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THE NUCLEUS FOR RESTRICTED LIE ALGEBRAS

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Abstract. The nucleus was a concept first developed in the cohomology theory for finite groups. In this paper the authors investigate the nucleus for restricted Lie algebras. The nucleus is explicitly described for several important classes of Lie algebras.

1. Introduction

1.1. The concept of the nucleus was first introduced in [BCR] in the context of the representation theory and cohomology of a finite group G over a field k of characteristic p dividing $|G|$. The authors were attempting to prove an old conjecture, which states that if S is a simple kG-module in the principal block, then $H^\bullet(G,S) \neq 0$. This conjecture remains open, but the nucleus has become an important part of the machinery for understanding the relationship between cohomology and representation theory. For example, it is an essential ingredient in the investigation of the thick subcategories of the stable module category [BCRi2, C2]. The nucleus also makes an appearance when we try to understand the annihilator of $\mathrm{Ext}^\bullet_{kG}(M,M')$ in terms of the varieties of M and M' [C1].

There are two different versions of the nucleus in the representation theory of finite groups. When the word "nucleus" is used without qualification, it refers to the group theoretical nucleus Y_G. The other version is called the representation theoretical nucleus θ_G. They are both defined as subsets of the cohomology variety V_G, and one of the main theorems of the subject [Ben2] states that these subsets are equal: $Y_G = \theta_G$. This theorem indicates further how the group structure is related directly to the representation theory and cohomology. It is easy to see from the definition that the subset Y_G is a closed homogeneous subvariety of V_G. The same is not at all true of θ_G, but it may be seen as a consequence of the above theorem.

To be more explicit, the cohomology variety V_G is defined to be the maximal ideal spectrum of the cohomology ring $H^\bullet(G,k)$ if $p=2$, and of the even degree part of the cohomology ring if $p>2$. The structure of this variety was completely described by Quillen [Qu] in terms of the set of elementary abelian p-subgroups E of G and the conjugations and inclusions between them. If M is a finitely generated

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kG-module, then $V_G(M)$ is defined to be the closed homogeneous subvariety of V_G determined by the kernel of the map

$$H^\bullet(G,k) = \mathrm{Ext}^\bullet_{kG}(k,k) \to \mathrm{Ext}^\bullet_{kG}(M,M)$$

given by tensoring exact sequences over k with M.

The nucleus Y_G is defined to be the union of the images of the maps $\mathrm{res}^*_{G,E}\colon V_E \to V_G$, where E runs over the set of elementary abelian p-subgroups of G with the property that the centralizer $C_G(E)$ is not p-nilpotent. The representation theoretic nucleus θ_G is defined to be the union of the subvarieties $V_G(M)$ as M runs over the finitely generated modules in the principal block such that $H^\bullet(G,M) = 0$ for $\bullet > 0$. A description which generalizes to other blocks is that θ_G is equal to the union of the subvarieties $V_G(M) \cap V_G(N)$ as M and N run over the finitely generated modules in the principal block such that $\mathrm{Ext}^\bullet_{kG}(M,N) = 0$ for $\bullet > 0$ (see [Ben3]).

1.2. Our main goal in this paper is to investigate the corresponding definitions for the p-restricted Lie algebras. In this case, the cohomology variety $V_{\mathfrak{g}}$ of $\mathfrak{g}$ was described by Suslin, Friedlander and Bendel [SFB1]. It is homeomorphic to the subvariety of the vector space $\mathfrak{g}$ given by the kernel of the p-restriction map $x \mapsto x^{[p]}$. If G is a semisimple algebraic group and $\mathfrak{g} = \mathrm{Lie}\ \mathrm{G}$ with $p > h$, the Coxeter number, then $V_{\mathfrak{g}}$ is the same as the variety $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$.

We begin by computing the representation theoretic nucleus in the case of a p-nilpotent Lie algebra, where the answer is $\{0\}$. This is analogous to the case of a finite p-group, where the trivial module is the only simple module. For a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, the nucleus is the entire cohomology variety. This is somewhat surprising because for finite groups, in odd characteristic the nucleus is always a proper subvariety of the cohomology variety [CR]. This example provides strong indications that there are significant qualitative differences between the group case and the Lie algebra case. Furthermore, in this case there is only one block, and the one-dimensional simple module which induces to the Steinberg module is an example of a simple module with no cohomology. So the original motivating question for finite groups has a negative answer in the case of p-restricted Lie algebras. In [CNP, Thm 4.3], it was shown that there exists non-projective indecomposable modules in the principal block with no cohomology as long as $\mathfrak{g}$ does not have underlying root system A_1. Our main theorem, which we prove in Section 3.5, extends this result. We provide an identification of the representation theoretic nucleus for the restricted Lie algebra of a semisimple simply connected algebraic group, under the hypothesis that $p > h+1$.

