THE \( p \)-EXPONENT OF THE \( K(1) \)-LOCAL SPECTRUM \( \Phi SU(n) \)

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(Communicated by Paul Goerss)

Abstract. Let \( p \) be a fixed odd prime. In this paper we prove an exponent conjecture of Bousfield, namely that the \( p \)-exponent of the spectrum \( \Phi SU(n) \) is \((n-1) + \nu_p((n-1)!)\) for \( n \geq 2 \). It follows from this result that the \( p \)-exponent of \( \Omega^p SU(n)(i) \) is at least \((n-1) + \nu_p((n-1)!)\) for \( n \geq 2 \) and \( i, q \geq 0 \), where \( SU(n)(i) \) denotes the \( i \)-connected cover of \( SU(n) \).

1. Introduction

Let \( p \) be a prime number and \( A \) be an object in an additive category. We define the \( p \)-exponent of \( A \) to be the smallest non-negative integer \( e \) such that the morphism \( p^e 1_A : A \to A \) is the zero morphism.

The purpose of this paper is to give a proof of a conjecture of Bousfield, namely that the \( p \)-exponent of the spectrum \( \Phi SU(n) \) is \((n-1) + \nu_p((n-1)!)\) for \( n \geq 2 \) and for \( p \) an odd prime. Here and throughout \( \nu_p \) denotes the exponent of \( p \) in an integer and \( \Phi \) is a \( v_1 \) telescope functor from the homotopy category of pointed CW-complexes to the category of \( K(1) \)-local spectra.

The functor \( \Phi \) was introduced by Bousfield and is described in [1, 2, 6]. A similar functor can also be found in [5]. Among the many intriguing properties of \( \Phi \) are the following: (i) for any spectrum \( E \), there is a natural equivalence \( \Phi(\Omega^\infty E) \simeq E_{K/p} \), (ii) \( \Phi \) preserves fibrations, and (iii) \( v_1^{-1} \pi_{*}(X; p) \cong \pi_{*}(\Phi X) \).

The functor \( \Phi \) is complicated enough to make actual calculations somewhat onerous. However, the following example is well known. It was shown in [4] that \( \Phi S^{2n+1} = v_1^{-1}M(p^n) \), where \( M(p^n) \) is the mod \( p^n \) Moore space.

One can also obtain \( \Phi S^{2n} \) from the fibration

\[
S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}.
\]

From here, using towers of fibrations with products of loop spaces on spheres, various Lie groups can be computed. The Lie group \( SU(n) \) is a natural first choice; it is interesting, yet tractable.

Given a 1-connected finite \( H \)-space \( X \), let \( M \cong \hat{Q}K^1(X; \widehat{\mathbb{Z}}_p) \cong PK^1(X; \widehat{\mathbb{Z}}_p) \), the \( p \)-adic Adams module of indecomposables or primitives. In [3], Bousfield proves, among other things, that if \( H_*(X; \mathbb{Q}) \) is associative and \( H_*(X; \mathbb{Z}(p)) \) is finitely generated over \( \mathbb{Z}(p) \), then \( M/\psi^p \) and \( \Phi X \) have the same \( p \)-exponent. For the case
\(X = SU(n)\) we have, via a result of Hodgkin [5],
\[ M_n \cong \tilde{Q}K^1(SU(n); \mathbb{Z}_p) \cong K^1(\Sigma \mathbb{CP}^{n-1}; \mathbb{Z}_p) \cong \tilde{K}^0(\mathbb{CP}^{n-1}; \mathbb{Z}_p). \]

Now, because \(\tilde{K}^0(\mathbb{CP}^{n-1}; \mathbb{Z}_p) \cong \mathbb{Z}_p[x]/(1, x^n)\) where \(x = \xi - 1\) and \(\xi\) is the canonical line bundle on \(\mathbb{CP}^{n-1}\), we have \(M_n\{x, x^2, \ldots, x^{n-1}\} \) with \(\psi^p x = \sum_{i=1}^{n-1} (p^i)x^i\) and \(\psi^p x^m = (\psi^p x)^m\) for \(2 \leq m \leq n - 1\). Hence to prove Bousfield’s conjecture, it suffices to prove the following lemma.

**Lemma 1.1.** The \(p\)-exponent of \(M_n/\psi^p\) is \((n - 1) + \nu_p((n - 1)!))\) for \(n \geq 2\).

From this we deduce our main theorem.

