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THE SPECTRUM OF SCHRÖDINGER OPERATORS WITH POSITIVE POTENTIALS IN RIEMANNIAN MANIFOLDS

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ABSTRACT. Let M be a noncompact complete Riemannian manifold. We consider the Schrödinger operator $-\Delta + V$ acting on $L^2(M)$, where V is a nonnegative, locally integrable function on M. We obtain some simple conditions which imply that $\inf \operatorname{Spec}(-\Delta + V)$, the bottom of the spectrum of $-\Delta + V$, is strictly positive. We also establish upper and lower bounds for the counting function $N(\lambda)$.

1. INTRODUCTION

Let M be a noncompact complete Riemannian manifold of dimension d. Let V be a nonnegative, locally integrable function on M. Consider the quadratic form

$$Q[\psi, \psi] = \int_M |\nabla \psi|^2 dx + \int_M V |\psi|^2 dx \tag{1.1}$$

on $L^2(M)$, where ∇ denotes the Riemannian gradient. Note that $\operatorname{Domain}(Q) = \{\psi \in H^1(M) : V^{1/2}\psi \in L^2(M)\}$. Clearly $Q[\,,\,]$ is a semibounded, symmetric closed form. It follows that there exists a unique self-adjoint operator, which we shall call $-\Delta + V$, such that

$$Q[\phi, \psi] = \langle (-\Delta + V)\phi, \psi \rangle_{L^2(M)} \tag{1.2}$$

for any $\phi \in \operatorname{Domain}(-\Delta + V)$ and $\psi \in \operatorname{Domain}(Q)$. Moreover, $C_0^\infty(M)$ is dense in $\operatorname{Domain}(Q)$.

In this paper we investigate certain spectral properties of $-\Delta + V$. In particular we obtain some simple conditions which imply that the bottom of spectrum

$$E = \inf \operatorname{Spec}(-\Delta + V) \tag{1.3}$$

is strictly positive. We also establish upper and lower bounds for the counting function $N(\lambda)$, the dimension of the spectral projection of $-\Delta + V$ on interval $[0, \lambda)$.

For a general Riemannian manifold, the question of positivity of E was recently studied by Ouhabaz [Ou] under the condition $V \in L^\infty(M)$. Let $B(x, r)$ denote the

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open ball in M centered at x with radius r. As in [Ou], we shall assume that there exist $r_0 > 0$ and $c_1, c_2 > 0$ such that

(a) The following L^2-Poincaré inequality holds on M:

$$\int_{B(x,r)} |u - \overline{u}_{B(x,r)}|^2 dy \le c_1 r^2 \int_{B(x,r)} |\nabla u|^2 dy, \quad \text{for all } x \in M \text{ and } r \le r_0 \tag{1.4}$$

where $u \in C^\infty(M)$ and

$$\overline{u}_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dy. \tag{1.5}$$

(b) M satisfies the local doubling condition

$$|B(x,2r)| \le c_2 \, |B(x,r)| \qquad \text{for all } r \le r_0 \text{ and } x \in M. \tag{1.6}$$

It is known that if Ricci curvature of M is bounded from below, then M satisfies conditions (1.4) and (1.6) for all $r_0 > 0$ (with c_1, c_2 depending on r_0) [Bu], [C-G-T].

Theorem A. *Let M be a noncompact complete Riemannian manifold. Suppose that there exists $r_0 > 0$ such that conditions* (1.4) *and* (1.6) *hold with constants c_1, c_2. Also assume that there exist constants c_3, c_4 and $p \in (1, \infty]$ such that*

$$\inf_{x \in M} \frac{1}{|B(x,r_0)|} \int_{B(x,r_0)} V(y) \, dy = c_3 > 0 \tag{1.7}$$

and

$$\left(\frac{1}{|B(x,r_0)|} \int_{B(x,r_0)} |V(y)|^p \, dy \right)^{1/p} \le \frac{c_4}{|B(x,r_0)|} \int_{B(x,r_0)} V(y) \, dy \tag{1.8}$$

for all $x \in M$. Then

$$E \ge \frac{c_3}{\max\left(2, 2^{\frac{1}{p-1}}\right) \left(2 c_1 c_3 c_4^{\frac{p}{p-1}} r_0^2 + 1\right) c_2^2}. \tag{1.9}$$

