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THE IMPEDANCE TOMOGRAPHY PROBLEM

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ABSTRACT. Using an operator theoretic framework and pseudo-spectral methods, we provide a simple and explicit formula for the conductivity coefficient in terms of the Dirichlet to Neumann map and the eigenvalues of the Laplacian operator.

1. Introduction

We are concerned with the reconstruction of the conductivity coefficient ρ of the impedance tomography problem appearing in

$$\nabla.(\rho\nabla u) = 0, \qquad x \in \Omega \Subset \mathbb{R}^n,\ n \geq 2, \tag{1.1}$$

from the knowledge of its Dirichlet to Neumann map, which we denote by T; that is,

$$u_{|\partial\Omega} \xrightarrow{T} \rho \frac{\partial u}{\partial \nu}_{|\partial\Omega}$$

where u satisfies (1.1) (see [2], [4] and [8]). Operator theory allows us to recast (1.1) in terms of infinite matrices. This simplifies the problem and provides a direct approach to finding explicit algorithms which could easily be implemented in practice (see [7]).

2. Notation

For the sake of simplicity, we first transform (1.1). Assume that ρ is differentiable, to obtain

$$\rho\Delta u + \nabla\rho.\nabla u = 0$$

or

$$\Delta u + \nabla a.\nabla u = 0, \qquad x \in \Omega \Subset \mathbb{R}^n, \tag{2.1}$$

where $a = \ln\rho$ is well defined. As usual for simplicity we assume that $\partial\Omega \in C^1$, $\inf_\Omega \rho(x) > 0$, and

$$\rho(x) = 1 \text{ if } x \in \partial\Omega.$$

Thus the corresponding Dirichlet to Neumann map T for (2.1) transforms into

$$u_{|\partial\Omega} \xrightarrow{T} \frac{\partial u}{\partial \nu}_{|\partial\Omega}$$

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and clearly recovering a is equivalent to finding ρ. For the direct problem (2.1), if $u \in H^1$ and $a \in L^2$, i.e. $\nabla a \in H^{-1}$, we would have $\Delta u + \nabla a.\nabla u \in H^{-1}$ and variational methods could be applied.

Let us denote by φ_n and ψ_n the eigenfunctions of the following Dirichlet and Neumann boundary value problems:

(2.2)
$$[D]\left\{\begin{array}{ll} -\Delta\varphi_n(x)+\varphi_n(x)=\lambda_n\varphi_n(x), & \\ \varphi_n(x)=0, & x\in\partial\Omega, \end{array}\right. \qquad [N]\left\{\begin{array}{ll} -\Delta\psi_n(x)+\psi_n(x)=\mu_n\psi_n(x), & \\ \frac{\partial\psi_n}{\partial\nu}(x)=0, & x\in\partial\Omega. \end{array}\right.$$

It is well known that $\varphi_n,\ \psi_n \in H^1(\Omega)$ and their traces $\frac{\partial\varphi_n}{\partial\nu} \in H^{-1/2}(\partial\Omega)$, $\psi_n \in H^{1/2}(\partial\Omega)$ if $\partial\Omega \in C^1$; see [1] and [5]. The eigenvalues form an increasing sequence and interlace

$$1=\mu_0\le\lambda_0\le\mu_1\le\lambda_1\le ...\le\lambda_n\le\mu_{n+1}\le ....$$

Both operators generated by $-\Delta+1$ and the Dirichlet or the Neumann boundary conditions are selfadjoint, and so the sets of eigenfunctions $\{\varphi_n\}_{n\ge 0}$ and $\{\psi_n\}_{n\ge 0}$ are orthogonal and form complete sets in $L^2(\Omega)$. In all that follows, we assume that both sets are normalized by the condition

$$||\varphi_n|| = ||\psi_n|| = 1 \tag{2.3}$$

and so the sets $\{\varphi_n\}_{n\ge 0}$ and $\{\psi_n\}_{n\ge 0}$ are orthonormal in $L^2(\Omega)$. Recall that when $\partial\Omega \in C^1$, the traces $\left\{\frac{\partial\varphi_n}{\partial\nu}\right\}_{n\ge 0}$ form a complete set of functionals in $H^{-1/2}(\partial\Omega)$ while the traces $\{\psi_n\}_{n\ge 0}$ are complete in $H^{1/2}(\partial\Omega)$. We now form a relation between the Fourier coefficients by simply using Green's identity

$$(\Delta u, v)-(u,\Delta v)=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\overline{v}d\sigma-\int_{\partial\Omega}u\frac{\partial\overline{v}}{\partial\nu}d\sigma \tag{2.4}$$

where $u,\ v\in H^1(\Omega)$.

