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SPECIAL VALUES OF ELLIPTIC FUNCTIONS AT POINTS OF THE DIVISORS OF JACOBI FORMS

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Abstract. The main result of the paper gives an explicit formula for the sum of the values of even order derivatives with respect to z of the Weierstrass $\wp$-function $\wp(\tau, z)$ for the lattice $\mathbf{Z}\tau \oplus \mathbf{Z}$ (where τ is in the upper half-plane) extended over the points in the divisor of $\phi(\tau, \cdot)$ (where $\phi(\tau, z)$ is a meromorphic Jacobi form) in terms of the coefficients of the Laurent expansion of $\phi(\tau, z)$ around $z = 0$.

1. Introduction

Classically, the theory of complex multiplication asserts that the value of the usual elliptic modular function j at an imaginary quadratic number τ is an algebraic integer and generates a ring class field of $\mathbf{Q}(\sqrt{d_\tau})$ where $d_\tau < 0$ is the discriminant of a non-trivial integral quadratic equation satisfied by τ. These special values also play an important role in modern number theory (cf. e.g. the papers [6, 7] by Gross-Zagier, [2] by Borcherds, [10] by Zagier and others).

A question not completely unreasonable then seems to be if there are other natural points τ in the upper half-plane $\mathcal{H}$ such that $j(\tau)$ is algebraic and has interesting arithmetic properties. Of course, by a well-known theorem of Schneider such points cannot be algebraic, and so at first sight a search appears to be rather hopeless. On the other hand, a little thought reveals that if f is a non-zero elliptic modular function of weight k with Fourier coefficients in a field $K \subset \mathbf{C}$, then $j(\tau)$ indeed is algebraic over K for any point τ in the divisor of f.

In [3], a weighted average sum of the values of $j_1 := j - 744$ (and of a sequence of modular functions j_n, $n \geq 2$, related to j) over such points was studied. In particular, among several other things an explicit formula for these average sums in terms of the Fourier coefficients of f was obtained.

On the other hand, the values of elliptic functions at division points of the associated lattice also are important in number theory. For example, for $\tau \in \mathcal{H}$ let $L_\tau := \mathbf{Z}\tau \oplus \mathbf{Z}$ be the lattice generated by τ and 1 and denote by $w(\tau, \cdot)$ the associated Weber function (i.e. essentially the Weierstrass $\wp$-function $\wp(\tau, \cdot)$ for L_τ properly normalized to be homogeneous of degree zero with respect to L_τ). Then it is a classical fact that for τ imaginary quadratic the value of $w(\tau, \cdot)$ at a non-trivial N-division point of L_τ is algebraic, and in fact together with $j(\tau)$ generates the ray

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class field of $\mathbf{Q}(\sqrt{d_\tau})$ of conductor N (cf. e.g. [9, chap. 10, sect. 1, Thm. 2 and sect. 2, Cor. to Thm. 7]).

In the above context, following [3] it appears to be quite reasonable to also look for points $z \in \mathbf{C}$ other than division points such that $\wp(\tau, z)$ is algebraic, and to study the corresponding values. Natural candidates here seem to be the points in the divisor of $\phi(\tau, \cdot)$, where $\phi(\tau, z)$ ($\tau \in \mathcal{H}$, $z \in \mathbf{C}$) is a Jacobi form.

In the present paper we will indeed give a formula analogous to the above-mentioned one of [3] in the case of elliptic functions, with j_n replaced by $\wp_n$ (essentially the $(2n-2)$-th partial derivative of $\wp$ with respect to z) and f replaced by a Jacobi form ϕ. More precisely, let $\phi(\tau, z)$ ($\tau \in \mathcal{H}$, $z \in \mathbf{C}$) be a meromorphic Jacobi form of weight $k \in \mathbf{Z}$ and index $m \in \mathbf{Z}$ with a K-rational q-expansion ($q := e^{2\pi i\tau}$). Fix $\tau \in \mathcal{H}$ and suppose that $\phi(\tau, \cdot)$ is a non-zero meromorphic function in z. Let $g_2(\tau)$ and $g_3(\tau)$ be the usual Weierstrass invariants attached to L_τ. Then the values $\wp_n(\tau, z)$ where z is in the divisor of $\phi(\tau, \cdot)$ and $z \notin L_\tau$ are algebraic over $K(g_2(\tau), g_3(\tau))$ (section 3, Proposition 1), and we express a weighted average sum of these values explicitly in terms of the Laurent series coefficients of $\phi(\tau, \cdot)$ around $z = 0$ and in terms of values of Eisenstein series at τ (section 4, Theorem).

