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NOTE ON THE RIEMANN-HURWITZ TYPE FORMULA FOR MULTIPLICATIVE SEQUENCES

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ABSTRACT. We give a formula of the Riemann-Hurwitz type for classes defined by multiplicative sequences as corollaries of the Chern number formula for ramified coverings.

1.

1.1. In the classical results on Riemann surfaces, we have the Riemann-Hurwitz formula which relates the topological Euler numbers of the covering space and base space. The difference of the Euler numbers can be expressed by local residues of ramification points. In the higher-dimensional case, we can see that the difference of the Euler numbers can be expressed by Euler numbers of degeneration loci ([Y]). In the simplest case that a covering map $f : X \longrightarrow Y$ between n-dimensional compact complex manifolds with covering multiplicity μ has the non-singular branch locus B, the above formula is

$$\chi(Y) - \mu \cdot \chi(X) = -\chi(B).$$

In such cases, we can also consider the genera of B determined by the multiplicative sequence $K_j(c_1, \cdots, c_j)$ of Chern classes. Thus we expect that the difference

$$K_n(c_1(X), \cdots, c_n(X)) - \mu K_n(c_1(Y), \cdots, c_n(Y))$$

can be explessed by datas of $\{K_j\}_{1\leq j\leq n-1}$ of B. But there are no results for this type of formulas.

In this note, we show that this type of formula follows immediately from the definition of the multiplicative sequence and the Chern number formula for ramified coverings ([Iz]).

1.2. First we recall the definition of multiplicative sequence. (For more detail, see [Hz1].) Let A be a commutative algebra with identity and consider $A[p_1, p_2, \cdots]$, the ring of polynomials in the p_i. Let $\{K_j\}$ be a sequence of polynomials in the

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indeterminates p_i and write

$$K(\sum_{j=0}^{\infty} p_j z^j) := \sum_{j=0}^{\infty} K_j(p_1, \cdot s, p_j) z^j.$$

The sequence $K = \{K_j\}$ is called a multiplicative sequence if every identity of the form

$$\sum_{j=0}^{\infty} p_j z^j = (\sum_{j=0}^{\infty} p'_j z^j)(\sum_{j=0}^{\infty} p''_j z^j)$$

implies an identity

$$K(\sum_{j=0}^{\infty} p_j z^j) = K(\sum_{j=0}^{\infty} p'_j z^j) K(\sum_{j=0}^{\infty} p''_j z^j).$$

Now let X be a n-dimensional compact complex manifold and $c_i(E) \in H^{2i}(X, \mathbf{Z})$ Chern classes of a holomorphic tangent bundle E over X. The (total) K-class of E is defined by

$$K(E) = \sum_{j=0}^{\infty} K_j(c_1(E) \cdots c_j(E))$$

where $\{K_j(c_1 \cdots c_j)\}$ is a multiplicative sequence. Then for vector bundles E and F, it follows from the definition of multiplicative sequence that

$$K(E \oplus F) = K(E)K(F).$$

1.3. Next we recall the Chern number formula for ramified coverings. For a ramified covering with good ramification between n-dimensional compact complex manifolds, we have the following formula ([Iz]).

Theorem 1.1 (Chern number formula)**.** *Let $f \colon Y \longrightarrow X$ be a ramified covering with covering multiplicity μ between compact complex manifolds of dimension n, $R_f = \sum_i r_i R_i$ the ramification divisor of f, and $B_f = \sum_i b_i B_i$ the branch locus of f. We set $f^* B_i = \sum_t r_{i_t} R_{i_t}$ where n_{i_t} denotes the mapping degree of the induced map $f|_{R_{i_t}} \colon R_{i_t} \to B_i$ with $b_i = \sum_t n_{i_t} r_{i_t}$. We assume that the ramification divisor and the irreducible components B_i of the branch locus B_f are all non-singular, and suppose that $n = \sum_{i+1}^{n} i \cdot N_i$. Then:*