Let u be a solution to (2.1) and let us agree to denote its traces by

$$u(x)=f(x) \text{ and } \frac{\partial u}{\partial\nu}(x)=g(x) \quad \text{if } x\in\partial\Omega.$$

Assuming that $f \in H^{1/2}(\partial\Omega)$ we deduce that $u \in H^1(\Omega)$ and thus multiplying (2.1) by φ_n and using (2.4) yields

$$(\Delta u,\varphi_n)+(\nabla a.\nabla u,\varphi_n)=0,$$
$$(u,\Delta\varphi_n)+(\nabla a.\nabla u,\varphi_n)=\int_{\partial\Omega}u\frac{\partial\varphi_n}{\partial\nu}d\sigma,$$
$$(1-\lambda_n)(u,\varphi_n)+(\nabla a.\nabla u,\varphi_n)=\int_{\partial\Omega}f\frac{\partial\varphi_n}{\partial\nu}d\sigma. \tag{2.5}$$

However if we used ψ_n instead of φ_n, we would end up with

$$(u,\Delta\psi_n)+(\nabla a.\nabla u,\psi_n) = -\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\psi_n d\sigma,$$
$$(1-\mu_n)(u,\psi_n)+(\nabla a.\nabla u,\psi_n) = -\int_{\partial\Omega}g\psi_n d\sigma. \tag{2.6}$$

Since the solution u belongs to $L^2(\Omega)$, it can be expanded in terms of both φ_n and ψ_n:

$$u = \sum_{k\geq 0} c_k \varphi_k = \sum_{n\geq 0} d_n \psi_n. \tag{2.7}$$

Let us agree to denote the following infinite matrices by

$$\begin{aligned}
\Lambda &= diag\,(1-\lambda_n)_{n\geq 0}\,, \\
M &= diag\,(1-\mu_n)_{n\geq 0}\,, \\
A_d &= (\nabla a.\nabla \varphi_k, \varphi_n)_{n,k\geq 0}\,, \\
A_n &= (\nabla a.\nabla \psi_k, \psi_n)_{n,k\geq 0}\,, \\
\alpha &= \left(\int_{\partial\Omega} f \frac{\partial \varphi_n}{\partial \nu} d\sigma\right)_{n\geq 0}, \\
\beta &= \left(\int_{\partial\Omega} g \psi_n d\sigma\right)_{n\geq 0}
\end{aligned}$$

where the subscripts d and n mean Dirichlet and Neumann bases respectively. Equations (2.5), (2.6) and (2.7) lead to the following system:

$$\begin{cases} \Lambda c + A_d c = \alpha, \\ M d + A_n d = -\beta. \end{cases} \tag{2.8}$$

3. Connections

In order to solve (2.8) for A_d or A_n by elimination or reduction, we need to connect some of the variables. For example the vectors c and d represent the same function u. From (2.7) and (2.3) we deduce the existence of a unitary operator P such that

$$d = Pc, \tag{3.1}$$

i.e. $d_n = \sum_{k\geq 0} (\varphi_k, \psi_n)\, c_k = \sum_{k\geq 0} p_{nk} c_k$.

Denote the operator $E : H^{1/2}(\partial\Omega) \longrightarrow \ell^2$ defined by $\alpha = E(f)$

$$\alpha_n = \int_{\partial\Omega} f \frac{\partial \varphi_n}{\partial \nu} d\sigma \qquad \text{for } n \geq 0$$

and similarly $\beta = F(g)$ where $F : H^{-1/2}(\partial\Omega) \longrightarrow \ell^2$ is defined by

$$\beta_n = \int_{\partial\Omega} g \psi_n d\sigma \qquad \text{for } n \geq 0.$$

The relation between the operators E, F and the Dirichlet-to-Neumann map T is described by

$$\begin{array}{ccc} f & \xrightarrow{E} & \alpha \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \widetilde{T}} \\ g & \xrightarrow{F} & \beta \end{array}$$

and by this we define a new operator $\widetilde{T}$, which is induced by T

$$\widetilde{T}E = FT$$

and $\beta = \widetilde{T}\alpha$.