Formally, in analogy with a corresponding result of [3], as is quite clear our Theorem can also be restated using the exponents of a product expansion of $\phi(\tau, \cdot)$ in terms of the functions $1 - z^n$ ($n \geq 1$). We shall make some short comments on this in section 5 and as an example explicitly state the result in the case of $\phi_{10,1}$ (section 5, Proposition 3).

Our method of proof is similar to that in [3]. In other words, we shall use a $\wp_n$-weighted version of the classical well-known valence formula for elliptic functions (respectively for Jacobi forms, in the latter case; cf. [5, chap. I, sect.1, Thm. 1.2]).

Notations. For $\tau \in \mathcal{H}$, $z \in \mathbf{C}$ we shall write $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi i z}$. We sometimes write $\rho = e^{2\pi i/3}$. The letter K denotes a subfield of $\mathbf{C}$, fixed throughout.

2. Preliminaries on meromorphic Jacobi forms and elliptic functions

We shall start by recalling the definition of a meromorphic Jacobi form (cf. [1, 5]). Let k and m be fixed integers. Then a function ϕ on $\mathcal{H} \times \mathbf{C}$ is called a meromorphic Jacobi form of weight k and index m if

i) ϕ is meromorphic on $\mathcal{H} \times \mathbf{C}$,

ii) $\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k \exp(2\pi i m \frac{cz^2}{c\tau+d})\, \phi(\tau, z)$ $(\forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 := SL_2(\mathbf{Z}))$ and $\phi(\tau, z + \lambda\tau + \mu) = \exp(-2\pi i m(\lambda^2\tau + 2\lambda z))\, \phi(\tau, z)$ $(\forall (\lambda, \mu) \in \mathbf{Z}^2)$,

iii) ϕ has a meromorphic q-expansion of the form

$$\phi(\tau, z) = \sum_{n \geq h} c_n(z) q^n \qquad (0 < |\zeta| < A,\ 0 < |q| < B|\zeta|^N)$$

where $A, B > 0$, $h \in \mathbf{Z}$ and $N \in \mathbf{N}$ are constants and the coefficients $c_n(z)$ for all n are contained in the field $\mathbf{C}(\zeta)$ of complex rational functions.

Applying ii) with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ equal to minus the identity we see that ϕ is either even or odd, depending on whether k is even or odd.

If all the coefficients $c_n(z)$ are contained in $K(\zeta)$, then we say that ϕ has a K-rational q-expansion.

We denote by

$$\wp(\tau,z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} \Big(\frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2}\Big) \qquad (\tau\in\mathcal{H},\, z\in\mathbf{C})$$

the Weierstrass $\wp$-function for the lattice $L_\tau = \mathbf{Z}\tau\oplus\mathbf{Z}$. This is a meromorphic Jacobi form of weight 2 and index zero; its q-expansion is given by

$$\wp(\tau,z) = (2\pi i)^2\Big(\frac{1}{12} + \frac{\zeta}{(1-\zeta)^2} + \sum_{n\geq 1}\Big(\sum_{d|n} d(\zeta^d+\zeta^{-d}-2)\Big)\,q^n\Big) \quad (|q|<|\zeta|<|q|^{-1})$$

(cf. e.g. [9, chap. 4, sect. 2]).