$$\begin{aligned} &c_1^{N_1}, \cdots, c_n^{N_n}(Y) - \mu \cdot c_1^{N_1}, \cdots, c_n^{N_n}(X) \\ &= \sum_i \sum_{\alpha=0}^{n-1} \left(\sum_t \frac{n_{i_t}(1 - (r_{i_t}+1)^{\alpha+1})}{(r_{i_t}+1)^{\alpha}} \right) P_\alpha(c_1(B_i), \cdots, c_{n-1}(B_i)) \cdot c_1(L_{B_i})^\alpha \frown [B_i] \end{aligned}$$

where we set

$$\begin{aligned} H_\xi^{(N_1 \cdots N_n)}(l) &= l^{-1} \left(\prod_{i=1}^{n} (c_i(\xi) + c_{i-1}(\xi) \cdot l)^{N_i} - c_1^{N_1} \cdots c_n^{N_n}(\xi) \right) \\ &= \sum_{\alpha=0}^{n-1} P_\alpha(c_1, \cdots, c_{n-1}) l^\alpha. \end{aligned}$$

Let $Q(x) = K(1 + X)$ be a characteristic power series of K. For the Chern polynomials determined by multiplicative sequences, we can write explicitly the coefficients $P_\alpha(c_1(B_i)\cdots c_{n-1}(B_i))$ in the above formula as follows. By applying the formula for K_n,

$$K_n(c_1(X)\cdots c_n(X)) - \mu K_n(c_1(Y)\cdots c_n(Y))$$

is expressed by the form

$$l^{-1}\{K_n(c_1+l, c_2+c_1l, \cdots, c_n+c_{n-1}l) - K_n(c_1,\cdots,c_n)\},$$

which is

$$c_1(L_B)^{-1}[K(TB\oplus L_B) - K(TB)]_n.$$

Thus we see that

$$\begin{aligned}[K(TB\oplus L_B) - K(TB)]_n &= [K(TB)K(L_B) - K(TB)]_n \\ &= \sum_{j=0}^{n-1} K_j(c_1,\cdots,c_j)K_{n-j}(l,0,\cdots,0).\end{aligned}$$

In the above we denote by $[\quad]_n$ the term of degree n. For the characteristic power series of K, we set

$$Q(x) := K(1+x) = 1 + \sum_{j=1}^{\infty} A_j x^j.$$

Then we set $l = c_1(L_B)$ and we have

$$K_j(c_1(L_B), 0, \cdots, 0) = A_j c_1(L_B)^j.$$

Therefore we obtain the following formula.

Theorem 1.2 (Riemann-Hurwitz formula for K-genus)**.** *The assumption on settings is the same as above. Let K be a multiplicative sequence and $Q(x) := K(1+x) = 1+\sum_{j=1}^{\infty} A_j x^j$ the characteristic power series of K. Then we have*

$$\begin{aligned}&K_n(c_1(X)\cdots c_n(X)) - \mu K_n(c_1(Y)\cdots c_n(Y)) \\ &\quad = \sum_i \sum_{\alpha=0}^{n-1} A_\alpha \left(\sum_t \frac{n_{i_t}(1-(r_{i_t}+1)^{\alpha+1})}{(r_{i_t}+1)^\alpha}\right) K_{n-\alpha-1}(c_1(B_i),\cdots,c_{n-1}(B_i)) \\ &\qquad\qquad \cdot c_1(L_{B_i})^\alpha \frown [B_i].\end{aligned}$$

2. Examples

2.1. **Euler number.** The result for the top Chern class implies the generalized Riemann-Hurwitz formula

$$\chi(X) - \mu\cdot\chi(Y) = -\sum_i b_i\cdot\chi(B_i),$$

which is a special case of the formula proved by Y. Yomdin [Y].

