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METRIZABILITY VS. FRÉCHET-URYSOHN PROPERTY

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ABSTRACT. In metrizable spaces, points in the closure of a subset A are limits of sequences in A; i.e., metrizable spaces are Fréchet-Urysohn spaces. The aim of this paper is to prove that metrizability and the Fréchet-Urysohn property are actually equivalent for a large class of locally convex spaces that includes (LF)- and (DF)-spaces. We introduce and study countable bounded tightness of a topological space, a property which implies countable tightness and is strictly weaker than the Fréchet-Urysohn property. We provide applications of our results to, for instance, the space of distributions $\mathfrak{D}'(\Omega)$. The space $\mathfrak{D}'(\Omega)$ is not Fréchet-Urysohn, has countable tightness, but its bounded tightness is uncountable. The results properly extend previous work in this direction.

1. Introduction

The *tightness* $t(X)$ [resp., *bounded tightness* $t_b(X)$] of a topological space X is the smallest infinite cardinal number m such that for any set A of X and any point $x \in \overline{A}$ (the closure in X) there is a set [resp., bounding set] $B \subset A$ for which $|B| \leq m$ and $x \in \overline{B}$. Recall that a subset B of X is *bounding* if every continuous real-valued function on X is bounded on B. The notion of countable tightness arises as a natural weakening of the Fréchet-Urysohn notion. Recall that X is *Fréchet-Urysohn* if for every set $A \subset X$ and every $x \in \overline{A}$ there is a sequence in A which converges to x. Clearly,

$$\text{Fréchet-Urysohn} \Rightarrow \text{countable bounded tightness} \Rightarrow \text{countable tightness}.$$

Franklin [9] recorded an example of a compact topological space with countable tightness, hence countable bounded tightness, which is not Fréchet-Urysohn.

In [5] Cascales and Orihuela introduced the class $\mathfrak{G}$ of those locally convex spaces (lcs) $E = (E, \mathfrak{T})$ for which there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets in the topological dual E' of E (called its $\mathfrak{G}$-representation) such that:

$$\text{(1)} \qquad \begin{aligned} &\text{(a)} \quad E' = \bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}; \\ &\text{(b)} \quad A_\alpha \subset A_\beta \text{ when } \alpha \leq \beta \text{ in } \mathbb{N}^{\mathbb{N}}; \\ &\text{(c)} \quad \text{in each } A_\alpha, \text{ sequences are } \mathfrak{T}\text{-equicontinuous.} \end{aligned}$$

In the set $\mathbb{N}^{\mathbb{N}}$ of sequences of positive integers the inequality $\alpha \leq \beta$ for $\alpha = (a_n)$ and $\beta = (b_n)$ means that $a_n \leq b_n$ for all $n \in \mathbb{N}$.

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The class $\mathfrak{G}$ is stable by the usual operations of countable type and contains many important spaces; e.g., all (LF)-spaces and the (DF)-spaces of Grothendieck. In [5] Cascales and Orihuela extended earlier results for (LM)- and (DF)-spaces by showing that if $E \in \mathfrak{G}$, its precompact sets are metrizable and both E and E with its weak topology $\sigma(E, E')$ are angelic spaces. In a very recent paper [4] we advanced the study started in [5], characterizing those spaces in class $\mathfrak{G}$ which have countable tightness when endowed with their weak topologies. We showed that quasi-barrelled spaces in $\mathfrak{G}$ have countable tightness for both the weak and original topologies [4, Theorem 4.8], a bold generalization of Kaplansky's classical theorem stating that the weak topology of metrizable spaces has countable tightness. On the other hand, we showed [4, Theorem 4.6] that for $E \in \mathfrak{G}$ the countable tightness of $(E, \sigma(E, E'))$ is equivalent to realcompactness of the weak dual $(E', \sigma(E', E))$.

The present article further advances our study of $\mathfrak{G}$: we show that in this class metrizability and the Fréchet-Urysohn property are actually equivalent, Theorem 2.2; moreover, we prove that for barrelled spaces E in $\mathfrak{G}$, metrizability and countable bounded tightness, as well as [E does not contain φ], are equivalent conditions, Theorem 2.5. These generalize earlier results of [11, 12, 14, 16] and have interesting applications. For example: the strong dual $E'_\beta := (E', \beta(E', E))$ of a regular (equivalently, locally complete) (LF)-space E has countable tightness provided E'_β is quasi-barrelled, but E'_β is metrizable if and only if it is Fréchet-Urysohn, if and only if E'_β is quasi-barrelled and $t_b(E'_\beta) \leq \aleph_0$. This applies to many concrete spaces, illustrated below for the space of distributions $\mathfrak{D}'(\Omega)$.