For an even integer $k\geq 4$ we denote by

$$G_k(\tau) = \sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^k} \qquad (\tau\in\mathcal{H}) \tag{1}$$

the Eisenstein series of weight k with respect to Γ_1. As usual, we let

$$g_2(\tau) := 60\,G_4(\tau),\ \ g_3(\tau) := 140\,G_6(\tau) \tag{2}$$

be the Weierstrass invariants attached to $\wp(\tau,\cdot)$.

We also write

$$G_2(\tau) = \frac{\pi^2}{3} + \sum_{m\neq 0}\Big(\sum_{n\in\mathbf{Z}} \frac{1}{(m\tau+n)^2}\Big) \qquad (\tau\in\mathcal{H}) \tag{3}$$

for the near-Eisenstein series of weight 2 with respect to Γ_1.

We sometimes also use the normalized functions

$$Q := 12(2\pi i)^{-4}\,g_2,\ \ R := -216(2\pi i)^{-6}\,g_3 \tag{4}$$

and

$$P := \frac{3}{\pi^2}\,G_2 \tag{5}$$

(Ramanujan's notation).

We let

$$\Delta(\tau) = (2\pi i)^{-12}(g_2^3(\tau) - 27g_3^2(\tau)) = q\prod_{n\geq 1}(1-q^n)^{24} \qquad (\tau\in\mathcal{H})$$

be the Ramanujan delta function which is a cusp form of weight 12 with respect to Γ_1.

We write

$$j(\tau) = \frac{12^3}{(2\pi i)^{12}}\,\frac{g_2^3(\tau)}{\Delta(\tau)} = \frac{1}{q} + 744 + 196884q + \ldots \qquad (\tau\in\mathcal{H})$$

for the usual modular invariant.

For each $n\in\mathbf{N}$ we define

$$\wp_n(\tau,z) := \frac{1}{(2n-1)!}\,\frac{\partial^{2n-2}}{\partial z^{2n-2}}\,\wp(\tau,z). \tag{6}$$

From the Laurent expansion of $\wp(\tau,\cdot)$ around $z=0$ it then easily follows that $\wp_n(\tau,\cdot)\,(n\geq 2)$ is the (uniquely determined) elliptic function with respect to L_τ which is holomorphic on $\mathbf{C}\setminus L_\tau$ and whose Laurent series around $z=0$ has the form

$$\wp_n(\tau,z) = \frac{1}{z^{2n}} + G_{2n}(\tau) + \mathcal{O}(z^2). \tag{7}$$

The transformation formulas ii) for $\wp$ with $k=2$ and $m=0$ imply that $\wp_n$ is a meromorphic Jacobi form of weight $2n$ and index zero. Clearly its q-expansion is $\mathbf{Q}$-rational.

We normalize $\wp$ with respect to τ by setting

$$w(\tau,z) := \left(-2^7\cdot 3^5\,\frac{g_2(\tau)g_3(\tau)}{(2\pi i)^{12}\Delta(\tau)}\right)\wp(\tau,z) \qquad (\tau\in\mathcal{H},\ z\in\mathbf{C})$$

so that w is a meromorphic Jacobi form of weight zero and index zero with a $\mathbf{Q}$-rational q-expansion.

The function $w(\tau,\cdot)$ classically is called the first Weber function attached to L_τ, at least if $\tau\neq i,\rho$ (cf. [9, chap. 6, sect. 2]). (The factor $-2^7\cdot 3^5$ is traditionally inserted so that certain power series expansions have integral coefficients; we do not need this.)

According to [1, Satz 1] the field of meromorphic Jacobi forms of weight zero and index zero with a K-rational q-expansion is equal to $K(j,w)$.

