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THE p-EXPONENT OF THE $K(1)_*$-LOCAL SPECTRUM $\Phi SU(n)$

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Abstract. Let p be a fixed odd prime. In this paper we prove an exponent conjecture of Bousfield, namely that the p-exponent of the spectrum $\Phi SU(n)$ is $(n-1)+\nu_p((n-1)!)$ for $n \geq 2$. It follows from this result that the p-exponent of $\Omega^q SU(n)\langle i\rangle$ is at least $(n-1)+\nu_p((n-1)!)$ for $n \geq 2$ and $i, q \geq 0$, where $SU(n)\langle i\rangle$ denotes the i-connected cover of $SU(n)$.

1. Introduction

Let p be a prime number and A be an object in an additive category. We define the p-exponent of A to be the smallest non-negative integer e such that the morphism $p^e 1_A : A \to A$ is the zero morphism.

The purpose of this paper is to give a proof of a conjecture of Bousfield, namely that the p-exponent of the spectrum $\Phi SU(n)$ is $(n-1)+\nu_p((n-1)!)$ for $n \geq 2$ and for p an odd prime. Here and throughout ν_p denotes the exponent of p in an integer and Φ is a v_1 telescope functor from the homotopy category of pointed CW-complexes to the category of $K(1)_*$-local spectra.

The functor Φ was introduced by Bousfield and is described in [1, 2, §6]. A similar functor can also be found in [5]. Among the many intriguing properties of Φ are the following: (i) for any spectrum E, there is a natural equivalence $\Phi(\Omega^\infty E) \simeq E_{K/p}$, (ii) Φ preserves fibrations, and (iii) ${v_1}^{-1}\pi_*(X;p) \cong \pi_*(\Phi X)$.

The functor Φ is complicated enough to make actual calculations somewhat onerous. However, the following example is well known. It was shown in [4] that $\Phi S^{2n+1} = {v_1}^{-1}M(p^n)$, where $M(p^n)$ is the mod p^n Moore space.

One can also obtain ΦS^{2n} from the fibration

$$S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}.$$

From here, using towers of fibrations with products of loop spaces on spheres, various Lie groups can be computed. The Lie group $SU(n)$ is a natural first choice; it is interesting, yet tractable.

Given a 1-connected finite H-space X, let $M \cong \hat{Q}K^1(X;\widehat{\mathbb{Z}}_p) \cong PK^1(X;\widehat{\mathbb{Z}}_p)$, the p-adic Adams module of indecomposables or primitives. In [3], Bousfield proves, among other things, that if $H_*(X;\mathbb{Q})$ is associative and $H_*(X;\mathbb{Z}_{(p)})$ is finitely generated over $\mathbb{Z}_{(p)}$, then M/ψ^p and ΦX have the same p-exponent. For the case

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$X = SU(n)$ we have, via a result of Hodgkin [6],

$$M_n \cong \hat{Q}K^1(SU(n);\widehat{\mathbb{Z}}_p) \cong K^1(\Sigma\mathbb{CP}^{n-1};\widehat{\mathbb{Z}}_p) \cong \tilde{K}^0(\mathbb{CP}^{n-1};\widehat{\mathbb{Z}}_p).$$

Now, because $\tilde{K}^0(\mathbb{CP}^{n-1};\widehat{\mathbb{Z}}_p) = \widehat{\mathbb{Z}}_p[x]/(1,x^n)$ where $x = \xi - 1$ and ξ is the canonical line bundle on $\mathbb{CP}^{n-1}$, we have $M_n\{x, x^2, \ldots, x^{n-1}\}$ with $\psi^p x = \sum_{i=1}^{n-1}\binom{p}{i}x^i$ and $\psi^p x^m = (\psi^p x)^m$ for $2 \le m \le n-1$. Hence to prove Bousfield's conjecture, it suffices to prove the following lemma.

Lemma 1.1. *The p-exponent of M_n/ψ^p is $(n-1) + \nu_p((n-1)!)$ for $n \ge 2$.*

From this we deduce our main theorem.

Theorem 1.2. *The p-exponent of $\Phi SU(n)$ is $(n-1) + \nu_p((n-1)!)$ for $n \ge 2$.*

Additionally, we obtain the following corollary since the functor Φ preserves loopings and since Φ carries i-connected coverings to equivalences.