Our notation and terminology are standard and we take [2, 15] as our basic reference texts.

2. A characterization of metrizable spaces

First we obtain a Makarov-type result (cf. [2, 8.5.20]) for spaces $E \in \mathfrak{G}$. Recall that an increasing sequence (A_n) of absolutely convex subsets of an lcs E is called *bornivorous* if for every bounded set B in E there exists A_m which absorbs the set B.

Lemma 1. *Let $E \in \mathfrak{G}$ and let $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-representation of E. For $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ put*

$$C_{n_1 \ldots n_k} = \bigcup \{A_\beta : \beta = (m_k) \in \mathbb{N}^\mathbb{N}, n_j = m_j,\ j = 1, 2, \ldots, k\},$$

$k \in \mathbb{N}$. Then the sequence of polars

$$C^o_{n_1} \subset C^o_{n_1, n_2} \subset \cdots \subset C^o_{n_1, n_2, \ldots, n_k} \subset \cdots$$

is bornivorous in E.

Proof. Assume that there exists a bounded set B in E such that $B \not\subset kC^o_{n_1 \ldots n_k}$ for every $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ there exists $x_k \in B$ such that $k^{-1} x_k \notin C^o_{n_1 \ldots n_k}$. Therefore for every $k \in \mathbb{N}$ there exists $f_k \in C_{n_1 \ldots n_k}$ such that $|f_k(x_k)| > k$. Then for every $k \in \mathbb{N}$ there exists $\beta_k = (m^k_n)_n \in \mathbb{N}^\mathbb{N}$ such that $f_k \in A_{\beta_k}$, where $n_j = m^k_j$ for $j = 1, 2, \ldots, k$.

Define $a_n = \max\{m^k_n : k \in \mathbb{N}\}$, $n \in \mathbb{N}$, and $\gamma = (a_n) \in \mathbb{N}^\mathbb{N}$. Clearly $\gamma \geq \beta_k$ for every $k \in \mathbb{N}$. Therefore, by property (b) in the definition of the $\mathfrak{G}$-representation of E, one gets $A_{\beta_k} \subset A_\gamma$, so $f_k \in A_\gamma$ for all $k \in \mathbb{N}$; by property (c) the sequence (f_k) is equicontinuous. Hence (f_k) is uniformly bounded on bounded sets in E, including B, a contradiction. $\square$

Recall that an lcs E is *barrelled* (resp. *quasi-barrelled*) if every closed and absolutely convex subset of E which is absorbing (resp. absorbs every bounded set of E) is a neighborhood of zero, or equivalently, if every weakly bounded (resp. strongly bounded) subset of E' is equicontinuous.

Along with the terminology of [19] a quasi-LB representation of an lcs F is a family $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of Banach discs in F satisfying the following conditions:

$$\text{(i)}\quad F = \bigcup \{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\},$$

$$\text{(ii)}\quad B_\alpha \subset B_\beta \text{ when } \alpha \leq \beta \text{ in } \mathbb{N}^\mathbb{N}.$$

An lcs is called a *quasi-LB space* if it admits a quasi-LB representation. The class of quasi-LB spaces is a large class: it contains all (LF)-spaces as well as their strong duals, and it is stable by closed subspaces, separated quotients, countable direct sums and countable topological products; cf. [19].

Now, we refine some of our ideas in Theorem 4.8 of [4] giving the characterization below.

Lemma 2. *For a quasi-barrelled space E the following statements are equivalent:*

i) *E is in $\mathfrak{G}$.*

ii) *$(E', \beta(E', E))$ is a quasi-LB space.*

iii) *There is a family of absolutely convex closed subsets*

$$\mathcal{F} := \{D_{n_1,n_2,\ldots,n_k} : k, n_1, n_2, \ldots, n_k \in \mathbb{N}\}$$

of E satisfying

a) *$D_{m_1,m_2,\ldots,m_k} \subset D_{n_1,n_2,\ldots,n_k}$, whenever $n_i \leq m_i$, $i = 1, 2, \ldots, k$.*

b) *For every $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ we have*

$$D_{n_1} \subset D_{n_1,n_2} \subset \cdots \subset D_{n_1,n_2,\ldots,n_k} \subset \cdots$$

and the sequence is bornivorous.