Remark 1.10. Under the conditions (1.4), (1.6), (1.7) and $V \in L^\infty(M)$, it is proved in [Ou] that $E > 0$. Clearly, if $V \in L^\infty(M)$ and satisfies (1.7), then V satisfies (1.8) with $p = \infty$ and $c_4 = \|V\|_\infty / c_3$. Thus Theorem A contains Theorem 1 in [Ou]. The main interest of our Theorem A lies in the much weaker assumption (1.8). We point out that in the Euclidean case it is possible to deduce (1.9) from the two-sided estimates in [Ma, Theorem 12.5.1, p. 465]. See also [A-B] for the special case $V \in L^\infty(\mathbb{R}^d)$.

Remark 1.11. The proof of Theorem A, which is different from that in [Ou], is fairly elementary. It is based on an idea of C. Fefferman-Phong [Fe, Main Lemma] for Schrödinger operators with nonnegative polynomial potentials in $\mathbb{R}^d$.

For $\lambda > 0$, let $N(\lambda)$ denote the dimension of the spectral projection of $-\Delta + V$ on interval $[0, \lambda)$. In this paper we also establish upper and lower bounds for $N(\lambda)$ under additional conditions on potential V. To state the result, we first introduce the basic length scale

$$\ell(x) = \inf \left\{ r > 0 : \quad \frac{r^2}{|B(x,r)|} \int_{B(x,r)} V(y) \, dy \ge r_0^2 c_3 \right\} \tag{1.12}$$

for $x \in M$, where c_3 is the same constant as in (1.7). Clearly $0 < \ell(x) \le r_0$. The effective potential $v(x)$ is then defined by

$$v(x) = \frac{1}{[\ell(x)]^2}. \tag{1.13}$$

The definition of v is similar to that of $\{m(x,V)\}^2$ in [Sh1].

In the place of (1.8), we will assume that there exist $p \in (1,\infty]$ and $c_5, c_6 > 0$ such that

$$\left\{ \frac{1}{|B|} \int_B |V(y)|^p dy \right\}^{1/p} \le c_5 \left\{ \frac{1}{|B|} \int_B V(y)\, dy + \frac{1}{r^2} \right\} \tag{1.14}$$

for all $B = B(x,r)$ with $0 < r \le r_0$ and

$$\frac{r^2}{|B(x,r)|} \int_{B(x,r)} V(y)\, dy \le c_6\, \eta\left(\frac{r}{R}\right) \left\{ \frac{R^2}{|B(x,R)|} \int_{B(x,R)} V(y)\, dy + 1 \right\} \tag{1.15}$$

for all $0 < r < R \le r_0$, where η is a positive function on $[0,1]$ such that $\eta(t) \to 0$ as $t \to 0$.

Theorem B. *Let M be a noncompact complete Riemannian manifold. Suppose that there exists $r_0 > 0$ such that conditions* (1.4) *and* (1.6) *hold for all $r \le r_0$ with constants c_1 and c_2. Let V be a nonnegative function on M satisfying* (1.7)*. Further assume that V satisfies conditions* (1.14) *and* (1.15) *for all $x \in M$, $0 < r < R \le r_0$ and some $c_5, c_6 > 0$, $p > 1$. Then there exist constants $C > 0$ and $c > 0$ depending only on c_i, $i = 1,2,3,5,6$, and p, η such that*

$$N(\lambda) \le C \int_{\{x\in M:\ v(x) \le C\lambda\}} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|}, \tag{1.16}$$

$$N(\lambda) \ge c \int_{\{x\in M:\ v(x) \le c\,\lambda\}} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|} \tag{1.17}$$

for λ sufficiently large.

If $M = \mathbb{R}^d$ and $p > d/2$, then condition (1.14) implies condition (1.15) with $\eta(t) = t^{2-(d/p)}$ by Hölder inequality. In particular, if V is in the reverse Hölder class, i.e.,

$$\left(\frac{1}{|B|} \int_B |V|^p dx \right)^{1/p} \le \frac{C}{|B|} \int_B V\, dx \quad \text{for every ball } B \text{ in } \mathbb{R}^d, \tag{1.18}$$