(1.2.1) Theorem. *Let G be a semisimple algebraic group over a field of characteristic $p > h+1$, with p-restricted Lie algebra $\mathfrak{g}$. Then the representation theoretic nucleus $\theta_{\mathfrak{g}}$ is equal to the subregular orbit.*

The subregular orbit is the (unique) largest closed orbit of G on $\mathcal{N}$ which is not equal to the whole of $\mathcal{N}$. This theorem provides evidence that for an arbitrary restricted Lie algebra $\mathfrak{g}$ there may be a closed subvariety $Y_{\mathfrak{g}}$ related to the structure of the Lie algebra which equals $\theta_{\mathfrak{g}}$. The question of the correct definition for $Y_{\mathfrak{g}}$ in general for p-restricted Lie algebras $\mathfrak{g}$ remains open. The proof of the above theorem uses in an essential way the recent calculation of the varieties of Weyl modules given in [NPV] as well as the prominent translation functors due to Jantzen.

(1.2.2) *Remark.* The paper was initiated at a chance meeting in 1998 at the Plan 9 Music Store in Charlottesville, Virginia.

We end with some questions suggested by our investigations of the representation theoretic nucleus.

2. Preliminaries

2.1. **Notation.** The conventions throughout this paper will closely follow those in [Jan]. Let G be a semisimple simply connected algebraic group defined and split over the finite field $\mathbb{F}_p$ with p elements. Moreover, let k be the algebraic closure of $\mathbb{F}_p$. We will often consider G as an algebraic group scheme over k. Let T be a maximal split torus and Φ be a root system associated to G for a Euclidean space $\mathbb{E}$. The inner product on $\mathbb{E}$ will be denoted by $\langle\ ,\ \rangle$ and the positive (resp. negative) roots will be denoted by Φ^+ (resp. Φ^-). We shall fix a base of simple roots Δ. Let $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ be the coroot corresponding to $\alpha\in\Phi$ and α_0 be the highest short root. Set ρ equal to the half sum of positive roots. For any $\alpha\in\Phi$, set $d_\alpha = \langle\alpha,\alpha\rangle/\langle\alpha_0,\alpha_0\rangle$. The Coxeter number associated to Φ is $h=\langle\rho,\alpha_0^\vee\rangle+1$. Let W be the Weyl group corresponding to Φ and W_p be the affine Weyl group. Let B be a Borel subgroup containing T corresponding to the positive roots and U be the unipotent radical of B.

For an affine group scheme H over k, let $F: H\to H^{(1)}$ be the Frobenius morphism and F^r be the composition of the Frobenius morphism r times with itself. Set $H_r=\ker F^r$. For $r=1$, it is well known that there exists a categorical equivalence between H_1-modules and restricted $\mathrm{Lie}(H)$-modules (or equivalently modules for the restricted enveloping algebra $u(\mathrm{Lie}(H))$). This fact will be used extensively throughout this paper. In particular, we will be interested in the cases when $H=G$, B, and U. For T a maximal split torus in G, set $G_rT=(F^r)^{-1}(T)$ where $F^r: G\to G$.

Let $X(T)$ be the integral weight lattice obtained from Φ. The set $X(T)$ has a partial ordering defined as follows: $\lambda\geq\mu$ if and only if $\lambda-\mu\in\sum_{\alpha\in\Delta}\mathbb{N}\alpha$. The set of dominant integral weights is defined by

$$X(T)_+ = \{\lambda\in X(T):\ 0\leq\langle\lambda,\alpha^\vee\rangle \text{ for all } \alpha\in\Delta\}.$$

For $\lambda\in X(T)_+$, let $H^0(\lambda)=\mathrm{ind}_B^G\lambda$ and $L(\lambda)=\mathrm{soc}_G H^0(\lambda)$. This provides a one-to-one correspondence between finite-dimensional simple G-modules (labelled by $L(\lambda)$) with $X(T)_+$. Furthermore, the set of restricted weights is

$$X_r(T)=\{\lambda\in X(T):\ 0\leq\langle\lambda,\alpha^\vee\rangle<p^r \text{ for all } \alpha\in\Delta\}.$$

For $\lambda\in X_r(T)$ let $L_r(\lambda)=L(\lambda)$. Furthermore, if $\lambda\in X(T)$ we can express $\lambda=\lambda_0+p^r\lambda'$ where $\lambda_0\in X_r(T)$ and $\lambda'\in X(T)$. Set $\widehat{L}_r(\lambda)=L(\lambda_0)\otimes p^r\lambda'$. A complete set of representatives for the isomorphism classes of simple G_r-modules (resp. G_rT-modules) is given by $\{L_r(\lambda)\colon \lambda\in X_r(T)\}$ (resp. $\{\widehat{L}_r(\lambda)\colon \lambda\in X(T)\}$). The Weyl modules are given by $V(\lambda)=H^0(-w_0\lambda)^*$ for $\lambda\in X(T)_+$. Here w_0 is the long element in W and * represents taking the dual module.

