)-set. To show that (4.15) holds, it is clearly sufficient to show that it holds in the special case when \( k = 1 \), i.e.,
\[ (4.16) \quad x \in Z \Rightarrow (px_1, qx_2) \in Z \mod 1 \text{ in each variable}. \]

Using the same argument that we used after (4.8), we see that to show that \( Z \) is a \( \mathcal{V} \)-set, we need only show that (4.16) holds for the special case when \( x = x^{j_n,n} \), a distinguished point in one of the closed rectangles \( I^{j_n,n} \) with sides \( (1/p)^n \) and \( (1/q)^n \).

If \( x = x^{j_n,1} \), then it follows from the enumeration of such points below (4.12) that (4.16) does indeed hold. Hence to show that \( Z \) is a \( \mathcal{V} \)-set, it only remains to establish the fact that (4.16) holds when \( x = x^{j_n,n} \) for \( n \geq 2 \). This will be accomplished if we show that the following fact holds:

Given \( x^{j_n,n} = (x_1^{j_n,n}, x_2^{j_n,n}) \), a distinguished point in an \( I^{j_n,n} \), then
\[ (4.17) \quad (px_1^{j_n,n}, qx_2^{j_n,n}) = x^{j_{n-1},n-1} \mod 1 \text{ in each variable} \]
for \( n \geq 2 \) where \( x^{j_{n-1},n-1} \) is a distinguished point in an \( I^{j_{n-1},n-1} \).

It is clear from the representation of \( x^{j_1,2} \) given above that
\[ x^{j_1,2} = \left( \frac{\varepsilon_1}{p}, \frac{\varepsilon_2}{q}, \delta_1 + \delta_2 \frac{q}{q} \right) \]
where \( \varepsilon_i \) and \( \delta_i \) run through the numbers \( 0, ..., p-1, \) and \( 0, ..., q-1 \), respectively, and we do not allow \( \varepsilon_i = 1, ..., p-2 \) and simultaneously \( \delta_i = 1, ..., q-2 \) for \( i = 1, 2 \). Exactly similar reasoning shows that
\[ (4.18) \quad x^{j_n,n} = \left( \sum_{i=1}^{n} \frac{\varepsilon_i}{p^i}, \sum_{i=1}^{n} \delta_i \right) \]
where $\varepsilon_i$ and $\delta_i$ are exactly as before, now with $i = 1, \ldots, n$. From (4.18), we see that
\[
(p_{x_1^{j_1}}, q_{x_2^{j_2}}) = \left(\varepsilon_1 + \sum_{i=1}^{n-1} \frac{\varepsilon_{i+1}}{p^i}, \delta_1 + \sum_{i=1}^{n-1} \frac{\delta_{i+1}}{q^i}\right).
\]
But $\varepsilon_i$ and $\delta_i$ are each non-negative integers, and we conclude from this last equality and (4.18) that (4.17) does indeed hold. Hence, $Z$ defined by (4.14) is a $\mathcal{V}$-set, and our example is complete.

In closing, we point out that the $H^J$-sets, defined in [Sh2] for dimension $N = 2$ and in [AW] for dimensions $N \geq 3$, can also be shown to be sets of uniqueness for the class $\mathcal{B}(T_N)$ with respect to distributions on the $N$-torus.

References


Department of Mathematics, University of California, Riverside, California 92521-0135

E-mail address: shapiro@math.ucr.edu