**Theorem 1.2.** The \(p\)-exponent of \(\Phi SU(n)\) is \((n - 1) + \nu_p((n - 1)!))\) for \(n \geq 2\).

Additionally, we obtain the following corollary since the functor \(\Phi\) preserves loopings and since \(\Phi\) carries \(i\)-connected coverings to equivalences.

**Corollary 1.3.** The \(p\)-exponent of \(\Phi! SU(n)/i\) is at least \((n - 1) + \nu_p((n - 1)!))\) for \(n \geq 2\) and \(i, q \geq 0\), where \(SU(n)/i\) denotes the \(i\)-connected cover of \(SU(n)\).

## 2. Proof of Lemma 1.1

The proof of Lemma 1.1 will proceed in two steps. Let \(e_i\) denote the \(p\)-exponent of \(x^i\) in \(M_n/\psi^p\), and let \(b = (n - 1) + \nu_p((n - 1)!))\). We will show \(e_1 = b\) and \(e_i \leq b\) for all \(i, 2 \leq i \leq n - 1\).

**Lemma 2.1.** Let \(a_1 = p^{b-1}\) and, for \(k > 1\),
\[
 a_k = \frac{(-1)^{k+1}}{k!} p^{b-k}(p-1)(2p-1)(3p-1)\cdots((k-1)p-1).
\]
Then \(\psi^p(\sum_{k=1}^{n-1} a_kx^k) = p^b x\) and \(\sum_{k=1}^{n-1} a_kx^k\) is the unique element of \(M_n\) taken to \(p^b x\) under the action of \(\psi^p\). Moreover \(e_1 = b\).