with exponent $p > d/2$ ($d \ge 2$), then V satisfies conditions (1.14) and (1.15). This is the case studied by the author in [Sh2], [Sh3] where estimates (1.16)-(1.17) are established. See also [Sh4]. We should point out that [Sh3], [Sh4] deal with the magnetic Schrödinger operator $(-i\nabla - \mathbf{a})^2 + V$ in $\mathbb{R}^d$. The results in [Sh2], [Sh3], [Sh4] extend some earlier work on nonclassical eigenvalue asymptotics. We refer the reader to [Sh3] for references. We remark that in many cases it is possible to read off the leading order of $N(\lambda)$ as $\lambda \to \infty$ from estimates like (1.16)-(1.17). This is especially useful in the case of degenerate potentials when the classical Cwickel-Lieb-Rosenbljum bound, which may be extended to manifolds satisfying the global Sobolev inequality [Li-Ya], [Le-So], fails to yield any information. In particular it follows that under the assumptions in Theorem B, the Schrödinger operator $-\Delta + V$ has a discrete spectrum if and only if for some $x_0 \in M$, $v(x) \to \infty$ as $\mathrm{dist}(x, x_0) \to \infty$. See Corollary 3.14.

Finally, we note that conditions (1.14)-(1.15), which allow V to vanish on certain open sets, are much weaker than the reverse Hölder condition (1.18). Theorem B is new even in the Euclidean case.

Throughout this paper we will assume that there exists $r_0 > 0$ such that the L^2-Poincaré inequality (1.4) and the doubling condition (1.6) hold for all $x \in M$ and $0 < r \le r_0$ with constants c_1, c_2.

2. Proof of Theorem A

For $x \in M$ and $0 < r \le r_0$, we define

$$V(x,r) = \frac{1}{2|B(x,r)|} \int_{B(x,r)} \frac{V(y)}{2c_1 r^2 V(y) + 1}\, dy. \tag{2.1}$$

The proof of Theorem A is based on the following lemma which is a direct extension of the Main Lemma in [Fe]. See also [Sh4, Theorem 3.1].

Lemma 2.2. *Let $B = B(x_0, r)$ where $x_0 \in M$ and $0 < r \le r_0$. Then, for any $u \in C^\infty(B)$,*

$$V(x_0,r) \int_B |u|^2 dx \le \int_B |\nabla u|^2 dx + \int_B V|u|^2 dx.$$

Proof. The proof is similar to that in the Euclidean case [Fe].

Using expansion and the Poincaré inequality (1.4), we obtain

$$\frac{1}{2|B|} \iint_{B\times B} |u(x)-u(y)|^2 dxdy = \int_B |u - \overline{u}_B|^2 dx \le c_1\, r^2 \int_B |\nabla u|^2 dx.$$

This, together with

$$\frac{1}{|B|} \iint_{B\times B} V(y)|u(y)|^2\, dxdy = \int_B V(x)|u(x)|^2\, dx$$

gives

$$\frac{1}{|B|} \iint_{B\times B} \left\{ \frac{1}{2c_1 r^2} |u(x)-u(y)|^2 + V(y)|u(y)|^2 \right\} dxdy \le \int_B |\nabla u|^2\, dx + \int_B V|u|^2 dx.$$

It follows that

$$\frac{1}{2|B|} \int_B \min\left(\frac{1}{2c_1 r^2}, V(y) \right) dy \cdot \int_B |u(x)|^2 dx \le \int_B |\nabla u|^2\, dx + \int_B V|u|^2 dx.$$

Since $\min(a,b) \ge ab/(a+b)$, we obtain

$$\frac{1}{2|B|} \int_B \min\left(\frac{1}{2c_1 r^2}, V(y) \right) dy \ge V(x_0, r).$$

The proof is complete.

Lemma 2.3. *Suppose that V satisfies* (1.7)-(1.8) *for some $p > 1$ with constants c_3, c_4. Then*

$$\inf_{x\in M} V(x, r_0) \ge \frac{c_3}{\max\left(2, 2^{\frac{1}{p-1}}\right) \left(2c_1 c_3 c_4^{\frac{p}{p-1}} r_0^2 + 1\right)}. \tag{2.4}$$

Proof. Let $q = p/(p-1)$ and $B = B(x, r_0)$. By Hölder's inequality,

$$\begin{aligned}\int_B V(y)\,dy &= \int_B \left(\frac{V(y)}{2c_1 r_0^2 V(y)+1}\right)^{1/q} \left(2c_1 r_0^2 V(y)+1\right)^{1/q} |V(y)|^{1/p}\,dy \\ &\le \left\{\int_B \frac{V(y)\,dy}{2c_1 r_0^2 V(y)+1}\right\}^{1/q} \left\{\int_B \left(2c_1 r_0^2 V(y)+1\right)^{p/q} V(y)\,dy\right\}^{1/p}.\end{aligned}$$