Corollary 1.3. *The p-exponent of $\Omega^q SU(n)\langle i\rangle$ is at least $(n-1)+\nu_p((n-1)!)$ for $n \ge 2$ and $i, q \ge 0$, where $SU(n)\langle i\rangle$ denotes the i-connected cover of $SU(n)$.*

2. Proof of Lemma 1.1

The proof of Lemma 1.1 will proceed in two steps. Let e_i denote the p-exponent of x^i in M_n/ψ^p, and let $b = (n-1) + \nu_p((n-1)!)$. We will show $e_1 = b$ and $e_i \le b$ for all i, $2 \le i \le n-1$.

Lemma 2.1. *Let $a_1 = p^{b-1}$ and, for $k > 1$,*

$$a_k = \frac{(-1)^{k+1}}{k!}p^{b-k}(p-1)(2p-1)(3p-1)\cdots((k-1)p-1).$$

Then $\psi^p(\sum_{k=1}^{n-1} a_k x^k) = p^b x$ and $\sum_{k=1}^{n-1} a_k x^k$ is the unique element of M_n taken to $p^b x$ under the action of ψ^p. Moreover $e_1 = b$.

Proof. Consider the matrix of ψ^p (over $\widehat{\mathbb{Z}}_p$) with respect to the basis $\{x, x^2, ..., x^{n-1}\}$:

$$[\psi^p] = \begin{bmatrix} c_{1,1} & 0 & 0 & \cdots & 0 \\ c_{2,1} & c_{2,2} & 0 & \cdots & 0 \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \cdots & c_{n-1,n-1} \end{bmatrix}$$

where $c_{i,j}$ = the coefficient of x^i in $((1+x)^p - 1)^j$. Note that

$$\sum_{i_1+i_2+\cdots+i_k=i}\binom{p}{i_1}\binom{p}{i_2}\cdots\binom{p}{i_k} = \binom{kp}{i}.$$

Thus, by the principle of inclusion and exclusion (see [7] for example),

$$c_{i,j} = \sum_{k=0}^{j-1}(-1)^k\binom{j}{j-k}\binom{(j-k)p}{i}.$$

For the time being, view $[\psi^p]$ as a linear transformation from $\mathbb{Q}^{n-1}$ to $\mathbb{Q}^{n-1}$. Then for $m \ge 0$, let $a'_1 = p^{m-1}$ and, for $k > 1$,

$$a'_k = \frac{(-1)^{k+1}}{k!}p^{m-k}(p-1)(2p-1)(3p-1)\cdots((k-1)p-1).$$

We will show that

$$\begin{bmatrix} c_{1,1} & 0 & 0 & \dots & 0 \\ c_{2,1} & c_{2,2} & 0 & \dots & 0 \\ c_{3,1} & c_{3,2} & c_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \dots & c_{n-1,n-1} \end{bmatrix} \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \\ \vdots \\ a'_{n-1} \end{bmatrix} = \begin{bmatrix} p^m \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Clearly $\sum_{j=1}^{n-1} c_{1,j}a'_j = p^m$ and $\sum_{j=1}^{n-1} c_{2,j}a'_j = 0$. We are left to show that $\sum_{j=1}^{n-1} c_{i,j}a'_j = 0$ for $i \geq 3$. Rearranging the sum $\sum_{j=1}^{n-1} c_{i,j}a'_j$ $(i \geq 3)$ yields

$$\begin{aligned} &= \binom{p}{i}\left(\binom{1}{1}a'_1 - \binom{2}{1}a'_2 + \binom{3}{1}a'_3 + \cdots + (-1)^{i-1}\binom{i}{1}a'_i\right) \\ &\quad + \binom{2p}{i}\left(\binom{2}{2}a'_2 - \binom{3}{2}a'_3 + \binom{4}{2}a'_4 + \cdots + (-1)^{i-2}\binom{i}{2}a'_i\right) \\ &\quad + \cdots + \binom{kp}{i}\left(\binom{k}{k}a'_k - \binom{k+1}{k}a'_{k+1} + \cdots + (-1)^{i-k}\binom{i}{k}a'_i\right) \\ &\quad + \cdots + \binom{ip}{i}\binom{i}{i}a'_i. \end{aligned} \tag{2.1}$$