c) *If $U_\alpha := \bigcup_k D_{n_1,n_2,\ldots,n_k}$, $\alpha \in \mathbb{N}^\mathbb{N}$, then $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a basis of neighborhoods of the origin in E.*

iv) *E has a basis of neighborhoods of the origin $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ satisfying the decreasing condition*

$$U_\beta \subset U_\alpha \text{ whenever } \alpha \leq \beta \text{ in } \mathbb{N}^\mathbb{N}. \tag{2}$$

Proof. Let us start by proving i)⇒ii). Fix a $\mathfrak{G}$-representation $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of E. Since E is quasi-barrelled, each A_α is equicontinuous. Thus $B_\alpha := A_\alpha^{oo}$ is strongly bounded and weakly compact (Alaoglu-Bourbaki), and thus is a $\beta(E', E)$-Banach disc. Therefore $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a quasi-LB representation of $(E', \beta(E', E))$.

The implication ii)⇒iii) uses the ideas of Theorem 4.8 in [4]. If $(E', \beta(E', E))$ is quasi-LB, [19, Proposition 2.2] applies to ensure us of a quasi-LB representation $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of $(E', \beta(E', E))$ with the extra property

$$\begin{aligned}&\text{for every } \beta(E', E)\text{-Banach disc } B \subset E' \text{ there is } \alpha \in \mathbb{N}^\mathbb{N}\\ &\text{with } B \subset A_\alpha.\end{aligned} \tag{3}$$

The above argument and condition (3) imply that the $\mathfrak{G}$-representation $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a fundamental family of $\mathfrak{T}$-equicontinuous subsets of E'. Hence the family of polars $\{A_\alpha^o : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a basis of neighborhoods of the origin in E.

Given $k, n_1, n_2, \dots, n_k \in \mathbb{N}$ we define $C_{n_1,n_2,\dots,n_k}$ as we did in Lemma 1 and taking polars we write

$$D_{n_1,n_2,\dots,n_k} := C^o_{n_1,n_2,\dots,n_k}.$$

The family $\{D_{n_1,n_2,\dots,n_k} : k, n_1, n_2, \dots, n_k \in \mathbb{N}\}$ matches our requirements. Indeed: a) follows from the fact that $C_{n_1,n_2,\dots,n_k} \subset C_{m_1,m_2,\dots,m_k}$ whenever $n_i \leq m_i$, $i = 1, 2, \dots, k$; b) is exactly the conclusion in Lemma 1; c) may be verified thusly: for every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ we have

$$V_\alpha := \overline{\bigcup_{k=1}^{\infty} D_{n_1,n_2,\dots,n_k}}^{\sigma(E,E')} \subset (\bigcap_{k=1}^{\infty} C_{n_1,n_2,\dots,n_k})^o \subset A^o_\alpha.$$

Observe now that V_α is closed, absolutely convex and bornivorous, thus V_α is a neighborhood of the origin. Use b) again and [2, Proposition 8.2.27] to obtain that for every $\varepsilon > 0$

$$V_\alpha = \overline{\bigcup_{k=1}^{\infty} D_{n_1,n_2,\dots,n_k}}^{\sigma(E,E')} \subset (1+\varepsilon) \bigcup_{k=1}^{\infty} D_{n_1,n_2,\dots,n_k} = (1+\epsilon) U_\alpha.$$

Thus $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of $\mathfrak{T}$-neighborhoods of the origin in E.

As iii)⇒iv) is obvious, it only remains to prove the implication iv)⇒i): if we take a basis of neighborhoods of the origin $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfying (2), then the family of polars $\{U^o_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of E. □

Clearly, then, every barrelled space in $\mathfrak{G}$ has a basis of 0-neighborhoods of size no more than $\mathfrak{c}$. Thus the reasoning of Proposition 1 of [18] gives the following partial positive solution to the (still open) barrelled countable enlargement (BCE) problem (cf. [18] and [2, Section 4.5]).

Corollary 2.1 (assume the Continuum Hypothesis)**.** *Every barrelled space in* $\mathfrak{G}$ *has a BCE, except those with the strongest locally convex topology.*

The previous lemmas naturally lead us to the characterization of metrizable spaces in class $\mathfrak{G}$, Theorem 2.2 below. This result non-trivially generalizes parts of [11, Theorem 5.1], [12, Theorem 2.1] and [16, Theorem 3].