2.2. **Support varieties.** Let A be a finite k-group and let

$$H(A,k)=\begin{cases} H^{2\bullet}(A,k) & \text{if char } k\neq 2,\\ H^{\bullet}(A,k) & \text{if char } k=2.\end{cases}$$

According to [FS], the cohomology ring $R = H(A,k)$ is a commutative, finitely generated k-algebra. For finite-dimensional $M, M' \in A$-mod, define the *relative support variety* $V_A(M,M')$ as follows. Yoneda composition defines an action of R on $\text{Ext}^\bullet_A(M,M')$. Let $J = J_A(M,M')$ be the annihilator ideal in $H(A,k)$ for this action. Set $V_K(M,M')$ equal to the maximum ideal spectrum of R/J. The *support variety* $V_A(M)$ is defined by setting $V_A(M) = V_A(M,M)$.

Let A_0 be the block of A containing the trivial module. Set

$$\begin{aligned}\mathcal{S} &= \{M \in \text{mod}(A_0) :\ H^\bullet(A,M) = 0, \quad \bullet > 0\}\\ &= \{M \in \text{mod}(A_0) :\ V_A(k,M) = \{0\}\}.\end{aligned}$$

The preceding equality follows by the self-injectivity of A.

The *representation theoretic nucleus* of A, θ_A, is defined as follows:

$$\theta_A = \bigcup_{M\in\mathcal{S}} V_A(M).$$

From the definition, it is not clear that θ_A is a closed set (i.e., a variety). For finite groups, the representation theoretic nucleus is a variety because one can identify this set with the union Y_G of the images of restriction maps from elementary abelian p-subgroups whose centralizers are not p-nilpotent [Ben2, Cor. 1.3]. In our computations involving infinitesimal Frobenius kernels we will show that the representation theoretic nucleus in these cases is also a variety.

Let H be an algebraic group scheme over k and $\mathfrak{h} = \text{Lie } H$. The Lie algebra $\mathfrak{h}$ is a restricted Lie algebra with p-mapping $x \to x^{[p]}$. According to [SFB1, (1.6), (5.11)], $V_{H_1}(k)$ is homeomorphic to $\mathcal{N}(\mathfrak{h}) := \{x \in \mathfrak{h} :\ x^{[p]} = 0\}$. Furthermore, under this identification, if M is an H_1-module, then $V_{H_1}(M)$ is homeomorphic to

$$\{x \in \mathfrak{h} :\ x^{[p]} = 0, \quad M_{\langle x\rangle} \text{ is not free}\}^{(1)} \cup \{0\}.$$

We remark that if M is an H-module, then $V_{H_1}(M)$ is an H-stable variety of $\mathfrak{h}$ under conjugation.

2.3. **Orbit theory.** Let G be a semisimple algebraic group with $\mathfrak{g} = \text{Lie } G$. Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g}$. For $p > h$, $\mathcal{N} = \mathcal{N}(\mathfrak{g})$. The group G acts on $\mathcal{N}$ by conjugation and $\mathcal{N}(\mathfrak{g})$ is a G-stable subvariety of $\mathcal{N}$.

The orbit theory of G on $\mathcal{N}$ has been well-studied. We refer the reader to [CM], [Hum] for details. The variety $\mathcal{N}$ has finitely many G-orbits. These orbits have been classified. For each $\alpha \in \Phi$, let x_α be a root vector. Set $x_{reg} = \sum_{\alpha\in\Delta} x_\alpha$. Then

$$\mathcal{N} = G \cdot \mathfrak{u} = \overline{G \cdot x_{reg}}$$

is called the *regular orbit*.

Let $I \subseteq \Delta$. Then $\mathfrak{g} = \mathfrak{u}_I^+ \oplus \mathfrak{l}_I \oplus \mathfrak{u}_I$ has a Levi decomposition. The set $G\cdot\mathfrak{u}_I$ can be expressed as the closure of an orbit. Orbits of this form are called *Richardson orbits*. We should remark that Levi factors which are conjugate under the action of G give rise to the same Richardson orbit. In a specific case one can take $J = \{\alpha\} \subseteq \Delta$. Then $\mathfrak{l}_J$ is of type A_1. The Richardson orbit in this special case is called the *subregular orbit*. The subregular orbit is the largest proper orbit (i.e. every proper orbit is contained in the subregular orbit). For $I \subseteq \Delta$, let Φ_I be the root system in Φ generated by I. Moreover, for $\lambda \in X(T)$, let

$$\Phi_{\lambda,p} = \{\alpha \in \Phi :\ d_\alpha\langle \lambda + \rho, \alpha^\vee\rangle \in p\mathbb{Z}\}.$$

The following result [NPV, (6.2.1) Thm.] due to Parshall, Vella and the second author involving Richardson orbits will be essential in our computations of the representation theoretic nucleus.

