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ISOSPECTRAL POLYGONS, PLANAR GRAPHS AND HEAT CONTENT

PATRICK MCDONALD AND ROBERT MEYERS

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Abstract. Given a pair of planar isospectral, nonisometric polygons constructed as a quotient of the plane by a finite group, we construct an associated pair of planar isospectral, nonisometric weighted graphs. Using the natural heat operators on the weighted graphs, we associate to each graph a heat content. We prove that the coefficients in the small time asymptotic expansion of the heat content distinguish our isospectral pairs. As a corollary, we prove that the sequence of exit time moments for the natural Markov chains associated to each graph, averaged over starting points in the interior of the graph, provides a collection of invariants that distinguish isospectral pairs in general.

1. Introduction

Let $D \subset \mathbb{R}^2$ be a piecewise smoothly bounded domain with compact closure. Let Δ be the Laplace operator acting on smooth functions on D with Dirichlet boundary conditions. We denote by $\mathrm{spec}(D)$ the Dirichlet spectrum of D, with nonnegative elements listed in increasing order, with multiplicity.

In his often cited 1965 paper, Mark Kac popularized a fundamental problem of planar spectral geometry: Does $\mathrm{spec}(D)$ determine D up to isometry? The problem was settled by Gordan, Webb, and Wolpert [GWW], who constructed a pair of nonisometric, isospectral planar polygons. In 1994 Buser, Conway, Doyle and Semmler [BCDS] gave an elegant and straightforward construction of families of isospectral nonisometric planar polygonal pairs (we will abbreviate references to such pairs by INIPP). Their constructions include a simplified version of the example of [GWW] as a special case, as well as the first example of a pair of isospectral planar domains all of whose normalized eigenfunctions agree at a pair of distinguished interior points (so-called *homophonic domains*). These examples are generated by a "seed" triangle together with a collection of congruent "reflection progeny" triangles produced by a sequence of reflections across edges (cf. section 3).

One might summarize the above by saying that, for piecewise smooth planar domains, the Dirichlet spectrum provides an incomplete collection of geometric invariants. Such a summary suggests that to construct a good collection of geometric invariants, one might be well served by finding invariants which distinguish

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INIPP. This paper is a first attempt to address such a search. More precisely, we show that in the category of weighted graphs and their associated combinatorial Laplacians, there exist weighted graph analogs of INIPP which are isospectral but not isomorphic, and that these graph pairs are distinguished by their *heat content asymptotics.*

To state our results more precisely requires the introduction of a significant amount of machinery, including the discrete analogs for boundary value problems (cf. section 2) as well as discrete analogs of INIPP (cf. section 3). Referencing these sections, we give concise statements of our results, the first of which is

Theorem 1.1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the underlying graphs associated to the INIPP given as the principal example in* [BCDS]*. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the weighted graphs obtained by applying the line graph construction (cf. section 3) to the pair $\mathcal{P}_1$ and $\mathcal{P}_2$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are isospectral weighted graphs that are not isomorphic.*

The existence of pairs of isospectral nonisomorphic weighted graphs is old news (cf. [H] and the text [CRS]). The content of Theorem 1.1 is that there is a natural method for producing combinatorial analogs of the examples produced by [BCDS]. Our second theorem provides a geometric method for distinguishing such pairs. We recall the necessary background:

Given a weighted graph G suppose that D is a (finite) subgraph of G with nonempty interior and nonempty boundary. Let Δ_{iD} be the Laplace operator associated to the interior of D and let $P_t = e^{t\Delta_{iD}}$ be the associated heat operator. Let $u(x,t)$ be defined by

$$u(x,t) \quad = \quad P_t(1) \tag{1.1}$$

where 1 is the indicator function for the interior of D. The heat content of D at time $t > 0$ is

$$q(t) \quad = \quad \sum_{x\in iD} u(x,t)W_V(x) \tag{1.2}$$

where $W_V(x)$ denotes the vertex weighting (cf. section 2). As $t \to 0^+$, there is an expansion of $q(t)$ of the form

$$q(t) \quad \simeq \quad \sum_{k=0}^{\infty} q_k t^k. \tag{1.3}$$

We call the coefficients which occur on the right-hand side of (1.3) the *heat content asymptotics of D* and we write

$$\operatorname{hca}(D) \quad = \quad \{q_k\}_{k=0}^{\infty}.$$

We can now state our second result:

Theorem 1.2. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be INIPP given in Theorem 1.1 and suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are the weighted graphs obtained by applying the line graph construction. Then* $\operatorname{hca}(\mathcal{G}_1)$ *and* $\operatorname{hca}(\mathcal{G}_2)$ *are different. More precisely,*

$$q_4(\mathcal{G}_1) \quad \neq \quad q_4(\mathcal{G}_2).$$

Associated to the graphs constructed above are natural Markov chains with generators given by the corresponding Laplace operators. From Theorem 1.2 and previous work involving the relationship between heat content and exit time for random walks associated to the combinatorial Laplace operator, we show that the moments of the exit time averaged over starting points in the domain determine the heat content asymptotics (Proposition 2.7). We deduce a corollary which states that average exit time moments distinguish the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ above (cf. Corollary 4.1).

Theorem 1.2 proves that the heat content can be used to distinguish the isospectral graphs which arise as analogs of isospectral polygons constructed in [BCDS]. This provides good reasons to believe that the same might be true in the continuous case. Preliminary results which clarify the relationship of the heat content to Dirichlet spectrum in the continuous case can be found in [MM2].

This paper is organized as follows: In the second section we recall the necessary facts from graph theory, discrete potential theory and the theory of Markov chains. In the third section we review the construction of [BCDS], introduce the line graph constructions cited in Theorem 1.1 and Theorem 1.2, and work through a specific example in detail. In the final section we provide proofs of our results.

2. Background and notation

In this section we fix notation and we develop the necessary background concerning discrete potential theory, discrete boundary value problems, and Markov chains. We make no claims to originality for the results of this section; the material has reached a high degree of development and there are many sources which provide proofs for the claims made in the text which follows. Proofs using the notation we adopt can be found in [MM1].

Let $G = (V, E)$ be a connected bidirected graph with vertex set V and edge set $E \subset V \times V$. Given $e \in E$, we will represent e as an ordered pair $e = (x, y)$ where $x,\ y \in V$ (y is the terminal point of e and x is the initial point of e and the ordering fixes a direction for e). We will say that a vertex x is incident with an edge e if x is a terminal point of e or x is an initial point of e. For x a vertex, we denote by $\deg(v)$ the degree of v: $\deg(v)$ = number of e such that x is incident with e. We will restrict our attention to graphs which admit no self-edges (i.e., edges of the form (x, x)). We will denote the statement "x is adjacent to y" by $x \sim y$.

Given an edge $e = (x, y)$, we will denote the *opposite edge* (y, x) by e^{-1}. An *orientation* of a bidirected graph G is a subset $\mathcal{O} \subset E$ satisfying $e \in \mathcal{O}$ if and only if $e^{-1} \notin \mathcal{O}$.

A vertex weighting (or volume) on a bidirected graph $G = (V, E)$ is a function $W_V : V \to \mathbb{R}^+$. An edge weighting on a bidirected graph $G = (V, E)$ is a function $W_E : V \times V \to \mathbb{R}^{\geq 0}$ which is supported on E.

Definition 2.1. Suppose that G is a connected bidirected graph with orientation $\mathcal{O}$, vertex weight W_V, and edge weight W_E. We say that the pair $W = (W_V, W_E)$ is a weighting for G if

$$W_E(x, y)W_V(x) = W_E(y, x)W_V(y) \tag{2.1}$$

for all $x, y \in V$. Given a weighting W, we associate to G a second vertex weighting, $w_V(x)$, called the natural auxiliary weighting, defined by

$$w_V(x) = \sum_{(x,y)\in E} W_E(x, y). \tag{2.2}$$

A triple $(G, \mathcal{O}, W)$ where G is a connected bidirected graph, $\mathcal{O}$ is an orientation of G and W is a weighting for G is called a *graph with geometry*.

We will denote by $C^0(G)$ the vector space of real valued functions on V and by $C^1(G)$ the vector space of real valued functions on E. There is a natural coboundary operator $d : C^0(G) \to C^1(G)$ defined by

$$df((x,y)) = f(y) - f(x).$$

Let $C_0^0(G) \subset C^0(G)$ and $C_0^1(G) \subset C^1(G)$ be the subspaces consisting of those functions with compact support.