We will need here the following notion introduced by Saxon and Ruess, respectively (cf. [2]): an lcs E is called *Baire-like* (resp. *b-Baire-like*) if for any increasing (and bornivorous) sequence (A_n) of absolutely convex closed subsets of E whose union is E, there exists $m \in \mathbb{N}$ such that A_m is a neighborhood of zero in E. Every b-Baire-like (Baire-like) space is quasi-barrelled (barrelled) and within metrizable spaces barrelledness and Baire-likeness are equivalent conditions.

Adapting an idea of Averbukh and Smolyanov, we proved [12, Proposition 1.2] that every Fréchet-Urysohn space is b-Baire-like (and bornological). We provide a direct proof below.

Theorem 2.2. *For a space* E *in* $\mathfrak{G}$ *the following statements are equivalent:*

i) *E is metrizable.*
ii) *E is Fréchet-Urysohn.*
iii) *E is b-Baire-like.*

Proof. The implication i)⇒ii) is clear and now we prove ii)⇒iii). Assume that there is in E an increasing and bornivorous sequence (A_n) of non-zero absolutely convex sets and no A_n is a 0-neighborhood. Then for each 0-neighborhood U and each $n \in \mathbb{N}$ there is $x_{U,n} \in U \setminus nA_n$, so 0 is in the closure of $\{x_{U,n}\}_U$ for each $n \in \mathbb{N}$. By assumption for each $n \in \mathbb{N}$ there is a sequence $\{U_n(k)\}_k$ of 0-neighborhoods such that $y_{k,n} := x_{U_n(k),n}$ converges to zero as k tends to infinity and

$$y_{k,n} \notin nA_n, \; n, k \in \mathbb{N}. \tag{4}$$

Take any sequence $x_n \in A_1$ of non-zero elements in E which converges to zero and put $A = \{y_{k,n} + x_n : k, n \in \mathbb{N}\}$. Then 0 is in the closure of A and by assumption there are two sequences (n_p) and (k_p) in $\mathbb{N}$ such that $y_{k_p,n_p} + x_{n_p}$ converges to zero. Note that (n_p) is unbounded. Indeed, otherwise, there exists a constant subsequence $n_{p_r} := L$ of (n_p). But then (k_{p_r}) must be unbounded; if not, it contains a subsequence (T) such that $y_{T,L} + x_L = 0$, so $y_{T,L} \in A_1 \subset LA_L$, a contradiction to condition (4). So (k_{p_r}) is unbounded. But then $y_{k_{p_r},L}$ converges to $-x_L$ (which is non-zero by assumption), a contradiction. We showed that indeed (n_p) is unbounded. Finally, $\{y_{k_p,n_p}\}_p \subset mA_m \subset n_pA_{n_p}$ for some $m \in \mathbb{N}$ and $n_p \geq m$. Again a contradiction to condition (4). This proves that E is b-Baire-like [and also bornological (take each $A_n = A$)].

Finally, we prove iii)⇒i). If E is b-Baire-like, then E is quasi-barrelled and therefore we can use Lemma 2 to produce a countable family

$$\mathcal{F} := \{D_{n_1,n_2,\ldots,n_k} : k, n_1, n_2, \ldots, n_k \in \mathbb{N}\},$$

as in iii) there. Since

$$D_{n_1} \subset D_{n_1,n_2} \subset \cdots \subset D_{n_1,n_2,\ldots,n_k} \subset \cdots$$

is bornivorous for every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, we have $E = \bigcup_{k=1}^{\infty} kD_{n_1,n_2,\ldots,n_k}$ and, again, since E is b-Baire-like, some $D_{n_1,n_2,\ldots,n_m}$ is a neighborhood of the origin for certain $m \in \mathbb{N}$. Thus the family

$$\mathcal{U} := \{D_{n_1,n_2,\ldots,n_k} \in \mathcal{F} : D_{n_1,n_2,\ldots,n_k} \text{ is } \mathfrak{T} - \text{neighborhood of } 0\}$$

is a countable basis of neighborhoods of the origin for E. □

The next corollary says in particular that the strong dual of a regular (LF)-space is metrizable if and only if it is Fréchet-Urysohn. An lcs E is an (LF)-space if E is the inductive limit of an increasing sequence of Fréchet, i.e. metrizable and complete lcs.