**Proof.** Consider the matrix of \(\psi^p\) (over \(\mathbb{Z}_p\)) with respect to the basis \(\{x, x^2, \ldots, x^{n-1}\}:
\[
[\psi^p] = [\begin{array}{cccccc}
 c_{1,1} & 0 & 0 & \ldots & 0 \\
 c_{2,1} & c_{2,2} & 0 & \ldots & 0 \\
 c_{3,1} & c_{3,2} & c_{3,3} & \ldots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \ldots & c_{n-1,n-1}
\end{array}]
\]
where \(c_{i,j}\) is the coefficient of \(x^i\) in \(((1 + x)^p - 1)^j\). Note that
\[
\sum_{i_1 + i_2 + \cdots + i_k = i} \binom{p}{i_1} \binom{p}{i_2} \cdots \binom{p}{i_k} = k^p \cdot \binom{kp}{i}.
\]
Thus, by the principle of inclusion and exclusion (see [4] for example),
\[
c_{i,j} = \sum_{k=0}^{j-1} (-1)^k \binom{j}{j-k} \binom{(j-k)p}{i}.
\]
For the time being, view \(\psi^p\) as a linear transformation from \(\mathbb{Q}^{n-1}\) to \(\mathbb{Q}^{n-1}\). Then for \(m \geq 0\), let \(a'_i = p^{m-1}\) and, for \(k > 1\),
\[
a'_k = \frac{(-1)^{k+1}}{k!} p^{m-k}(p-1)(2p-1)(3p-1)\cdots((k-1)p-1).
\]
We will show that
\[
\begin{bmatrix}
c_{1,1} & 0 & 0 & \cdots & 0 \\
c_{2,1} & c_{2,2} & 0 & \cdots & 0 \\
c_{3,1} & c_{3,2} & c_{3,3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \cdots & c_{n-1,n-1} \\
\end{bmatrix}
\begin{bmatrix}
a_1' \\
a_2' \\
a_3' \\
\vdots \\
a_{n-1}' \\
\end{bmatrix}
= \begin{bmatrix} p^m \
0 \\
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}.
\]
Clearly \(\sum_{j=1}^{n-1} c_{i,j}a_j' = p^m\) and \(\sum_{j=1}^{n-1} c_{2,j}a_j' = 0\). We are left to show that \(\sum_{j=1}^{n-1} c_{i,j}a_j'\) \((i \geq 3)\) yields
\[
= \binom{p}{i} \left( \binom{1}{1} a_1' - \binom{2}{1} a_2' + \binom{3}{1} a_3' + \cdots + (-1)^{i-1} \binom{i}{1} a_i' \right) + \binom{2p}{i} \left( \binom{2}{1} a_2' - \binom{3}{2} a_3' + \frac{4}{2} a_4' + \cdots + (-1)^{i-2} \binom{i}{2} a_i' \right) + \cdots + \binom{kp}{i} \left( \binom{k}{1} a_k' - \binom{k+1}{k} a_{k+1}' + \cdots + (-1)^{i-k} \binom{k}{k} a_i' \right) + \cdots + \binom{ip}{i} \binom{i}{i} a_i'.
\] (2.1)
By induction one can see that for \(l = 1, \ldots, i,\)
\[
\sum_{k=l}^{i} (-1)^{k-l} \binom{k}{l} a_k' = \frac{(-1)^{i+1}}{i!} p^{m-i} \binom{i}{l} (p-1)(p-2) \cdots (ip-1)
\]
where \(\sim\) means leave out. Therefore (2.1) becomes
\[
\binom{p^{m-i}}{i} \binom{i}{l} (p-1)(p-2) \cdots (ip-1) \sum_{l=1}^{i} (-1)^{l+1} \binom{i}{l} \binom{lp}{i} \frac{1}{lp-1}.
\]
So it suffices to show that
\[
\sum_{l=1}^{i} (-1)^{l+1} \binom{i}{l} \binom{lp}{i} \frac{1}{lp-1} = 0.
\]
Notice that
\[
\sum_{l=1}^{i} (-1)^{l+1} \binom{i}{l} \binom{lp}{i} \frac{1}{lp-1} = \frac{p}{(i-1)!} \sum_{l=1}^{i} (-1)^{l+1} \binom{i-1}{l-1} (lp-2) \cdots (lp-i+1).
\]
Let \(f(t) = \sum_{l=1}^{i} (-1)^{l+1} \binom{i-1}{l-1}(lp-2) \cdots (lp-i+1) t^{lp-i}.\) Then
\[
f(t) = \sum_{l=1}^{i} (-1)^{l-1} \binom{i-1}{l-1} \frac{d}{dt} t^{lp-2} = \left( \frac{d}{dt} \right)^{i-2} t^{lp-2} \sum_{l=1}^{i} (-1)^{l-1} \binom{i-1}{l-1} t^{(l-1)p}.
\]
Hence \(f(t) = (\frac{d}{dt})^{i-2} t^{lp-2} (1-t^p)^{i-1}.\) Thus \(f(1) = 0\) since all terms will be divisible by \((1-t^p).\) Therefore \(\sum_{j=1}^{n-1} c_{i,j}a_j' = 0\) for \(i \geq 3.\)
Note that \(\ker[\nu_p] = 0\) over \(\mathbb{Q}.\) Thus \(\langle a_1', a_2', a_3', \ldots, a_{n-1}' \rangle\) is the unique vector in \(\mathbb{Q}^{n-1}\) that is taken to \((p^m, 0, 0, \ldots, 0)\) by the transformation \([\nu_p].\)
Now notice that the \(a_k', 1 \leq k \leq n-1,\) are integral, hence also elements of \(\hat{\mathbb{F}}_p,\) only when \(m-k \geq \nu_p(k!),\) i.e., \(m \geq n-1 + \nu_p((n-1)!) = b.\)
Let \( a_1 = p^{b-1} \) and, for \( k > 1 \),
\[
    a_k = \frac{(-1)^{k+1}}{k!} p^{b-k} (p-1)(2p-1)(3p-1) \cdots ((k-1)p-1).
\]

Then, since \( \ker[\psi^p] = 0 \) over \( \mathbb{Z}_p \), \( (a_1, a_2, a_3, \ldots, a_{n-1}) = \sum_{k=1}^{n-1} a_k x^k \) is the unique element of \( M_n \) such that \( \psi^p(\sum_{k=1}^{n-1} a_k x^k) = p^b x \).

To see that there does not exist \( w \in M_n \) such that \( \psi^p(w) = p^{b-\varepsilon} x, \varepsilon \in \mathbb{Z}^+ \), consider the following. Suppose such a \( w = \sum_{k=1}^{n-1} q_k x^k \) existed. Then at least one of the \( q_k \) has to be in \( \mathbb{Z}^+ \). But then \( \psi^p(\psi^p(w)) = p^b x \). Since \( p^b q_k = a_k \) by uniqueness, we get the contradiction \( p^b q_k \in \mathbb{Z}^+ \) and \( p^b q_k \in \mathbb{Z} \).

The next lemma will finish the proof of Lemma 1.1.

**Lemma 2.2.** For \( 2 \leq i \leq n - 1 \), let \( e_i \) denote the \( p \)-exponent of \( x^i \) in \( M_n/\psi^p \). Then \( e_i \leq b \).