If $p \ge 2$, then $p/q = p-1 \ge 1$. By Minkowski's inequality,

$$\begin{aligned}&\left\{\int_B \left(2c_1 r_0^2 V(y)+1\right)^{p/q} V(y)dy\right\}^{q/p} \\ &\quad\le 2c_1 r_0^2 \left\{\int_B |V(y)|^p\,dy\right\}^{q/p} + \left\{\int_B V(y)\,dy\right\}^{q/p}.\end{aligned}$$

It follows that

$$\begin{aligned}V(x, r_0) &= \frac{1}{2|B|}\int_B \frac{V(y)\,dy}{2c_1 r_0^2 V(y)+1} \\ &\ge \frac{\left(\frac{1}{|B|}\int_B V(y)\,dy\right)^q}{2\left\{\frac{1}{|B|}\int_B (2c_1 r_0^2 V(y)+1)^{\frac{p}{q}} V(y)\,dy\right\}^{q/p}} \\ &\ge \frac{\left(\frac{1}{|B|}\int_B V(y)\,dy\right)^q}{4c_1 r_0^2\left(\frac{1}{|B|}\int_B |V(y)|^p dy\right)^{q/p} + 2\left(\frac{1}{|B|}\int_B V(y)\,dy\right)^{q/p}} \\ &\ge \frac{\left(\frac{1}{|B|}\int_B V(y)\,dy\right)^q}{4c_1 r_0^2 c_4^q\left(\frac{1}{|B|}\int_B V(y)dy\right)^{q} + 2\left(\frac{1}{|B|}\int_B V(y)\,dy\right)^{q/p}} \\ &= \frac{1}{4c_1 r_0^2 c_4^q + 2\left(\frac{1}{|B|}\int_B V(y)\,dy\right)^{-1}} \\ &\ge \frac{c_3}{4c_1 c_3 c_4^q r_0^2 + 2}.\end{aligned}$$

If $1 < p < 2$, then $p/q = p-1 < 1$. In place of Minkowski's inequality, we use

$$\begin{aligned}&\left\{\int_B \left(2c_1 r_0^2 V(y)+1\right)^{\frac{p}{q}} V(y)\,dy\right\}^{q/p} \\ &\quad\le 2^{\frac{q}{p}-1}\left\{2c_1 r_0^2\left(\int_B |V(y)|^p dy\right)^{q/p} + \left(\int_B V(y)\,dy\right)^{q/p}\right\}.\end{aligned}$$

The rest is the same as in the case $p \ge 2$. The proof is finished.

We are now in a position to give the

Proof of Theorem A. Let α denote the constant in right-hand side of (2.4). By Lemma 2.2, we have

$$\alpha\int_{B(x_0,r_0)} |\psi|^2 dx \le \int_{B(x_0,r_0)} |\nabla\psi|^2 dx + \int_{B(x_0,r_0)} V|\psi|^2 dx \tag{2.5}$$

for any $\psi \in C_0^\infty(M)$. We divide both sides of (2.5) by $|B(x_0, r_0)|$ and then integrate the resulting inequality with respect to x_0 over M. By Fubini's Theorem, we obtain

$$\alpha \int_M |\psi(x)|^2 h(x)\, dx \le \int_M |\nabla \psi(x)|^2 h(x)\, dx + \int_M V(x)|\psi(x)|^2 h(x)\, dx$$

where

$$h(x) = \int_{B(x,r_0)} \frac{dx_0}{|B(x_0, r_0)|}. \tag{2.6}$$

Note that, if $x_0 \in B(x, r_0)$, then $B(x_0, r_0) \subset B(x, 2r_0)$ and $B(x, r_0) \subset B(x_0, 2r_0)$. By the doubling condition (1.6), this implies that

$$h(x) \ge \int_{B(x,r_0)} \frac{dx_0}{|B(x, 2r_0)|} \ge \frac{1}{c_2},$$
$$h(x) \le c_2 \int_{B(x,r_0)} \frac{dx_0}{|B(x_0, 2r_0)|} \le c_2 \int_{B(x,r_0)} \frac{dx_0}{|B(x, r_0)|} = c_2.$$

It follows that

$$\frac{\alpha}{c_2^2} \int_M |\psi|^2 dx \le \int_M |\nabla \psi|^2 dx + \int_M V|\psi|^2 dx$$

for any $\psi \in C_0^\infty(M)$. By the minimax principle, $E = \inf \operatorname{Spec}(-\Delta + V) \ge \alpha/c_2^2$. This completes the proof.