Multiplying the first equation in (2.8) by $\widetilde{T}$ together with (3.1) leads to

$$\left\{\begin{array}{l} \widetilde{T}\Lambda c + \widetilde{T}A_d c = \widetilde{T}\alpha, \\ MPc + A_n Pc = -\widetilde{T}\alpha. \end{array}\right. \tag{3.2}$$

To connect A_d with A_n recall that the change of basis operator P satisfies the diagram

$$\begin{array}{ccc} L_d^2 & \xrightarrow{P} & L_n^2 \\ \Big\downarrow A_d & & A_n \Big\downarrow \\ L_d^2 & \xrightarrow{P} & L_n^2 \end{array}$$

which leads to

$$A_n P = P A_d.$$

Thus (3.2) reduces to

$$\left\{\begin{array}{l} \widetilde{T}\Lambda c + \widetilde{T}A_d c = \widetilde{T}\alpha, \\ MPc + PA_d c = -\widetilde{T}\alpha \end{array}\right.$$

from which it follows that

$$PA_d c + \widetilde{T}A_d c = -\widetilde{T}\Lambda c - MPc.$$

Denote by s_ρ the subspace of ℓ^2 corresponding to the space of solutions u of (1.1), and denote by $\Pi_\rho : \ell^2 \to \ell^2$ the projection operator onto the subspace s_ρ. Then using the projection onto s_ρ, we deduce

$$\left(P + \widetilde{T}\right) A_d \Pi_\rho = -\left(\widetilde{T}\Lambda + MP\right) \Pi_\rho.$$

Thus we have proved

Theorem 1. *Assume that $\partial\Omega \in C^1$ and $P + \widetilde{T}$ is invertible. Then A_d is given explicitly by*

$$A_d \Pi_\rho = -\left(P + \widetilde{T}\right)^{-1} \left(\widetilde{T}\Lambda + MP\right) \Pi_\rho.$$

Having A_d, we now reconstruct the impedance coefficient ρ.

4. The coefficient ρ

A few steps remain to obtain a explicitly from the recovered matrix

$$A_d = \left(\nabla a . \nabla \varphi_k, \varphi_n\right)_{n,k \geq 0}.$$

It is readily seen that

$$\nabla a . \nabla \varphi_k = \sum_{n \geq 0} \left(\nabla a . \nabla \varphi_k, \varphi_n\right) \varphi_n = \gamma_k. \tag{4.1}$$

Now choose a direction to integrate (4.1), say x_1, and denote the remaining directions by $\widehat{x_1}$, i.e. $x = (x_1, \widehat{x_1})$. Finding the expansion of x_1 in terms of φ_n, i.e.

$$x_1 = \sum_{k \geq 0} s_k \varphi_k$$

leads to

$$\begin{aligned}\frac{\partial a}{\partial x_1} &= \nabla a.\nabla x_1 = \nabla a.\nabla \sum_{k\geq 0} s_k\varphi_k \\ &= \sum_{k\geq 0} s_k \nabla a.\nabla \varphi_k = \sum_{k\geq 0} s_k\gamma_k.\end{aligned}$$

If we denote it by $g(x) = \sum_{k\geq 0} s_k\gamma_k$ then from

$$\frac{\partial a}{\partial x_1} = g(x)$$

we deduce that

$$a(x_1, \widehat{x_1}) = \int_{p(\widehat{x_1})}^{x_1} g(\eta, \widehat{x_1}) d\eta$$

where $p(\widehat{x_1})$ is the x_1 coordinate of one of the points on the boundary where integration was started and along the direction x_1. Once we have a, recall that

$$\rho(x) = \exp(a(x)).$$

Remark. The fact that we used matrices instead of Green's functions and integral equations allowed us to unify the cases where $n \geq 3$ and $n = 2$; see [8] and [6]. Observe that in the case $\rho = 1$, i.e. $a = 0$, we have $A_d = A_n = 0$ and it follows that the Dirichlet to Neumann map $\widetilde{T}$ is explicitly given by

$$\widetilde{T}\alpha = -MP\Lambda^{-1}\alpha$$

which shows the dependence of $\widetilde{T}$ on the eigenvalues of the Laplacian and so on the geometry of Ω. The important nonuniqueness cases are now obvious as they depend essentially from the null space associated with the matrix $P + \widetilde{T}$.

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