(2.3.1) Theorem. *For $\lambda \in X(T)_+$ and p good, choose $I \subseteq \Delta$ so that $w(\Phi_{\lambda,p}) = \Phi_I$ for some $w \in W$. Then $V_{G_1}(H^0(\lambda)) = G \cdot \mathfrak{u}_I$.*

2.4. **Translation functors.** Let $\overline{C}_{\mathbb{Z}} = \{\lambda \in X(T) : \ 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p \ \text{ for all } \alpha \in \Phi^+\}$. Moreover, let M be a G-module and let $\text{pr}_\sigma(M)$ be the sum of all submodules of M with G-composition factors with high weight in $W_p \cdot \sigma$. If Z is a set of W_p-orbits in $X(T)$ (under the dot action), then

$$M \cong \bigoplus_{\mu \in Z} \text{pr}_\mu(M). \tag{2.4.1}$$

We will use the translation functors [Jan, II 7.6] first constructed by Jantzen. If $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$, let $\nu_1 \in W(\mu - \lambda) \cap X(T)_+$. For any G-module M, let

$$T_\lambda^\mu(M) = \text{pr}_\mu(\text{pr}_\lambda(M) \otimes H^0(\nu_1)).$$

The functor T_λ^μ is exact. Moreover, T_λ^μ and T_μ^λ are adjoint to each other. The functor T_λ^μ commutes with the forgetful functor from G-modules to G_rT-modules. Consequently, T_λ^μ and T_μ^λ are adjoint in the category of G_rT-modules. Many of the results for G-modules (in [Jan, II Ch. 7]) have analogous statements in the category of G_rT-modules (see [Jan, II 9.19]). We will rely on these results in our computations. Finally observe that since $V_{G_1}(M_1 \otimes M_2) = V_{G_1}(M_1) \cap V_{G_1}(M_2)$ and $V_{G_1}(M_1 \oplus M_2) = V_{G_1}(M_1) \cup V_{G_1}(M_2)$, we have

$$V_{G_1}(T_\lambda^\mu(M)) \subseteq V_{G_1}(M) \cap V_{G_1}(H^0(\nu_1)) \subseteq V_{G_1}(M).$$

3. Computation of the representation theoretic nucleus

3.1. Carlson and Robinson [CR, Thm. 4.1] have demonstrated that for group algebras of a finite group, the nucleus must be a proper subvariety of the support variety of the trivial module if the underlying field does not have characteristic two. For fields of characteristic two there are examples where the nucleus is not proper [CR, Prop. 5.2]. In our first result we show that for the Borel subalgebra of a classical Lie algebra the representation theoretic nucleus need not be proper. This result is independent of the characteristic of the field.

(3.1.1) Proposition. *Let G be a semisimple algebraic group. Then*

(a) $\theta_{U_1} = \{0\}$;
(b) $\theta_{B_1} = \mathcal{N}(\mathfrak{u})$.

In particular if $p > h$, then $\theta_{B_1} = \mathfrak{u}$.

Proof. (a) Since $\mathfrak{u}$ is a p-nilpotent Lie algebra, the only simple U_1-module is the trivial module k. If $H^\bullet(U_1, M) = 0$ for $\bullet > 0$, then M is a projective U_1-module; thus $\theta_{U_1} = \{0\}$.

(b) Let $(p-1)\rho \in X(T)_+$ be the dominant weight corresponding to the Steinberg module. We have

$$L((p-1)\rho) = \text{ind}_{B_1}^{G_1}(p-1)\rho.$$

Since $L((p-1)\rho)$ is a projective G_1-module, it follows that

$$H^\bullet(B_1, (p-1)\rho) = H^\bullet(G_1, L((p-1)\rho)) = 0$$

for $\bullet > 0$. The group scheme B_1 has one block. Moreover, $(p-1)\rho$ is a one-dimensional B_1-module so $V_{B_1}((p-1)\rho) = \mathcal{N}(\mathfrak{b}) = \mathcal{N}(\mathfrak{u})$. Consequently, $\theta_{B_1} = \mathcal{N}(\mathfrak{u})$. $\square$

3.2. Let $\uparrow$ be the Strong Linkage relation on $X(T)$ as defined in [Jan, II 6.4]: $\lambda \uparrow \mu$ if and only if there exists $\mu_1, \mu_2, \dots, \mu_t \in X(T)$ and reflections $s_1, s_2, \dots, s_{t+1} \in W_p$ such that

$$\lambda \leq s_1 \cdot \lambda = \mu_1 \leq s_2 \cdot \mu_1 = \mu_2 \leq \cdots \leq s_t \cdot \mu_{t-1} = \mu_t \leq s_{t+1} \cdot \mu_t = \mu.$$

Here $\cdot$ denotes the "dot action" of the affine Weyl group.

Let $\mathcal{B}_\lambda(G_1)$ (resp. $\mathcal{B}_\lambda(G_1T)$) be the block for G_1 (resp. G_1T) that contains $L_1(\lambda)$ (resp. $\widehat{L}_1(\lambda)$). For brevity we will write $\mu \in \mathcal{B}_\lambda(G_1)$ to indicate that the simple G_1-module $L_1(\mu)$ is in the block $\mathcal{B}_\lambda(G_1)$. A similar statement can be made for $\lambda \in \mathcal{B}_0(G_1T)$. The principal block for G_1 (resp. G_1T) is $\mathcal{B}_0(G_1)$ (resp. $\mathcal{B}_0(G_1T)$). From [Jan, II 9.19 (1)], $\lambda \in \mathcal{B}_0(G_1T)$ if and only if $\lambda \in W \cdot 0 + pX(T)$. Moreover, $\lambda \in \mathcal{B}_0(G_1)$ $(\lambda \in X_1(T))$ if and only if $\lambda \equiv W \cdot 0 \pmod{pX(T)}$.