Remark 2.7. For a manifold M with bounded geometry, Kondrat'ev and Shubin [K-S] gave a necessary and sufficient condition on V for the spectrum of $-\Delta + V$ to be discrete. This extends a theorem of A. M. Molchanov for the case $M = \mathbb{R}^d$ [Ma, Theorem 12.5.4, p. 466]. The necessary and sufficient condition involves the Newtonian capacity in geodesic coordinates. We point out that if M is of bounded geometry, one may deduce Theorem A from Proposition 5.3 in [K-S] on the lower bound for the first Neumann eigenvalue of $-\Delta + V$ on a ball.

3. Proof of Theorem B

Let $\lambda > 0$. Suppose $1/\sqrt{\lambda} \le r_0$. Since M satisfies the doubling condition (1.6), there exist a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in M and a positive number K depending on c_2 such that for $B_j = B(x_j, \frac{1}{\sqrt{\lambda}})$, we have $M = \bigcup_j B_j$, $\frac{1}{2}B_j \cap \frac{1}{2}B_k = \emptyset$ if $j \ne k$, and each $x \in M$ is contained in at most K balls B_j. We refer the reader to [Ou] for a proof of above statement.

Let Λ denote the set of all $j \in \mathbb{N}$ such that

$$\frac{1}{|B_j|} \int_{B_j} V\, dx \le \lambda. \tag{3.1}$$

We begin with a lower bound for $N(\lambda)$.

Lemma 3.2. *There exists $C > 0$ depending only on c_2 such that*

$$N(C\lambda) \ge \textit{the number of elements in } \Lambda$$

for all $\lambda \ge 1/r_0^2$.

Proof. For each $j \in \Lambda$, let

$$\psi_j(x) = \begin{cases} \frac{1}{2\sqrt{\lambda}} - \operatorname{dist}(x, x_j), & x \in \frac{1}{2}B_j, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\psi_j \in \text{Domain}(Q)$. It is easy to see that

$$Q[\psi_j, \psi_j] \le C\,|B_j| \le C\,\lambda\,\|\psi_j\|^2_{L^2(M)}. \tag{3.3}$$

Since $\{\frac{1}{2}B_j\}_{j\in\Lambda}$ are mutually disjoint, $\{\psi_j\}_{j\in\Lambda}$ is an orthogonal set in $L^2(M)$. This, together with (3.3), implies that $N(C\lambda) \ge$ the number of elements in Λ by the minimax principle.

Lemma 3.4. *Suppose that V satisfies condition* (1.15) *for all $0 < r < R \le r_0$. Also assume that V satisfies* (1.7). *Then there exist $C > 0$, $c > 0$ depending only on c_2, c_3, c_6, r_0 and η such that*

$$N(C\lambda) \ge c \int_{\{x\in M:\ v(x)\le c\,\lambda\}} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|}$$

for λ sufficiently large.

Proof. By the doubling condition (1.6),

$$\int_{B_j} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|} \approx 1. \tag{3.5}$$

It follows from Lemma 3.2 that

$$\begin{aligned} N(C\lambda) &\ge \text{the number of elements in } \Lambda \\ &\ge c \sum_{j\in\Lambda} \int_{B_j} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|} \\ &\ge c \int_{\bigcup_j B_j} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|}. \end{aligned}$$