2.2. **Signature of surface.** From the fact that the signature of the surface is expressed by L_1, we see that the residue of L_1 consists only of the self-intersection number of B. Thus we have the formula for signature:

Theorem 2.1 (The formula for signature for ramified coverings)**.** *Let $f\colon X \longrightarrow Y$ be a ramified covering between compact complex analytic surfaces with covering multiplicity μ, $R_f = \sum_i r_i R_i$ the ramification divisor of f, and $B_f = \sum_i b_i B_i$ the branch locus of f. We assume that the ramification divisor and irreducible components B_i of the branch locus B_f are all non-singular. Then*

$$\begin{aligned} Sign(X) - \mu \cdot Sign(Y) &= \frac{1}{3}(p_1(X) - \mu \cdot p_1(Y)) \\ &= -\sum_i \frac{n_{i_t} r_{i_t}(r_{i_t}+2)}{3(r_{i_t}+1)} \int_{B_i} c_1^2(B_i) \\ &= -\sum_i \frac{n_{i_t} r_{i_t}(r_{i_t}+2)}{3(r_{i_t}+1)} B_i \cdot B_i\ . \end{aligned}$$

Originally, the formula for signature for cyclic coverings is formulated for 4-manifolds as follows.

Theorem 2.2 (Hirzebruch [Hz2])**.** *Let X be a compact oriented differentiable manifold of dimension 4 without boundary on which the cyclic groups G_n of order n act by orientation-preserving diffeomorphisms. Suppose that Y is a differential submanifold of X, not necessarily connected, and has codimension 2, and that G_n operates freely on $X - Y$. Then*

$$Sign(X) - n \cdot Sign(X/G_n) = -\frac{n^2-1}{3n} Y' \cdot Y'$$

where Y' is the branch locus in X/G_n.

This is a particular case of the above formula for signature for ramified coverings, the case that $r = b = n - 1$.

2.3. **Todd genus.** The Todd polynomials T_j are determined by the characteristic power series

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} B_k x^{2k}$$

where B_k are the Bernoulli numbers. As an example, let us compute the formula for T_6. The first few terms of the above series are:

$$Q(lx) = 1 + \frac{l}{2}x + \frac{l^2}{12}x^2 - \frac{l^4}{720}x^4 + \frac{l^6}{27240}x^6 - \cdots ,$$

which implies the following formula:

$$\begin{aligned} T_6(Y) - \mu \cdot T_6(X) &= -\sum_i n_{i_t} r_{i_t} \frac{T_5\,(B_i)}{2} \\ &+ \sum_i \sum_t \frac{n_{i_t} r_{i_t} (1 - (r_{i_t}+1)^2)}{(r_{i_t}+1)} \int_{B_i} \frac{T_4(B_i) \smile c_1(N_{B_i})}{12} \\ &- \sum_i \sum_t \frac{n_{i_t} r_{i_t} (1 - (r_{i_t}+1)^4)}{(r_{i_t}+1)^3} \int_{B_i} \frac{T_2(B_i) \smile c_1^3(N_{B_i})}{720} \\ &+ \sum_i \sum_t \frac{n_{i_t} r_{i_t} (1 - (r_{i_t}+1)^6)}{(r_{i_t}+1)^5} \int_{B_i} \frac{c_1^5(N_{B_i})}{27240}. \end{aligned}$$

2.4. **L-genus.** The L polynomials L_j are determined by the characteristic power series:

$$Q(x) = \frac{\sqrt{x^2}}{\tanh \sqrt{x^2}} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} B_k}{(2k)!} B_k x^{2k}.$$

We also compute the formula for the 6-dimensional case, that is, for L_3. The first few terms of the expansion are:

$$Q(lx) = 1 + \frac{l^2}{3} x^2 - \frac{l^4}{45} x^4 + \frac{2l^6}{945} x^6 + \cdots.$$

Thus we have

$$\begin{aligned} L_3(X) - \mu \cdot L_3(Y) &= -\sum_i \sum_t \frac{n_{i_t} r_{i_t} (1 - (r_{i_t}+1)^2)}{(r_{i_t}+1)} \int_{B_i} \frac{L_2(B_i)}{3} \smile c_1(L_{B_i}) \\ &- \sum_i \sum_t \frac{n_{i_t} r_{i_t} (1 - (r_{i_t}+1)^4)}{(r_{i_t}+1)^3} \int_{B_i} \frac{L_1(B_i)}{45} \smile c_1(L_{B_i})^3 \\ &- \sum_i \sum_t \frac{n_{i_t} r_{i_t} (1 - (r_{i_t}+1)^6)}{(r_{i_t}+1)^5} \int_{B_i} \frac{2c_1(L_{B_i})^5}{945}\ . \end{aligned}$$

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