**Proof.** First note that the relations of \( M_n/\psi^p \) are given by the following equations:
\[
    \begin{align*}
    \alpha_{1,1} x + \alpha_{1,2} x^2 + \alpha_{1,3} x^3 + \cdots + \alpha_{1,n-2} x^{n-2} + \alpha_{1,n-1} x^{n-1} &= 0, \\
    \alpha_{2,2} x^2 + \alpha_{2,3} x^3 + \cdots + \alpha_{2,n-2} x^{n-2} + \alpha_{2,n-1} x^{n-1} &= 0, \\
    \alpha_{3,3} x^3 + \cdots + \alpha_{3,n-2} x^{n-2} + \alpha_{3,n-1} x^{n-1} &= 0, \\
    & \vdots \\
    \alpha_{n-2,n-2} x^{n-2} + \alpha_{n-2,n-1} x^{n-1} &= 0, \\
    \alpha_{n-1,n-1} x^{n-1} &= 0,
    \end{align*}
\]
where \( \alpha_{i,j} = \sum_{k=0}^{i-1} (-1)^k \binom{i}{k} \binom{(i-k)p}{j} \) (these relations can be obtained from the transpose of the matrix \( [\psi^p] \)). Notice that \( \alpha_{i,i} = \binom{i}{1}^2 \) and \( \alpha_{i,i+1} = \binom{i}{1} \binom{i}{1} \binom{i}{1} \binom{i}{1}^{-1} \).

Since \( \alpha_{n-1,n-1} = p^{n-1} \) we know that the \( p \)-exponent of \( x^{n-1} \) is \( n-1 \). Via back-substitution, we are then able to find the \( p \)-exponent of \( x^{n-2}, x^{n-3} \), and so on, all the way up to \( x \). This line of thinking leads us to the formula
\[
e_i = d_{i,i} + \max\{ e_j - d_{i,j} : j = i+1, \ldots, n-1 \},
\]
where \( d_{i,j} = \nu_p(\alpha_{i,j}) \). (Note: \( d_{i,i} = i \).) It follows that \( e_i \geq e_{i+1} + (i - d_{i,i+1}) \).

Next we see that \( i - d_{i,i+1} = -\nu_p(i) \), since \( d_{i,i+1} = \nu_p(\alpha_{i,i+1}) = \nu_p(\binom{i}{1}^2 \binom{i}{1}^2 \binom{i}{1}^2 \binom{i}{1}^{-1}) = \nu_p(\binom{p-2}{2} i^p) \). Thus \( e_i \geq e_{i+1} - \nu_p(i) \). Hence if we can show that \( e_{ip} \leq b - \nu_p(ip) \) for all \( i \) such that \( 1 \leq i \leq q \), where \( qp \leq n-1 < (q+1)p \), we will be done.

By Lemma 2.1 and the relation
\[
\alpha_{1,1} x + \alpha_{1,2} x^2 + \cdots + \alpha_{1,n-1} x^{n-1} = \sum_{i=1}^{n-1} \binom{p}{i} x^i = 0
\]
we have \( e_p \leq b - 1 \). Now choose the smallest \( k \) such that \( e_{kp} > b - \nu_p(kp) \). Then the relation
\[
\alpha_{k,k} x^k + \alpha_{k,k+1} x^{k+1} + \cdots + \alpha_{k,kp-1} x^{kp-1} + \alpha_{k,kp} x^{kp} = p^k x^k + \cdots + x^{kp} = 0
\]
implies that \( e_k \geq k - \nu_p(kp) + 1 = (b+1) + (k - \nu_p(kp)) \).
Since $k - \nu_p(kp) \geq 0$ for all $k$ and $p$, we choose $i$ so that $(i + 1)p > k \geq ip$ and get the contradiction

$$b + (1 + k - \nu_p(kp)) \leq e_k \leq e_{k-1} \leq \cdots \leq e_{ip} \leq b.$$ 

Therefore it must be the case that $e_{ip} \leq b - \nu_p(ip)$ for all $1 \leq i \leq q$. \hfill $\square$

The results in this paper are part of the author’s Lehigh University thesis, written under the direction of Donald M. Davis.

**Acknowledgements**

The author would like to thank Donald M. Davis and Gilbert A. Stengle for helpful insights that led to the final proof of Lemma 1.1. The author would also like to thank the referee for suggesting that Corollary 1.3 be added.

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