Let $(G, \mathcal{O}, W)$ be a graph with geometry. The weighting W gives rise to a pair of inner products

$$\langle \cdot, \cdot \rangle_V : C_0^0(G) \times C_0^0(G) \to \mathbb{R},$$
$$\langle \cdot, \cdot \rangle_E : C_0^1(G) \times C_0^1(G) \to \mathbb{R}$$

defined by

$$\langle f, g \rangle_V = \sum_{x \in V} f(x) g(x) W_V(x), \tag{2.3}$$

$$\langle F, G \rangle_E = \sum_{x \in V} \sum_{(x,y) \in \mathcal{O}} F(x,y) G(x,y) W_E(x,y) W_V(x). \tag{2.4}$$

These inner products give rise to L^2 spaces of functions on vertices and edges. In addition, there is an adjoint map

$$d_W^* : C_0^1(G) \to C_0^0(G)$$

which in turn gives rise to the (vertex) Laplace operator $\Delta : C_0^0(G) \to C_0^0(G)$ defined by

$$\Delta = d_W^* d.$$

The action of the Laplacian can be characterized as a weighted average: If $f \in C^0(G)$ and $x \in V$, then

$$-\Delta f(x) = \sum_{y \in V} f(y) W_E(x,y) - w_V(x) f(x)$$

where W_E is the edge weight function and w_V is the auxiliary vertex weight function (2.2).

There is a natural theory of boundary value problems for graphs and subgraphs. In fact, suppose that $G = (V, E)$ is a graph, $(G, \mathcal{O}, W)$ a graph with geometry. If $A \subset V$, we define

$$C^0(G, A) = \{ f \in C^0(G) : f|_{G \setminus A} = 0 \}.$$

There are natural isomorphisms $J_A : C^0(A) \to C^0(G, A)$ which induce inclusions:

$$I_A : C^0(A) \to C^0(G)$$

and projections:

$$P_A : C^0(G) \to C^0(A).$$

The vertex weighting W_V induces a weighting on A and an inner product on compactly supported functions:

$$\langle f, g \rangle_A = \langle I_A f, I_A g \rangle_V .$$

Definition 2.2. Suppose that $G = (V, E)$ is a bidirected graph. Suppose that $D = (V', E')$ is a bidirected subgraph of G. We say that $x \in V'$ is an interior vertex of D if for all $y \in V$, $(x, y) \in E, \Rightarrow y \in V'$ and $(x, y) \in E'$. We denote the collection of all interior vertices of D by iD. We call all vertices of D which are not interior vertices of D *boundary vertices* of D. We denote the collection of all boundary vertices of D by ∂D. A *domain* of G is a finite connected bidirected subgraph of G with nonempty interior vertex set.

Note that if $(G, \mathcal{O}, W)$ is a graph with geometry and D is a domain of G, then $\mathcal{O}$ induces (by restriction) an orientation, $\mathcal{O}'$, of D, and W induces (by restriction) a weighting, W', for D which coincides with W at all vertices of D. The triple $(D, \mathcal{O}', W')$ gives rise to a Laplace operator, $\Delta_D : C^0(D) \to C^0(D)$, the induced Laplace operator associated to D. There is a natural description of the action of the Laplacian: If $f \in C^0(D)$ and x is an interior vertex x of D, then

$$\Delta_D f(x) = \Delta I_D f(x) \tag{2.5}$$

where I_D is the natural inclusion, $I_D : C^0(D) \to C^0(G)$.