(3.2.1) Proposition. *If $\lambda \in \mathcal{B}_0(G_1)$ $(\lambda \in X_1(T))$, then there exists $\nu \in X(T)$ such that*

(i) $0 \uparrow \lambda + p\nu$;
(ii) $\lambda + p\nu \in X(T)_+$ *with* $\nu \in X(T)_+$.

Proof. Since $\lambda \in \mathcal{B}_0(G_1)$, there exists $w \in W$, $\zeta \in X(T)$ such that $w \cdot \lambda + p\zeta = 0$, so $w \cdot \lambda = -p\zeta$. From [Jan, II 6.4(5)], we have $w \cdot \lambda \uparrow \lambda$ for all $w \in W$. This implies that $-p\zeta \uparrow \lambda$ or $0 \uparrow \lambda + p\zeta$. Let β be the highest (long) root. By applying [Jan, II 6.4(3)(4)], $\lambda + p\zeta \uparrow \lambda + p\zeta + ps\beta$ for $s \in \mathbb{Z}^+$. Therefore, $0 \uparrow \lambda + p\zeta + ps\beta$ for $s \in \mathbb{Z}^+$. For s sufficiently large $\zeta + s\beta \in X(T)_+$ with $\lambda + p(\zeta + s\beta) \in X(T)_+$ because $\langle \beta, \alpha^\vee \rangle > 0$ for all $\alpha \in \Delta$. $\square$

3.3. The following theorem shows that for classical Lie algebras for $p \geq h$, the representation theoretic nucleus is contained in the subregular nilpotent orbit, and is thus a proper subvariety of the support variety of the trivial module.

(3.3.1) Theorem. *Let G be a semisimple algebraic group with $p \geq h$. Then $\theta_{G_1} \subseteq G \cdot \mathfrak{u}_J$ where $J = \{\alpha\} \subseteq \Delta$.*

Proof. Let $\mathcal{D} = \{\sigma \in X(T)_+ :\ 0 \uparrow \sigma\}$ and $\mathcal{S} = \{M \in \text{mod}(\mathcal{B}_0(G_1)) :\ H^\bullet(G_1, M) = 0,\ \bullet > 0\}$. We will first show that $V_{G_1}(H^0(\sigma), M) \subseteq G \cdot \mathfrak{u}_J$ for all $\sigma \in \mathcal{D}$ and $M \in \mathcal{S}$. This will be accomplished by using induction on the ordering in $\mathcal{D}$. First observe that $\text{Ext}^\bullet_{G_1}(H^0(0), M) = \text{Ext}^\bullet_{G_1}(k, M) = 0$ for $\bullet > 0$; thus $V_{G_1}(H^0(0), M) = \{0\} \subseteq G \cdot \mathfrak{u}_J$.

Now let $\sigma \in \mathcal{D}$, $\sigma \neq 0$ and assume that for all $\eta \in \mathcal{D}$ with $0 \uparrow \eta$ and $\eta < \sigma$, $V_{G_1}(H^0(\eta), M) \subseteq G \cdot \mathfrak{u}_J$. Since $\sigma \neq 0$ and $0 \uparrow \sigma$, there exists a reflection s in W_p such that $s \cdot \sigma < \sigma$ and $0 \uparrow s \cdot \sigma$. Let μ' be such that $\text{Stab}_{W_p} \mu' = \{1, s\}$. There exists a $w \in W_p$ such that for some $\mu \in \overline{C}_{\mathbb{Z}}$, $w \cdot 0 = \sigma$ and $w \cdot \mu = \mu'$. According to [Jan, II 7.13 Prop.], the module $T^\lambda_\mu(H^0(\mu')) = T^\lambda_\mu(H^0(w \cdot \mu))$ has composition factors with multiplicity at most one of the form $H^0(\sigma)$ and $H^0(s \cdot \sigma)$.

Without loss of generality we may assume that there exists an exact sequence of the form

$$0 \to H^0(s \cdot \sigma) \to T^\lambda_\mu(H^0(\mu')) \to H^0(\sigma) \to 0.$$

We remark that the order of the composition factors may be reversed, but the argument is the same in either case. Also, note that it is possible to have $s \cdot \sigma \notin X(T)_+$ in which case $H^0(s\cdot\sigma) = 0$. The short exact sequence above induces a long exact sequence in cohomology:

$$\cdots \to \operatorname{Ext}^i_{G_1}(H^0(\sigma), M) \to \operatorname{Ext}^i_{G_1}(T^\lambda_\mu(H^0(\mu')), M) \to \operatorname{Ext}^i_{G_1}(H^0(s\cdot\sigma)), M) \to \cdots.$$

This implies that

$$\begin{aligned} V_{G_1}(H^0(\sigma), M) &\subseteq V_{G_1}(H^0(s\cdot\sigma), M) \cup V_{G_1}(T^\lambda_\mu(H^0(\mu')), M) \\ &\subseteq G\cdot \mathfrak{u}_J \cup V_{G_1}(T^\lambda_\mu(H^0(\mu')), M). \end{aligned}$$

By using the definition of the translation functor and Theorem 2.3.1 with $\Phi_{\mu',p}$ of type A_1, we have

$$\begin{aligned} V_{G_1}(T^\lambda_\mu(H^0(\mu')), M) &\subseteq V_{G_1}(T^\lambda_\mu(H^0(\mu'))) \\ &\subseteq V_{G_1}(H^0(\mu')) \\ &\subseteq G\cdot \mathfrak{u}_J. \end{aligned}$$

Hence, $V_{G_1}(H^0(\sigma), M) \subseteq G\cdot\mathfrak{u}_J$.