To finish the proof, it suffices to show that

$$\{x \in M:\ v(x) \le c\,\lambda\} \subset \bigcup_{j\in\Lambda} B_j \tag{3.6}$$

for some $c > 0$. To this end, we note that by (1.15), if $x \in B_j$ for some $j \notin \Lambda$ and $R = \frac{2}{\delta\sqrt{\lambda}}$, then

$$\begin{aligned} \frac{R^2}{|B(x,R)|} \int_{B(x,R)} V(y)\,dy &\ge \frac{1}{c_5\eta(\delta)} \frac{\left(\frac{2}{\sqrt{\lambda}}\right)^2}{|B(x, \frac{2}{\sqrt{\lambda}})|} \int_{B(x,\frac{2}{\sqrt{\lambda}})} V(y)\,dy - 1 \\ &\ge \frac{c}{\eta(\delta)} - 1 > r_0^2 c_3 \end{aligned}$$

if δ is small. It follows from the definition that $\ell(x) < \frac{2}{\delta\sqrt{\lambda}}$. Thus $v(x) > c\lambda$ for all $x \in \bigcup_{j\notin\Lambda} B_j$. This implies that

$$\{x \in M:\ v(x) \le c\,\lambda\} \subset M \setminus \bigcup_{j\notin\Lambda} B_j \subset \bigcup_{j\in\Lambda} B_j.$$

The proof is complete.

Next we estimate $N(\lambda)$ from above.

Lemma 3.7. *Suppose that V satisfies* (1.14). *Then there exists $c > 0$ depending only on c_1, c_2, c_5 and p such that*

$$N(c\lambda) \leq \textit{the number of elements in } \Lambda$$

for all $\lambda \geq 1/r_0^2$.

Proof. Let $r = 1/\sqrt{\lambda} \leq r_0$. Note that by (1.14), if $j \notin \Lambda$,

$$c\,\lambda \leq \left(\frac{1}{|B_j|} \int_{B_j} |V|^p dx \right)^{1/p} \leq \frac{C}{|B_j|} \int_{B_j} V\,dx. \tag{3.8}$$

It then follows from the proof of Lemma 2.3 that, if $j \notin \Lambda$,

$$\begin{aligned}
V(x_j, r) &\geq \frac{c\left(\frac{1}{|B_j|}\int_{B_j} V\,dx\right)^q}{\frac{1}{\lambda}\left(\frac{1}{|B_j|}\int_{B_j}|V|^p dx\right)^{q/p} + \left(\frac{1}{|B_j|}\int_{B_j} V\,dx\right)^{q/p}} \\
&\geq \frac{c\left(\frac{1}{|B_j|}\int_{B_j} V\,dx\right)^q}{\frac{1}{\lambda}\left(\frac{1}{|B_j|}\int_{B_j} V dx\right)^{q} + \left(\frac{1}{|B_j|}\int_{B_j} V\,dx\right)^{q/p}} \\
&= \frac{c\left(\frac{1}{|B_j|}\int_{B_j} V\,dx\right)}{\frac{1}{\lambda}\left(\frac{1}{|B_j|}\int_{B_j} V\,dx\right) + 1} \\
&\geq c\,\lambda.
\end{aligned}$$

By Lemma 2.2, this implies that

$$c\,\lambda \int_{B_j} |\psi|^2 dx \leq \int_{B_j} |\nabla\psi|^2 dx + \int_{B_j} V|\psi|^2 dx \tag{3.9}$$

for all $j \notin \Lambda$ and $\psi \in C_0^\infty(M)$.

For $j \in \Lambda$, we use Poincaré inequality (1.4) to obtain

$$c\,\lambda \int_{B_j} |\psi|^2 dx \leq \int_{B_j} |\nabla\psi|^2 dx + \int_{B_j} V|\psi|^2 dx \tag{3.10}$$

if $\psi \perp \chi_{B_j}$. Combining (3.9) and (3.10), we see that if $\psi \in C_0^\infty(M)$ and $\psi \perp \chi_{B_j}$ for all $j \in \Lambda$, then

$$c\,\lambda \int_{M} |\psi|^2 dx \leq \int_{M} |\nabla\psi|^2 dx + \int_{M} V|\psi|^2 dx \tag{3.11}$$

where we have also used the fact that $1 \leq \sum_j \chi_{B_j} \leq K$. Lemma 3.7 now follows from (3.11) by the minimax principle.

Lemma 3.12. *Suppose that V satisfies the conditions* (1.7), (1.14) *and* (1.15). *Then there exist $C > 0$ and $c > 0$ depending on $c_1, c_2, c_3, c_5, c_6, \eta$ and p such that*

$$N(c\lambda) \leq C \int_{\{x \in M:\ v(x) \leq C\,\lambda\}} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|}$$

for λ sufficiently large.