Corollary 2.3. *Let E be a locally complete quasi-LB space. Then the strong dual $(E', \beta(E', E))$ belongs to $\mathfrak{G}$ and the following statements are equivalent:*

i) *$(E', \beta(E', E))$ is metrizable.*
ii) *$(E', \beta(E', E))$ is Fréchet-Urysohn.*
iii) *$(E', \beta(E', E))$ is b-Baire-like.*

Proof. Since E is locally complete, then every $\mathfrak{T}$-bounded subset is contained in a Banach disc. Use [19, Proposition 2.2] to produce a quasi-LB representation of $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of E with the extra property

$$\begin{aligned} &\text{for every } \mathfrak{T}\text{-bounded set } B \subset E \text{ there is } \alpha \in \mathbb{N}^{\mathbb{N}} \\ &\text{with } B \subset A_\alpha. \end{aligned} \tag{5}$$

For each $\alpha \in \mathbb{N}^{\mathbb{N}}$ consider the polar $U_\alpha := A_\alpha^o$. The family $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighborhoods of the origin in $(E', \beta(E', E))$ satisfying the decreasing condition (2) in iv) of Lemma 2. Hence the polars of U_α in E'' form a $\mathfrak{G}$-representation for $(E', \beta(E', E))$. Thus $(E', \beta(E', E))$ is in $\mathfrak{G}$ and the equivalences here immediately follow from Theorem 2.2 above. □

Since every quasi-barrelled space $E \in \mathfrak{G}$ has countable tightness [4, Theorem 4.8], our corollary applies as follows.

Corollary 2.4. *Let E be a locally complete quasi-LB-space. If $(E', \beta(E', E))$ is quasi-barrelled, then*

$$t(E', \beta(E', E)) \leq \aleph_0.$$

In particular, if E is an (LF)-space which is locally complete (equivalently, regular) and $(E', \beta(E', E))$ is quasi-barrelled, then $t(E', \beta(E', E)) \leq \aleph_0$.

Recall that in [4] we provided an example of a Fréchet space for which its strong dual does not have countable tightness.

Now we will show that *bounded countable tightness* characterizes metrizability for barrelled spaces in class $\mathfrak{G}$. We need the following lemma.

Lemma 3. *Let φ be an $\aleph_0$-dimensional vector space endowed with the finest locally convex topology. Then $t(\varphi) \leq \aleph_0$ but $t_b(\varphi)$ is uncountable.*

Proof. Since φ is an (LF)-space and the tightness of any (LF)-space is countable, by [4, Corollary 4.3] we get that $t(\varphi) \leq \aleph_0$. On the other hand, since φ is non-metrizable it is not a Fréchet-Urysohn space after Theorem 2.2 above. Therefore there exists a subset A in φ such that $0 \in \overline{A}$, but 0 is not the limit of a sequence in A. Assume now that there is a countable and bounding set $B \subset A$ such that $0 \in \overline{B}$. Since B is also bounded and every bounded set in φ is finite-dimensional, 0 belongs to the sequential closure of B which gives us the contradiction that finishes the proof. □

Noting that a barrelled space is b-Baire-like if and only if it is Baire-like, we have the following generalization of Theorem 3 of [16].

Theorem 2.5. *Let $E \in \mathfrak{G}$ be barrelled. The following five statements are equivalent:*

i) *E is metrizable.*
ii) *E is Fréchet-Urysohn.*
iii) *E is Baire-like.*
iv) *$t_b(E) \leq \aleph_0$.*
v) *E does not contain φ.*

Proof. By Theorem 2.2, the first three conditions are equivalent. If E is metrizable, then clearly the bounded tightness of E is countable; i.e., iv) holds. If iv) holds, then E cannot contain φ by Lemma 3. If E does not contain φ, then E is Baire-like by [17, Theorem 2.1]. □

We also refer the reader to [13] for more information concerning the Fréchet-Urysohn property and its relation with various barrelledness conditions.

As a consequence of the last theorem we obtain for duals of quasi-LB spaces the following characterization.

Corollary 2.6. *If a quasi-LB space E and its strong dual $(E', \beta(E', E))$ are both locally complete, then the following assertions are equivalent:*

(i) $(E', \beta(E', E))$ *is metrizable.*
(ii) $(E', \beta(E', E))$ *is quasi-barrelled and* $t_b((E', \beta(E', E))) \leq \aleph_0$.