Finally, following [5] we put

$$\phi_{10,1} = \frac{1}{144}(RE_{4,1} - QE_{6,1}) \tag{8}$$

where Q and R are defined by (4) and $E_{4,1}$ resp. $E_{6,1}$ are the usual Jacobi-Eisenstein series of weight 4 resp. 6 and index 1 [5, chap. I, sects. 2 and 3]. Then $\phi_{10,1}$ is the unique Jacobi cusp form of weight 10 and index 1 whose Taylor expansion around $z=0$ has the form

$$(2\pi i)^2\Delta(\tau)z^2 + \mathcal{O}(z^4)$$

[5, chap. I, sect. 3]. The function $\phi_{10,1}(\tau,\cdot)$ vanishes doubly at $z=0$ and hence nowhere else on $\mathbf{C}\setminus L_\tau$ [5, chap. I, sect. 1, Thm. 1.2]. Also, $\phi_{10,1}$ has a $\mathbf{Q}$-rational q-expansion [5, chap. I, sect. 3, formula (17)].

3. Simple rationality results

Proposition 1. *Let ϕ be a meromorphic Jacobi form which has a K-rational q-expansion. Fix $\tau\in\mathcal{H}$ and suppose that $\phi(\tau,\cdot)$ is a non-zero meromorphic function on $\mathbf{C}$. Suppose that z is a point in the divisor of $\phi(\tau,\cdot)$ and $z\notin L_\tau$. Then $\wp_n(\tau,z)$ is algebraic over $K(g_2(\tau),g_3(\tau))$, for all $n\geq 1$, where g_2 and g_3 are defined by* (2).

Proof. Replacing ϕ by $\frac{1}{\phi}$ it is sufficient to consider the case where z is a zero of $\phi(\tau,\cdot)$.

It also suffices to consider the case $n=1$. Indeed, from the Laurent expansion of $\wp(\tau,z)$ at $z=0$ involving the Eisenstein series and from (7) it is easy to see by induction that $\wp_n$ is a monic polynomial in $\wp$ of degree n and that the coefficient at $\wp^\ell$ $(\ell<n)$ is in $\mathbf{Q}[g_2,g_3]$ (in fact, a modular form of weight $2n-2\ell$).

Let k be the weight and m be the index of ϕ. Then

$$\psi := \frac{\phi^{24}}{\phi_{10,1}^{24m}\Delta^{2k-20m}}$$

with $\phi_{10,1}$ defined by (8) is a meromorphic Jacobi form of weight zero and index zero which outside of L_τ has the same zero divisor as ϕ. Also ψ has a K-rational q-expansion.

Therefore, by [1, Satz 1] (cf. section 2) one can write ψ as a quotient of two polynomials in w with coefficients in $K[j]$. Since $\{(\tau,z)\,|\,z\in\mathbf{C}\}$ by hypothesis is not contained in the zero or pole curve of ϕ, it follows that $w(\tau,z)$ is algebraic over

$K(j(\tau))$. Since $w(i,\cdot)=0$ and $w(\rho,\cdot)=0$, the latter hypothesis also guarantees that $\tau\neq i,\rho$, hence $g_2(\tau)g_3(\tau)\neq 0$, and our assertion follows.

Remark. For $n\in\mathbf{N}$ put

$$w_n(\tau,z) := \left(-2^7\cdot 3^5\,\frac{g_2(\tau)g_3(\tau)}{(2\pi i)^{12}\Delta(\tau)}\right)^n\,\wp_n(\tau,z) \qquad (\tau\in\mathcal{H},\ z\in\mathbf{C}).$$

Then w_n is a meromorphic Jacobi form of weight zero and index zero, and the same argument as above shows that $w_n(\tau,z)$ is algebraic over $K(j(\tau))$.

The Theorem in the next section gives an identity between the values of $\wp_n(\tau,z)$ occurring in Proposition 1 and the Laurent coefficients of $\phi(\tau,\cdot)$ at $z=0$. It might therefore be worthwhile to note the following result which for the sake of simplicity we state only in the holomorphic case.

Proposition 2. *Let ϕ be a holomorphic Jacobi form with a K-rational q-expansion. Denote by $\chi_n(\tau)\,(n\geq 0)$ its Taylor coefficients around $z=0$. Then χ_n is in $K[g_2,g_3,G_2]$ for all n, where G_2 is defined by* (3).