By induction one can see that for $l = 1, ..., i$,

$$\sum_{k=l}^{i}(-1)^{k-l}\binom{k}{l}a'_k = \frac{(-1)^{l+1}}{i!}p^{m-i}\binom{i}{l}(p-1)(2p-1)\cdots\widehat{(lp-1)}\cdots(ip-1)$$

where $\widehat{\ }$ means leave out. Therefore (2.1) becomes

$$\left(\frac{p^{m-i}}{i!}(p-1)(2p-1)\cdots(ip-1)\right)\sum_{l=1}^{i}(-1)^{l+1}\binom{i}{l}\binom{lp}{i}\frac{1}{lp-1}.$$

So it suffices to show that

$$\sum_{l=1}^{i}(-1)^{l+1}\binom{i}{l}\binom{lp}{i}\frac{1}{lp-1} = 0.$$

Notice that

$$\sum_{l=1}^{i}(-1)^{l+1}\binom{i}{l}\binom{lp}{i}\frac{1}{lp-1} = \frac{p}{(i-1)!}\sum_{l=1}^{i}(-1)^{l+1}\binom{i-1}{l-1}(lp-2)\cdots(lp-i+1).$$

Let $f(t) = \sum_{l=1}^{i}(-1)^{l-1}\binom{i-1}{l-1}(lp-2)\cdots(lp-i+1)t^{lp-i}$. Then

$$f(t) = \sum_{l=1}^{i}(-1)^{l-1}\binom{i-1}{l-1}\left(\frac{d}{dt}\right)^{i-2}t^{lp-2} = \left(\frac{d}{dt}\right)^{i-2}t^{p-2}\sum_{l=1}^{i}(-1)^{l-1}\binom{i-1}{l-1}t^{(l-1)p}.$$

Hence $f(t) = (\frac{d}{dt})^{i-2}t^{p-2}(1-t^p)^{i-1}$. Thus $f(1) = 0$ since all terms will be divisible by $(1-t^p)$. Therefore $\sum_{j=1}^{n-1} c_{i,j}a'_j = 0$ for $i \geq 3$.

Note that $\ker[\psi^p] = 0$ over $\mathbb{Q}$. Thus $\langle a'_1, a'_2, a'_3, ..., a'_{n-1}\rangle$ is the unique vector in $\mathbb{Q}^{n-1}$ that is taken to $\langle p^m, 0, 0, ..., 0\rangle$ by the transformation $[\psi^p]$.

Now notice that the a'_k, $1 \leq k \leq n-1$, are integral, hence also elements of $\widehat{\mathbb{Z}}_p$, only when $m - k \geq \nu_p(k!)$, i.e., $m \geq n - 1 + \nu_p((n-1)!) = b$.

Let $a_1 = p^{b-1}$ and, for $k > 1$,

$$a_k = \frac{(-1)^{k+1}}{k!} p^{b-k}(p-1)(2p-1)(3p-1)\cdots((k-1)p-1).$$

Then, since $\ker[\psi^p] = 0$ over $\widehat{\mathbb{Z}}_p$, $\langle a_1, a_2, a_3, ..., a_{n-1}\rangle = \sum_{k=1}^{n-1} a_k x^k$ is the unique element of M_n such that $\psi^p(\sum_{k=1}^{n-1} a_k x^k) = p^b x$.

To see that there does not exist $w \in M_n$ such that $\psi^p(w) = p^{b-\epsilon}x$, $\epsilon \in \mathbb{Z}^+$, consider the following. Suppose such a $w = \sum_{k=1}^{n-1} q_k x^k$ existed. Then at least one of the q_k has to be in $\widehat{\mathbb{Z}}_p - \mathbb{Z}$. But then $\psi^p(p^\epsilon w) = p^\epsilon \psi^p(w) = p^b x$. Since $p^\epsilon q_k = a_k$ by uniqueness, we get the contradiction $p^\epsilon q_k \in \widehat{\mathbb{Z}}_p - \mathbb{Z}$ and $p^\epsilon q_k \in \mathbb{Z}$. □

The next lemma will finish the proof of Lemma 1.1.

Lemma 2.2. *For $2 \leq i \leq n-1$, let e_i denote the p-exponent of x^i in M_n/ψ^p. Then $e_i \leq b$.*