Next we will show that $V_{G_1}(L(\lambda), M) \subseteq G\cdot\mathfrak{u}_J$ for all $\lambda\in\mathcal{D}$ and $M\in\mathcal{S}$. Again we will use induction on the ordering of weights in $\mathcal{D}$. For $M\in\mathcal{S}$, one has

$$V_{G_1}(k, M) = V_{G_1}(L(0), M) = \{0\} \subseteq G\cdot\mathfrak{u}_J.$$

If $\lambda\in\mathcal{D}$, there exists a short exact sequence of G-modules of the form

$$0 \to L(w\cdot 0) \to H^0(w\cdot 0) \to N \to 0$$

with all composition factors in N of the form $L(\sigma)$ with $\sigma\uparrow\lambda$ and $\sigma<\lambda$. By the same argument as in the preceding paragraph (using the long exact sequence), we have

$$\begin{aligned} V_{G_1}(L(w\cdot 0), M) &\subseteq V_{G_1}(H^0(w\cdot 0), M) \cup V_{G_1}(N, M) \\ &\subseteq G\cdot\mathfrak{u}_J \cup G\cdot\mathfrak{u}_J \\ &\subseteq G\cdot\mathfrak{u}_J. \end{aligned}$$

The second-to-last line follows by using the induction hypothesis and the assertion proved in the preceding paragraph.

Finally, let $\lambda\in X_1(T)$ and $L_1(\lambda)$ be a simple G_1-module in $\mathcal{B}_0(G_1)$. We claim that $V_{G_1}(L_1(\lambda), M) \subseteq G\cdot\mathfrak{u}_J$ for all $M\in\mathcal{S}$. Since $\lambda\in\mathcal{B}_0(G_1)$, by Proposition 3.2.1, there exists $\nu\in X(T)_+$ such that $0\uparrow\lambda+p\nu$ and $\lambda+p\nu\in X(T)_+$. Therefore,

$$L(\lambda+p\nu) \cong L(\lambda)\otimes L(\nu)^{(1)} \cong L_1(\lambda)\otimes L(\nu)^{(1)}.$$

This implies that $\operatorname{Ext}^\bullet_{G_1}(L(\lambda+p\nu), M) \cong \operatorname{Ext}^\bullet_{G_1}(L_1(\lambda), M)\otimes L(\nu)^{(1)}$. Therefore, from the claim in the preceding paragraph,

$$V_{G_1}(L_1(\lambda), M) = V_{G_1}(L(\lambda+p\nu), M) \subseteq G\cdot\mathfrak{u}_J$$

for all $M\in\mathcal{S}$. Now suppose that $M\in\mathcal{S}$. Then

$$V_{G_1}(M) = V_G(\bigoplus_{\lambda\in\mathcal{B}_0(G_1)} L_1(\lambda), M) \subseteq G\cdot\mathfrak{u}_J.$$

Consequently, $\theta_{G_1}\subseteq G\cdot\mathfrak{u}_J$. □

3.4. In this section we will investigate the properties of support varieties of induced modules relative to the translation functor.

(3.4.1) Proposition. *Let G be a semisimple algebraic group with p good. Let $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$ such that μ belongs to the closure of the facet containing λ. If $w \cdot \mu \in X(T)_+$ where $w \in W_p$, then $V_{G_1}(T_\mu^\lambda(H^0(w\cdot\mu))) = V_{G_1}(H^0(w\cdot\mu))$.*

Proof. Set $\mu' = w \cdot \mu$. Since $T_\mu^\lambda(H^0(\mu'))$ is a G_1T-module, it follows by [Jan, I 6.9 (4)(5)] for $\sigma \in X_1(T)$,

$$\begin{aligned}\operatorname{Ext}^\bullet_{G_1}(L_1(\sigma), T_\mu^\lambda(H^0(\mu'))) &\cong \bigoplus_{\nu\in X(T)} \operatorname{Ext}^\bullet_{G_1T}(\widehat{L}_1(\sigma+p\nu), T_\mu^\lambda(H^0(\mu')))\\ &\cong \bigoplus_{\nu\in X(T)} \operatorname{Ext}^\bullet_{G_1T}(T_\lambda^\mu(\widehat{L}_1(\sigma+p\nu)), H^0(\mu')).\end{aligned}$$