Proof. Since ϕ is holomorphic, the coefficients of the q-expansion of ϕ are Laurent polynomials in ζ with coefficients in K. Therefore, taking derivatives with respect to z and evaluating at $z=0$ shows that $(2\pi i)^{-n}\chi_n$ is a Fourier series in q with coefficients in K, and the same is true for $(2\pi i)^{-n-\ell}\chi_n^{(\ell)}\,(n\geq 0,\ \ell\geq 0)$.

By [5, chap. I, sect. 3] there exists a sequence $(\xi_n)_{n\geq 0}$ of holomorphic modular forms of weight $k+n$ with respect to Γ_1 such that χ_n for every $n\geq 0$ can be expressed as a $\mathbf{Q}$-linear combination of $(2\pi i)^\ell\,\xi_{n-2\ell}^{(\ell)}\,(0\leq\ell\leq\frac{n}{2})$, and conversely $\xi_n\,(n\geq 0)$ can be expressed as a $\mathbf{Q}$-linear combination of $(2\pi i)^\ell\,\chi_{n-2\ell}^{(\ell)}\,(0\leq\ell\leq\frac{n}{2})$. It follows that $(2\pi i)^{-n}\xi_n$ has Fourier coefficients in K, hence $(2\pi i)^k\xi_n\in K[g_2,g_3]$ for all n.

We now observe the well-known formulas

$$(2\pi i)^{-1}\,Q' = \frac{1}{3}\,(PQ-R),$$

$$(2\pi i)^{-1}\,R' = \frac{1}{2}\,(PR-Q^2)$$

and

$$(2\pi i)^{-1}\,P' = \frac{1}{12}\,(P^2-Q)$$

(with Q,R defined by (4) and P by (5)). These formulas imply that the differential operators $(2\pi i)^\ell\,\frac{d^\ell}{d\tau^\ell}$ leave $K[g_2,g_3,G_2]$ invariant. Our claim now follows.

4. Average sums of special values

The main result of the paper is the following

Theorem. *Let ϕ be a meromorphic Jacobi form of index $m\in\mathbf{Z}$. Fix $\tau\in\mathcal{H}$ and suppose that $\phi(\tau,\cdot)$ is a non-zero meromorphic function on $\mathbf{C}$. Write*

$$\frac{z\frac{\partial}{\partial z}\phi(\tau,z)}{\phi(\tau,z)} = h_\tau - \sum_{n\geq 1} b_n(\tau)z^n$$

with $h_\tau = ord_{z=0}\,\phi(\tau,\cdot)$. *Let* $\wp_n\,(n\geq 1)$ *be the functions defined by* (6), *with* $\wp_1=\wp$. *Then one has*

$$b_{2n}(\tau) = \sum_{z\in\mathbf{C}/L_\tau, z\notin L_\tau} ord_z\,\phi(\tau,\cdot)\,\wp_n(\tau,z) + \begin{cases} 2m\,G_2(\tau), & \text{if } n=1,\\ h_\tau\,G_{2n}(\tau), & \text{if } n>1,\end{cases} \tag{9}$$

where $G_2, G_4, \ldots$ *are the Eisenstein series defined by* (1) *and* (3).

Remarks. i) Note that clearly $b_n(\tau)$ is a polynomial with integral coefficients in

$$\frac{\chi_1(\tau)}{\chi_0(\tau)}, \ldots, \frac{\chi_n(\tau)}{\chi_0(\tau)}$$

where

$$\phi(\tau,z) = \sum_{n\geq 0}\chi_n(\tau)z^{n+h_\tau}$$

is the Laurent expansion of $\phi(\tau,\cdot)$ at $z=0$. In fact, according to [3, proof of Theorem 3] one has the *explicit* formula

$$b_n(\tau) = F_n\Big(\frac{\chi_1(\tau)}{\chi_0(\tau)}, \ldots, \frac{\chi_n(\tau)}{\chi_0(\tau)}\Big)$$