Note that it is not necessarily the case that (2.5) holds at boundary vertices: the respective auxiliary weightings at a boundary vertex may not coincide. We can give a description of the action of the domain Laplacian as follows: enumerate the vertices of D with the first m vertices interior, the remaining $N - m$ vertices boundary. For $x \in D$, let δ_x be the indicator function of the vertex x. Then $\{\delta_{x_i}\}_{i=1}^N$ is a basis for $C^0(D)$ and the domain Laplacian with respect to this basis has the form

$$\Delta_D = \begin{pmatrix} \Delta_{iD,iD} & \Delta_{\partial D,iD} \\ \Delta_{iD,\partial D} & \Delta_{\partial D,\partial D} \end{pmatrix} \tag{2.6}$$

where the action of the operators $\Delta_{A,B}$ is given by

$$\Delta_{A,B} = P_B \Delta_D I_A; \tag{2.7}$$

here I_A and P_B are the natural inclusion and projection maps. Of particular interest is the interior Laplace operator acting on functions on iD, $\Delta_{iD,iD} : C^0(iD) \to C^0(iD)$. For notational ease, we will write $\Delta_{iD,iD} = \Delta_{iD}$. Again, there is a natural description for the action of the interior Laplacian: If $f \in C^0(iD)$ and $x \in iD$, then

$$-\Delta_{iD} f(x) = \sum_{y \in iD} f(y) W_E(x, y) - w_V(x) f(x). \tag{2.8}$$

It is straightforward to check that the interior Laplacian is self-adjoint.

There is a natural Markov chain associated to any graph with geometry, with the Laplace operator serving as the infinitesimal generator of the chain. The reversibility condition given in (2.1) of Definition 2.1 insures that the vertex weighting W_V is a stationary distribution for this process. A straightforward computation gives that the expected time required to leave a vertex x is $\frac{1}{w_V(x)}$, where w is the auxiliary weighting (cf. Definition 2.2). Similarly, the probability of moving from vertex x to vertex y, given that vertex x has just been vacated, is $\frac{W_E(x,y)}{w_V(x)}$.

Suppose that D is a domain in G with nonempty boundary. Let P_t be the corresponding transition operator for the natural Markov chain on the interior of

D, killed on exiting iD:

$$\begin{aligned} P_t &= e^{t\Delta_{iD}} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\Delta_{iD}^n. \end{aligned} \tag{2.9}$$

We can use the transition operator P_t given by (2.9) to solve the discrete version of the heat equation on D. More precisely, let 1 be the indicator function of the interior of the domain D and set

$$u(x,t) \quad = \quad P_t 1. \tag{2.10}$$

Then u solves the initial value problem

$$\begin{aligned} \Delta u &= \frac{\partial u}{\partial t} \text{ on } iD \times (0,\infty), \\ u(x,0) &= 1 \text{ on } iD. \end{aligned} \tag{2.11}$$

Definition 2.3. Let $(G,\mathcal{O},W)$ be a graph with geometry and suppose that $D \subset G$ is a domain with nonempty boundary. Let $u(x,t)$ be as in (2.10). The heat content of D is the function $q : [0,\infty) \to \mathbb{R}$ defined by

$$q(t) \quad = \quad \langle 1, u(\cdot,t)\rangle_{iD}. \tag{2.12}$$

There is a small time expansion:

$$q(t) \quad \simeq \quad \sum_{n=0}^{\infty} q_n t^n. \tag{2.13}$$

We call the coefficients q_n appearing in (2.13) the heat content asymptotics of D.

Let δ_x be the indicator function for the vertex x and set

$$p(x,t) \quad = \quad \left\langle \frac{\partial u}{\partial t}, \delta_x \right\rangle, \tag{2.14}$$

the pairing being the inner product described in (2.3). Then $p(x,t)$ is the probability density function for the exit time from iD, denoted by τ, of the natural Markov chain, given that we start at the point x. We define a sequence of invariants associated to the domain D by computing exit time moments and integrating over the domain:

Definition 2.4. Let $(G,\mathcal{O},W)$ be a graph with geometry, and D a domain of G with nonempty boundary. For k a natural number, set

$$A_k \quad = \quad \left\langle 1, \int_0^{\infty} t^k p(x,t)\,dt \right\rangle_{iD}. \tag{2.15}$$

We call the sequence $\{A_k\}$ the moment spectrum associated to the domain D, and we write $\{A_k\} = \mathrm{mspec}(D)$.

Repeated integration by parts exhibits the elements of the moment spectrum in terms of iterated solutions of Poisson problems (cf. [MM1]). We record this as a proposition:

Proposition 2.5. *Let $(G,\mathcal{O},W)$ be a graph with geometry, and D a domain of G with nonempty boundary. For k a natural number and A_k as in* (2.15),

$$A_k \quad = \quad \left\langle 1, (-1)^k k! \Delta^{-k} 1 \right\rangle_{iD}.$$

From Proposition 2.5, it is immediate that the elements A_k are invariant under automorphisms of the ambient graph G.