Therefore,

$$\begin{aligned}&\operatorname{Ext}^\bullet_{G_1}(\bigoplus_{\sigma\in\mathcal{B}_\lambda(G_1)} L(\sigma), T_\mu^\lambda(H^0(\mu')))\\ &\qquad\cong \bigoplus_{\sigma\in\mathcal{B}_\lambda(G_1)}\bigoplus_{\nu\in X(T)} \operatorname{Ext}^\bullet_{G_1T}(T_\lambda^\mu(\widehat{L}_1(\sigma+p\nu)), H^0(\mu')).\end{aligned} \tag{3.4.2}$$

Let $\gamma \in X(T)$ be such that $\widehat{L}_1(\gamma) \in \widehat{\mathcal{B}}_\mu(G_1T)$. The proof of [Jan, II 7.15] can be adapted for G_1T-modules to show that there exists $\widehat{L}_1(\eta)$ with $\widehat{L}_1(\eta) \in \widehat{B}_\lambda(G_1T)$ such that

$$T_\lambda^\mu(\widehat{L}_1(\eta)) = \widehat{L}_1(\gamma). \tag{3.4.3}$$

Hence, by using (3.4.2),

$$\begin{aligned}\operatorname{Ext}^\bullet_{G_1}(\bigoplus_{\sigma\in\mathcal{B}_\lambda(G_1)} L(\sigma), T_\mu^\lambda(H^0(\mu'))) &\cong \bigoplus_{\eta\in\mathcal{B}_\lambda(G_1T)} \operatorname{Ext}^\bullet_{G_1T}(T_\lambda^\mu(\widehat{L}_1(\eta)), H^0(\mu'))\\ &\cong \operatorname{Ext}^\bullet_{G_1T}(\bigoplus_{\eta\in\mathcal{B}_\lambda(G_1T)} T_\lambda^\mu(\widehat{L}_1(\eta)), H^0(\mu'))\\ &\supseteq \operatorname{Ext}^\bullet_{G_1T}(\bigoplus_{\gamma\in\mathcal{B}_\mu(G_1T)} \widehat{L}_1(\gamma), H^0(\mu'))\\ &\cong \operatorname{Ext}^\bullet_{G_1}(\bigoplus_{\zeta\in\mathcal{B}_\mu(G_1)} L_1(\zeta), H^0(\mu')).\end{aligned}$$

Therefore, $\dim V_{G_1}(H^0(\mu')) \leq \dim V_{G_1}(T_\mu^\lambda(H^0(\mu')))$ from the inclusion above and [NPV, (2.2.2) Thm.]. In Section 2.4 we have seen that

$$V_{G_1}(T_\mu^\lambda(H^0(\mu'))) \subseteq V_{G_1}(H^0(\mu')).$$

Since $V_{G_1}(H^0(\mu'))$ is irreducible [NPV, (6.3.1) Cor.], it follows that

$$V_{G_1}(T_\mu^\lambda(H^0(\mu'))) = V_{G_1}(H^0(\mu')).$$ □

3.5. An open question in [CNP, (5.1)] was to determine when there exist indecomposable G-modules M in the principal block for G_r (not projective as G_r-modules) such that $H^\bullet(G_r, M) = 0$. An affirmative answer was given for $r \geq 2$ in [BN, (4.1.1) Thm.]. The following theorem provides an affimative answer for $r = 1$ as long as G does not have underlying root system A_1.

(3.5.1) Theorem. *Let G be a semisimple algebraic group with $p > h+1$. There exists a G-module M such that*

(a) *M is in the principal block of G_1;*
(b) *$H^\bullet(G_1, M) = 0$ for $\bullet > 0$;*
(c) *$V_{G_1}(M) = G \cdot \mathfrak{u}_J$ where $J = \{\alpha\} \subseteq \Delta$.*

Proof. Let $\mu' = w \cdot \mu \in X(T)_+$ where $w \in W_p$ and $\mu \in \overline{C}_{\mathbb{Z}}$. Also assume that $\Phi_{\mu,p}$ is of type A_1. Consider $M = T_\mu^0(H^0(\mu'))$. By construction M is in $\mathcal{B}_0(G_1)$. Moreover, by Proposition 3.4.1 (with $\lambda = 0$) and Theorem 2.3.1, $V_{G_1}(M) = G \cdot \mathfrak{u}_J$ where $J = \{\alpha\} \subseteq \Delta$. We need to show that $H^\bullet(G_1, M) = 0$ for $\bullet > 0$. Now

$$\begin{aligned} H^\bullet(G_1, M) &\cong \operatorname{Ext}^\bullet_{G_1}(k, T_\mu^0(H^0(\mu'))) \\ &\subseteq \operatorname{Ext}^\bullet_{G_1}(k, H^0(\mu') \otimes H^0(w(-\mu))) \\ &\cong \operatorname{Ext}^\bullet_{G_1}(H^0(w(-\mu))^*, H^0(\mu')) \\ &\cong \operatorname{Ext}^\bullet_{G_1}(V(w_0 w\mu), H^0(\mu')). \end{aligned}$$

Let $L(\lambda)$ be a G-composition factor of $H^0(\sigma)$. Then $\lambda \uparrow \sigma$ and $\lambda \in W \cdot \sigma + pX(T)$. Decompose $\lambda = \lambda_0 + p\nu$ and $\sigma = \sigma_0 + p\nu'$ where $\lambda_0, \sigma_0 \in X_1(T)$ and $\nu, \nu' \in X(T)_+$. Observe that $L(\lambda) = L_1(\lambda_0) \otimes L(\nu)^{(1)}$. Therefore, if $L_1(\lambda_0)$ is a G_1-composition factor of $H^0(\lambda)$, then $\lambda_0 \in W \cdot \sigma + pX(T)$. We can conclude that if $H^\bullet(G_1, M) = 0$, then $\operatorname{Ext}^\bullet_{G_1}(V(w_0 w\mu), H^0(\mu')) \neq 0$ for some $\bullet \geq 0$ which implies that $w_0 w\mu \in W \cdot \mu + pX(T)$.