Proof. The implication (i)⇒(ii) is obvious and the implication (ii)⇒(i) immediately follows from Theorem 2.5 applied to $(E', \beta(E', E))$. Indeed, Corollary 2.3 says that $(E', \beta(E', E))$ is in $\mathfrak{G}$; besides this, as $(E', \beta(E', E))$ is locally complete and quasi-barrelled it is barrelled, [2, 5.1.10], hence Theorem 2.5 applies and we are done. □

If $\Omega \subset \mathbb{R}^n$ is an open set, then the space of test functions $\mathfrak{D}(\Omega)$ is a complete Montel (LF)-space, so its strong dual, the space of distributions $\mathfrak{D}'(\Omega)$, is a quasi-complete ultrabornological (hence quasi-barrelled) non-metrizable space. We consequently have:

Corollary 2.7. *If $\Omega \subset \mathbb{R}^n$ is an open set, then $\mathfrak{D}'(\Omega)$ has countable tightness for the original and weak topologies but $t_b(\mathfrak{D}'(\Omega))$ is uncountable.*

Proof. By Corollary 2.3 we have $\mathfrak{D}'(\Omega) \in \mathfrak{G}$. As $\mathfrak{D}'(\Omega)$ is quasi-barrelled, we can apply [4, Theorem 4.8] to obtain that $\mathfrak{D}'(\Omega)$ has countable tightness for the original and weak topologies. On the other hand, that $t_b(\mathfrak{D}'(\Omega))$ is uncountable now follows from the fact that $\mathfrak{D}'(\Omega)$ is non-metrizable and from Corollary 2.6. □

Professor Bonet and the referee kindly pointed out that the same reasoning applies to the space $A(\Omega)$ of real analytic functions on Ω via the work [7, Theorem 1.6 and Proposition 1.7] of Domanski and Vogt, who also showed that this space, the subject of much recent attention, has no basis [8].

In addition, note that if $E \in \mathfrak{G}$, then any lcs which contains E as a dense subspace also belongs to $\mathfrak{G}$. Therefore Theorem 2.2 also applies to show the following, where, as usual, $C_p(X)$ denotes the space $C(X)$ of continuous real functions on the topological space X endowed with the topology of pointwise convergence on X.

Corollary 2.8. *The space $C_p(X)$ belongs to the class $\mathfrak{G}$ if and only if X is countable (if and only if $C_p(X)$ is metrizable).*

Proof. Indeed, $C_p(X)$ is a dense subspace of the product $\mathbb{R}^X$ which is a Baire space [2, 1.2.13], hence b-Baire-like. If $C_p(X) \in \mathfrak{G}$, then $\mathbb{R}^X \in \mathfrak{G}$ and Theorem 2.2 applies. □

This extends the main result of [14] which states that $C_p(X)$ is an (LM)-space if and only if X is countable. Let us remark that, alternatively, Corollary 2.8 can be proved from the fact that quasi-barrelled spaces in class $\mathfrak{G}$ have countable tightness, [4, Proposition 4.7]. Indeed, if $C_p(X) \in \mathfrak{G}$, then its completion, the Baire space $\mathbb{R}^X$ is also in $\mathfrak{G}$, and so we have that $t(\mathbb{R}^X) \leq \aleph_0$; but this is the case if and only if X is countable as the reader can easily check.

Let E be a locally convex space and let us write $E_\sigma := (E, \sigma(E, E'))$, $E'_\sigma := (E', \sigma(E', E))$. Note that when E'_σ is K-analytic (see [6, 10] for a definition), $t(E_\sigma) \leq \aleph_0$ because $(E'_\sigma)^n$ is still K-analytic $n \in \mathbb{N}$ (hence Lindelöf), [1, Theorem II.1.1] tells us that $t(C_p(E'_\sigma)) \leq \aleph_0$, and thus E_σ (as a subspace of $C_p(E'_\sigma)$) has countable tightness.

Conversely, if $E \in \mathfrak{G}$ and $t(E_\sigma) \leq \aleph_0$, then E'_σ is K-analytic as we showed in [4, Theorem 4.6]. Corollary 2.8 allows us to now provide an example showing that $E \in \mathfrak{G}$ cannot be dropped when proving this implication. Indeed, let X be an uncountable Lindelöf P-space. Since X^n is Lindelöf for any $n \in \mathbb{N}$, [1, Theorem II.1.1] applies again to obtain that $t(C_p(X)) \leq \aleph_0$. By Corollary 2.8 the space $C_p(X)$ does not belong to $\mathfrak{G}$. Now if we assume that $F := C_p(X)'_\sigma$ is K-analytic, then F has an ordered family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of compact sets in F covering it (cf. [3, Corollary 1.2]), i.e., satisfying conditions (a) and (b) in (1). Since X is a P-space (i.e., every G_δ set in X is open), every bounding set in X is finite and by [2, 10.1.20] the space $C_p(X)$ is barrelled. Hence every set A_α is equicontinuous, so condition (c) holds in (1) too, and consequently the space $C_p(X)$ belongs to $\mathfrak{G}$, which is a contradiction.

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