where

$$F_n(X_1,\ldots,X_n) := n \sum_{\substack{m_1,\ldots,m_n\geq 0\\ m_1+2m_2+\cdots+nm_n=n}} (-1)^{m_1+\cdots+m_n}\,\frac{(m_1+\cdots+m_n-1)!}{m_1!\ldots m_n!} \cdot X_1^{m_1}\ldots X_n^{m_n}.$$

ii) Notice that evaluating the integral (10) below in the same way with $\wp_n$ replaced by a partial derivative of $\wp$ with respect to z of odd order would lead only to an obvious zero identity, since for ϕ a Jacobi form the function $\frac{\frac{\partial}{\partial z}\phi(\tau,z)}{\phi(\tau,z)}$ is always odd with respect to z and the corresponding $\wp_n$ also would be odd.

iii) In the proof of the Theorem, only the relevant transformation formula of ϕ under $\mathbf{Z}^2$ and not under Γ_1 will enter. Hence a corresponding statement would, e.g., also be true with ϕ replaced by $L_m\,\phi$, where $L_m := 8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}$ is the heat operator (which in general does not map Jacobi forms to themselves). Note that L sends even (odd) functions to even (odd) functions with respect to z.

Proof of the Theorem. The argument is similar to the proof of Theorem 1.2 on p. 10 of [5]. Let $\mathcal{F}_\tau$ be a closed fundamental parallelogram for the action of L_τ on $\mathbf{C}$ whose boundary does not contain any zeros or poles of $\phi(\tau,\cdot)$ nor any points of L_τ. For convenience, let us also suppose that $0\in\mathcal{F}_\tau$. We consider the integral

$$\frac{1}{2\pi i}\int_{\partial\mathcal{F}_\tau}\frac{\frac{\partial}{\partial z}\phi(\tau,z)}{\phi(\tau,z)}\,\wp_n(\tau,z)\,dz. \tag{10}$$

Since

$$\frac{1}{2\pi i}\frac{\frac{\partial}{\partial z}\phi(\tau,z)}{\phi(\tau,z)}$$

is invariant under $z \mapsto z+1$ and changes by $2m$ when one replaces z by $z+\tau$, and $\wp_n$ is invariant under these substitutions, the value of (10) is equal to

$$2m \int_{z_0}^{z_0+1} \wp_n(\tau, z)\, dz \tag{11}$$

where z_0 is an appropriate point in $\mathbf{C} \setminus L_\tau$ and the integration is along the line between z_0 and z_0+1 with positive orientation.

Suppose that $n > 1$. Then by the definition of $\wp_n$ and since $\wp(\tau, \cdot)$ has period 1, we obtain for (11) the value zero.

On the other hand, if $n = 1$, we get for (11) the value $-2m\eta_\tau(1)$. Here

$$\eta_\tau : L_\tau \to \mathbf{C},\ \omega \mapsto \zeta(\tau, z+\omega) - \zeta(\tau, z) \qquad (\text{any } z \in \mathbf{C} \setminus L_\tau)$$

is the Weierstrass eta function for L_τ, and $\zeta(\tau, z)\,(\tau \in \mathcal{H},\ z \in \mathbf{C})$ is the usual Weierstrass zeta function for L_τ so that

$$\frac{\partial}{\partial z}\, \zeta(\tau, z) = -\wp(\tau, z).$$

We can also evaluate (10) by the residue theorem, taking into account that $\wp_n(\tau, \cdot)$ is holomorphic outside of L_τ and that for $n \geq 2$ its Laurent series at $z = 0$ is given by (7) (if $n = 1$, of course, we have $\wp(\tau, \cdot) = \frac{1}{z^2} + \mathcal{O}(z^2)$). We then obtain for (10) the value

$$\sum_{z \in \mathbf{C}/L_\tau, z \notin L_\tau} \operatorname{ord}_z \phi(\tau, \cdot)\, \wp_n(\tau, z) \;+\; \big(\delta_n\, h_\tau G_{2n}(\tau) \;-\; b_{2n}(\tau)\big)$$

where δ_n is 1 or zero according as $n > 1$ or $n = 1$.