For D as above, let $\operatorname{spec}(D)$ denote the spectrum of Δ_{iD}. For $\lambda \in \operatorname{spec}(D)$, let $\mathcal{E}_\lambda(1)$ be the orthogonal projection of the function 1 on the eigenspace corresponding to λ. Let

$$a_\lambda^2 \quad = \quad \langle \mathcal{E}_\lambda(1), \mathcal{E}_\lambda(1) \rangle_{iD} \tag{2.16}$$

and define the set $\operatorname{spec}^*(D)$ by

$$\operatorname{spec}^*(D) \quad = \quad \{\lambda \in \operatorname{spec}(D) : a_\lambda^2 \neq 0\}. \tag{2.17}$$

Decomposing $u_k = (-1)^k k! \Delta_{iD}^{-k} 1$ in terms of an orthonormal basis for Δ_{iD} we have (as in [MM1])

$$A_k \quad = \quad (-1)^k k! \sum_{\lambda \in \operatorname{spec}^*(D)} a_\lambda^2 \left(\frac{1}{\lambda}\right)^k . \tag{2.18}$$

We conclude, using the theory of classical moment problems (as in [MM1], [MM2]),

Proposition 2.6. *Let $D,\ D'$ be domains in G with nonempty boundary. Then*

$$\operatorname{mspec}(D) = \operatorname{mspec}(D') \quad \textit{implies} \quad \operatorname{spec}^*(D) = \operatorname{spec}^*(D'). \tag{2.19}$$

Similarly, $\operatorname{mspec}(D)$ *determines the constants* $\{a_\lambda^2 : \lambda \in \operatorname{spec}^*(D)\}$.

As a consequence of Proposition 2.6, we have (as in [MM1]):

Proposition 2.7. *Let $D,\ D'$ be domains in G with nonempty boundary. Then*

$$\operatorname{mspec}(D) = \operatorname{mspec}(D') \quad \textit{implies} \quad \operatorname{hca}(D) = \operatorname{hca}(D'). \tag{2.20}$$

The moment spectrum contains a great deal of information related to the spectral (and, in the smooth case, Riemannian) geometry of D. For example, in the smooth case it is a theorem that the moment spectrum determines the heat content, and for generic domains in Euclidean space, the moment spectrum determines the Dirichlet spectrum [MM2].

3. Construction of examples

We begin by recalling the construction of the isospectral polygons following [BCDS].

Fix an equilateral triangle $\mathcal{T}$ of side length l, orient $\mathcal{T}$ in the plane, and label the edges of $\mathcal{T}$ as $a,\ b,\ c$. Reflect $\mathcal{T}$ across each of its edges creating three new triangles, each congruent to $\mathcal{T}$, each bordering $\mathcal{T}$, and each labelled via the reflection. For each new triangle, there are two edges which do not border $\mathcal{T}$. Using the orientation, for each new triangle, choose the edge following the border edge and reflect, creating another triple of labelled triangles, each congruent to $\mathcal{T}$. Note that the procedure (a choice of seed and a choice of a sequence of edges) determines a planar polygon, $\mathcal{P}_1$. If we reverse the orientation of the seed triangle and follow the same procedure, we obtain a second planar polygon, $\mathcal{P}_2$, which is a mirror image of $\mathcal{P}_1$. Summarizing, our procedure defines a pair of isometric planar polygons, each constructed as a collection of congruent *defining triangles* (in fact, seven such triangles - see Figure 1).