We can find a fundamental weight ω_α for some $\alpha \in \Delta$ such that $-2 \leq \langle \omega_\alpha, \gamma^\vee \rangle \leq 2$ for all $\gamma \in \Phi$. Let $\mu = -\sum_{\beta \in \Delta - \{\alpha\}} \omega_\beta$. Then $\mu \in \overline{C}_{\mathbb{Z}}$. Furthermore, $\Phi_{\mu,p}$ is of type A_1.

Now choose $\mu' = w \cdot \mu \in X(T)_+$ where $w \in W_p$ such that $w_0 w \neq \mathrm{id} + p\sigma$ for some $\sigma \in X(T)$. From our analysis above we can actually say that $w_0 w\mu \in W \cdot \mu + pX(T)$ for $w \in W$ with $w_0 w \neq \mathrm{id}$. Express $w_0 w\mu = w_1 \cdot \mu + p\nu_1$ for some $w_1 \in W$ and $\nu_1 \in X(T)$. Let $\gamma \in \Delta$. Then

$$\begin{aligned} |p\langle \nu_1, \gamma^\vee \rangle| &= |\langle w_0 w\mu - w_1 \cdot \mu, \gamma^\vee \rangle| \\ &= |\langle w_0 w\mu - w_1(\mu + \rho) + \rho, \gamma^\vee \rangle| \\ &= |\langle \mu, (w_0 w)^{-1} \gamma^\vee \rangle - \langle \omega_\alpha, w_1^{-1} \gamma^\vee \rangle + 1| \\ &\leq h + 1. \end{aligned}$$

It follows that for $p > h+1$, $\nu_1 = 0$; thus $w_0 w\mu \in W \cdot \mu$. One can now apply [Jan, II (7.7b) Lemma] to conclude that $w_0 w = \mathrm{id}$. This is a contradiction. Consequently, $w_0 w\mu \notin W \cdot \mu + pX(T)$ and $H^\bullet(G_1, T_\mu^0(H^0(\mu'))) = 0$. □

We can now show that the representation theoretic nucleus for G_1 for $p > h+1$ is precisely the subregular orbit.

(3.5.2) Corollary. *Let G be a semisimple algebraic group with $p > h+1$. Then $\theta_{G_1} = G \cdot \mathfrak{u}_J$ where $J = \{\alpha\} \subseteq \Delta$.*

Proof. From Theorem 3.3.1, we have $\theta_{G_1} \subseteq G \cdot \mathfrak{u}_J$. On the other hand, by Theorem 3.5.1, there exists an $M \in \mathcal{S}$ with $V_{G_1}(M) = G \cdot \mathfrak{u}_J$, thus $\theta_{G_1} = G \cdot \mathfrak{u}_J$. □

In view of the equivalence of categories between the G_1-modules and the $u(\mathfrak{g})$-modules, Theorem 1.2.1 follows immediately from the above corollary.

4. Further directions

4.1. In this section, we describe some questions opened up by this work. Let $\mathfrak{g}$ be a p-restricted Lie algebra.

(4.1.1) Question. What is $\theta_{\mathfrak{g}}$ in the case of the p-restricted Lie algebra of a semisimple algebraic group with $p \leq h+1$?

(4.1.2) Question. What is $\theta_{\mathfrak{g}}$ for $\mathfrak{g}$ a restricted Lie algebra of Cartan type?

(4.1.3) Question. What is the correct definition of the nucleus $Y_{\mathfrak{g}}$ in general? It should be defined just in terms of the Lie structure, without mentioning cohomology. There should then be a theorem stating that it is equal to the representation theoretic nucleus $\theta_{\mathfrak{g}}$.

(4.1.4) Question. Is it true that if M and M' are finitely generated modules in the principal block of $u(\mathfrak{g})$, then $V_{\mathfrak{g}}(M) \cap V_{\mathfrak{g}}(M') \subseteq V_{\mathfrak{g}}(M, M') \cup \theta_{\mathfrak{g}}$? This was proved in the finite group case in [C1], but the proof does not obviously translate.

(4.1.5) Question. Describe the thick subcategories of the stable category of finitely generated modules in the principal block of $\mathfrak{g}$. If the situation is analogous to the finite group case [BCRi2], then modulo modules whose variety is contained in the nucleus, one should be able to answer the question in terms of varieties. Each line through the origin in the nucleus should then support an infinite number of thick subcategories.

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