We therefore find the identity

$$b_{2n}(\tau) = \sum_{z \in \mathbf{C}/L_\tau, z \notin L_\tau} \operatorname{ord}_z \phi(\tau, \cdot)\, \wp_n(\tau, z) + \begin{cases} 2m\, \eta_\tau(1), & \text{if } n = 1, \\ h_\tau\, G_{2n}(\tau), & \text{if } n > 1. \end{cases} \tag{12}$$

This proves the assertion of the Theorem for $n > 1$. To prove it for $n = 1$, it suffices to show that

$$\eta_\tau(1) = G_2(\tau). \tag{13}$$

Identity (13) for example is given in [8, pp. 166-167]. It is also an easy consequence of (12) in the special case of the Jacobi cusp form $\phi_{10,1}$ of weight 10 and index 1 defined by (8). Indeed, recall that $\phi_{10,1}(\tau, \cdot)$ vanishes doubly at $z = 0$ and nowhere else on $\mathbf{C} \setminus L_\tau$, and that its Taylor expansion around $z = 0$ is given by

$$\phi_{10,1}(\tau, z) = -(2\pi)^2 \Delta(\tau) z^2 \;+\; \frac{4}{3}\pi^4 P(\tau)\Delta(\tau) z^4 \;+\; \mathcal{O}(z^6),$$

where P is defined by (5) (the latter formula follows from the Taylor expansions of $E_{4,1}$ and $E_{6,1}$ given in [5, chap. III, sect. 8, p. 94]). Since

$$F_2(X_1, X_2) = X_1^2 - 2X_2,$$

this implies (13).

This finishes the proof of the Theorem.

5. Product expansions

We shall make some short remarks about product expansions of Jacobi forms in terms of the functions $1-z^n$ $(n \geq 1)$.

Let

$$f(z) = \sum_{n\geq 0} a_n z^{n+h}$$

be a power series holomorphic in a small punctured neighborhood of $z = 0$, with $h \in \mathbf{Z}$ and $a_0 \neq 0$. Then as is easy to see [4, sect. 2] (cf. also [3, Propos. 2.1]), f has a product expansion

$$f(z) = a_0 z^h \prod_{n\geq 1} (1-z^n)^{c_n}$$

convergent in a small ϵ-neighborhood of $z = 0$ (contained in the unit disc). Here we use the convention that complex powers are defined by the principal branch of the complex logarithm. Moreover, for $|z| < \epsilon$ one has the identity

$$\frac{zf'(z)}{f(z)} = h - \sum_{n\geq 1} \big(\sum_{d|n} dc_d\big) z^n.$$

Therefore, bearing in mind Remark i) after the Theorem, the left-hand side of (9) could also be rewritten in terms of the exponents $c_n(\tau)$ of the z-product expansion of $\phi(\tau,\cdot)$. Note that when τ ranges over $\mathcal{H}$, $\phi(\tau,\cdot)$ in general will have zeros arbitrarily close to $z = 0$, and so the convergence in z of this product for each fixed τ will depend on the chosen τ.

As an example, let us again consider the case $\phi = \phi_{10,1}$. For $\tau \in \mathcal{H}$ let

$$\delta_\tau := \min\{|m\tau + n| \,|\, (m,n) \in \mathbf{Z}^2 \setminus \{(0,0)\}\}.$$

Then from the Theorem one easily finds

Proposition 3. *For the function* $\phi_{10,1}$ *defined by* (8)*, one has*

$$\phi_{10,1}(\tau,z) = -(2\pi)^2 \Delta(\tau) z^2 \prod_{n\geq 1} (1-z^{2n})^{c_n(\tau)} \qquad (\tau \in \mathcal{H},\ |z| < \min\{\delta_\tau, 1\})$$

where

$$c_n(\tau) := \frac{1}{n} \sum_{d|n} \mu(\frac{n}{d}) G_{2d}(\tau),$$

μ *is the Möbius function and the Eisenstein series* $G_2, G_4, \ldots$ *are defined by* (1) *and* (3).

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