Proof. First note that the relations of M_n/ψ^p are given by the following equations:

$$\begin{aligned}
\alpha_{1,1}x + \alpha_{1,2}x^2 + \alpha_{1,3}x^3 + \cdots + \alpha_{1,n-2}x^{n-2} + \alpha_{1,n-1}x^{n-1} &= 0,\\
\alpha_{2,2}x^2 + \alpha_{2,3}x^3 + \cdots + \alpha_{2,n-2}x^{n-2} + \alpha_{2,n-1}x^{n-1} &= 0,\\
\alpha_{3,3}x^3 + \cdots + \alpha_{3,n-2}x^{n-2} + \alpha_{3,n-1}x^{n-1} &= 0,\\
&\vdots\\
\alpha_{n-2,n-2}x^{n-2} + \alpha_{n-2,n-1}x^{n-1} &= 0,\\
\alpha_{n-1,n-1}x^{n-1} &= 0
\end{aligned}$$

where $\alpha_{i,j} = \sum_{k=0}^{i-1}(-1)^k\binom{i}{i-k}\binom{(i-k)p}{j}$ (these relations can be obtained from the transpose of the matrix $[\psi^p]$). Notice that $\alpha_{i,i} = \binom{p}{1}^i$ and $\alpha_{i,i+1} = \binom{i}{1}\binom{p}{2}\binom{p}{1}^{i-1}$.

Since $\alpha_{n-1,n-1} = p^{n-1}$ we know that the p-exponent of x^{n-1} is $n-1$. Via back-substitution, we are then able to find the p-exponent of x^{n-2}, x^{n-3}, and so on, all the way up to x. This line of thinking leads us to the formula

$$e_i = d_{i,i} + \max\{e_j - d_{i,j} : j = i+1, ..., n-1\},$$

where $d_{i,j} = \nu_p(\alpha_{i,j})$. (Note: $d_{i,i} = i$.) It follows that $e_i \geq e_{i+1} + (i - d_{i,i+1})$.

Next we see that $i - d_{i,i+1} = -\nu_p(i)$, since $d_{i,i+1} = \nu_p(\alpha_{i,i+1}) = \nu_p(\binom{i}{1}\binom{p}{2}\binom{p}{1}^{i-1})$ $= \nu_p(\frac{p-1}{2}ip^i)$. Thus $e_i \geq e_{i+1} - \nu_p(i)$. Hence if we can show that $e_{ip} \leq b - \nu_p(ip)$ for all i such that $1 \leq i \leq q$, where $qp \leq n-1 < (q+1)p$, we will be done.

By Lemma 2.1 and the relation

$$\alpha_{1,1}x + \alpha_{1,2}x^2 + \cdots + \alpha_{1,n-1}x^{n-1} = \sum_{i=1}^{n-1}\binom{p}{i}x^i = 0$$

we have $e_p \leq b-1$. Now choose the smallest k such that $e_{kp} > b - \nu_p(kp)$. Then the relation

$$\alpha_{k,k}x^k + \alpha_{k,k+1}x^{k+1} + \cdots + \alpha_{k,kp-1}x^{kp-1} + \alpha_{k,kp}x^{kp} = p^k x^k + \cdots + x^{kp} = 0$$

implies that $e_k \geq k + b - \nu_p(kp) + 1 = (b+1) + (k - \nu_p(kp))$.

Since $k - \nu_p(kp) \geq 0$ for all k and p, we choose i so that $(i+1)p > k \geq ip$ and get the contradiction

$$b + (1 + k - \nu_p(kp)) \leq e_k \leq e_{k-1} \leq \cdots \leq e_{ip} \leq b.$$

Therefore it must be the case that $e_{ip} \leq b - \nu_p(ip)$ for all $1 \leq i \leq q$. □

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References

[1] A.K. Bousfield: Uniqueness of infinite deloopings for K-theoretic spaces, *Pacific J. Math.* **129** (1987), 1–31. MR **89g:**55017
[2] A.K. Bousfield: Unstable localization and periodicity, In *Algebraic Topology: New Trends in Localization and Periodicity*, Birkhauser, Basel (1996), 33–50. MR **98c:**55014
[3] A.K. Bousfield: The K-theory localizations and v_1-periodic homotopy groups of H-spaces, *Topology* **38** (1999), 1239–1264. MR **2000d:**55022
[4] D.M. Davis: Odd primary *bo*-resolutions and K-theory localization, *Illinois J. Math.* **30** (1986), 79–100. MR **87g:**55026
[5] D.M. Davis and M. Mahowald: v_1-localizations of finite torsion spectra and spherically resolved spaces, *Topology* **32** (1993), 543–550. MR **94h:**55018
[6] L. Hodgkin: On the K-theory of Lie groups, *Topology* **6** (1967), 1–35. MR **35:**4950
[7] F.S. Roberts: *Applied Combinatorics*, Prentice Hall (1984), 252–280. MR **85h:**05001

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