Proof. By Lemma 3.7,

$$\begin{aligned} N(c\lambda) &\leq \text{ the number of elements in } \Lambda \\ &\leq C \sum_{j\in\Lambda} \int_{\frac{1}{2}B_j} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|} \\ &\leq C \int_{\bigcup_{j\in\Lambda} \frac{1}{2}B_j} \frac{dx}{|B(x, \frac{1}{\sqrt{\lambda}})|}. \end{aligned}$$

Thus it suffices to show that

$$\bigcup_{j\in\Lambda} \frac{1}{2}B_j \subset \{x \in M : \ v(x) \leq C\,\lambda\} \tag{3.13}$$

for some $C > 0$. To see (3.13), we use condition (1.14). Note that if $x \in \frac{1}{2}B_j$ for some $j \in \Lambda$, then $B(x, \frac{1}{2\sqrt{\lambda}}) \subset B_j$ and

$$\begin{aligned} &\frac{\left(\frac{\delta}{2\sqrt{\lambda}}\right)^2}{\left|B(x, \frac{\delta}{2\sqrt{\lambda}})\right|} \int_{B(x, \frac{\delta}{2\sqrt{\lambda}})} V(y)\, dy \\ &\qquad \leq c_6\, \eta(\delta) \left\{ \frac{\left(\frac{1}{2\sqrt{\lambda}}\right)^2}{\left|B(x, \frac{1}{2\sqrt{\lambda}})\right|} \int_{B(x, \frac{1}{2\sqrt{\lambda}})} V(y)\, dy + 1 \right\} \\ &\qquad \leq C\, \eta(\delta) < c_3 r_0^2 \end{aligned}$$

if $\delta \in (0, \delta_0)$ and δ_0 is small. It follows from the definition that $\ell(x) \geq \delta_0/(2\sqrt{\lambda})$. Thus $v(x) \leq C\,\lambda$ for all $x \in \bigcup_{j\in\Lambda} \frac{1}{2}B_j$. (3.13) is then proved and the proof of Lemma 3.12 is complete.

Proof of Theorem B. By the doubling condition (1.6), it is easy to see that the upper bound (1.16) follows from Lemma 3.12, while the lower bound (1.17) follows from Lemma 3.4.

Corollary 3.14. *Under the same assumptions as in Theorem* B*, the spectrum of* $-\Delta + V$ *is discrete if and only if for some* $x_0 \in M$*,* $v(x) \to \infty$ *as* $dist(x, x_0) \to \infty$.

Proof. Suppose that for some $x_0 \in M$, $v(x) \to \infty$ as $\text{dist}(x, x_0) \to \infty$. For any λ sufficiently large, there exists $R_\lambda > 0$ such that $\{x \in M : \ v(x) < C\,\lambda\} \subset B(x_0, R_\lambda)$. It follows from (1.16) that

$$N(\lambda) \leq C\,|B(x_0, R_\lambda)| \sup_{x\in B(x_0,R_\lambda)} \frac{1}{|B(x, \frac{1}{\sqrt{\lambda}})|} < \infty.$$

Thus $N(\lambda) < \infty$ for any $\lambda > 0$. It follows that the spectrum of $-\Delta + V$ is discrete.

Next suppose that for some $x_0 \in M$, $v(x) \nrightarrow \infty$ as $\text{dist}(x, x_0) \to \infty$. Then there exist $\varepsilon > 0$ and a sequence $\{x_j\}$ of points in M such that $v(x_j) < 1/\varepsilon^2$ and $\text{dist}(x_j, x_0) \to \infty$ as $j \to \infty$. It follows that $\ell(x_j) > \varepsilon$. Consequently, by definition,

$$\frac{1}{|B(x_j, \varepsilon)|} \int_{B(x_j,\varepsilon)} V(y)\, dy < \frac{c_3 r_0^2}{\varepsilon^2}.$$

By the proof of Lemma 3.2, there exists $\psi_j \in \text{Domain}(Q)$ such that $\|\psi_j\|_{L^2(M)} = 1$, $\text{supp}\psi_j \subset B(x_j, \varepsilon)$ and $Q[\psi_j, \psi_j] \leq C/\varepsilon^2$. Since $\text{dist}(x_j, x_0) \to \infty$ as $j \to \infty$, by the minimax principle, we have $N(C/\varepsilon^2) = \infty$. This completes the proof.

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