We will perturb the construction of the polygons $\mathcal{P}_1$ and $\mathcal{P}_2$ to produce polygons which are isospectral but not isometric. The perturbation is straightforward: allow

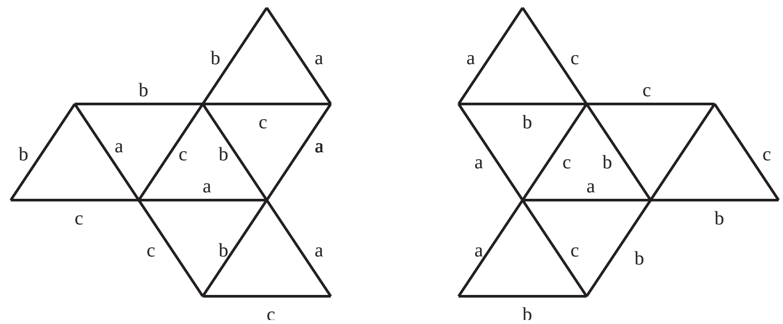

FIGURE 1. Isospectral planar polygons

the labels $a,\ b,\ c$ to be edge lengths for an acute scalene triangle $\mathcal{T}_{abc}$ which we orient in the plane. We construct polygons consisting of seven congruent copies of $\mathcal{T}_{abc}$ by reflection across edges as outlined above. Using transplantation, it is then easy to see that $\mathcal{P}_{1abc}$ and $\mathcal{P}_{2abc}$ are isospectral.

As it is clear that the seed triangles of each configuration are translations of each other, it is clear that the polygons are not isometric (cf. [BCDS]).

We now construct weighted graphs for each isospectral polygon in Figure 1. We begin by noting that each polygon has the structure of a weighted graph: vertices and edges are those which appear in the figure, edge weights are as labeled, and vertex weights are all set to the same constant. We follow an algorithm which we label *the construction of the associated weighted line graph*:

Definition 3.1. Given a polygon, $\mathcal{P}_{iabc}$, generated by reflections of a seed triangle as above, construct a weighted graph $\mathcal{G}_{iabc}$ as follows:

- For each edge present in the construction of $\mathcal{P}_{iabc}$, introduce a vertex.
- If x and y are vertices in $\mathcal{G}_{iabc}$ with x arising from edge l_x and y arising from edge l_y, then $e = (x, y)$ defines an edge in $\mathcal{G}_{iabc}$ if and only if the edges l_x and l_y in $\mathcal{P}_{iabc}$ are sides of the same triangle.
- If $e = (x, y)$ is an edge in $\mathcal{G}_{iabc}$ with x arising from edge l_x and y arising from edge l_y, then e carries the weight equal to the length of the remaining edge of the triangle in $\mathcal{P}_{iabc}$ associated to l_x and l_y.
- The vertex weighting is constant.

We note that the condition defining the edge structure insures that the edge weighting is always well defined.

Applying the line graph construction to the examples in Figure 1, we obtain the weighted graphs of Figure 2. We note that the weighted graphs $\mathcal{G}_{iabc}$ carry the structure of weighted graphs described in section 2 above. In particular, each graph carries a Laplace operator and thus each graph admits an associated Dirichlet spectrum for its interior.

Figure 3 gives the corresponding interiors of $\mathcal{G}_{iabc}$. References (4.1) and (4.2) give the corresponding interior Laplace operators, with vertex labelings given by Figure 3.

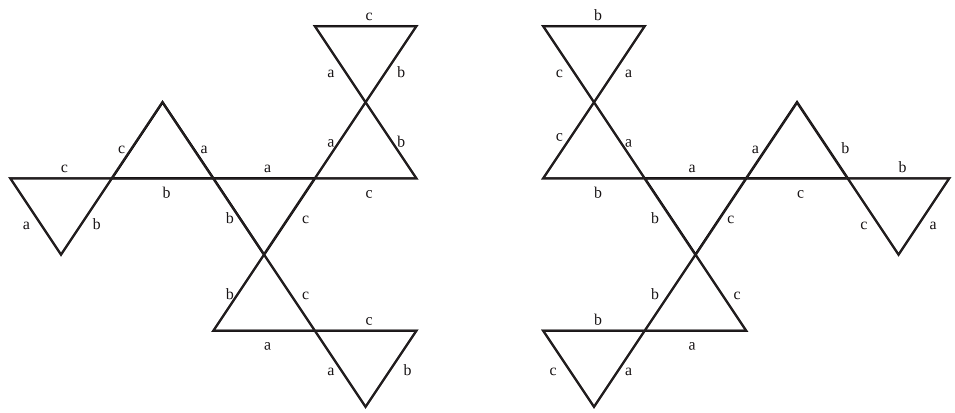

FIGURE 2. Isospectral graphs associated to isospectral polygons

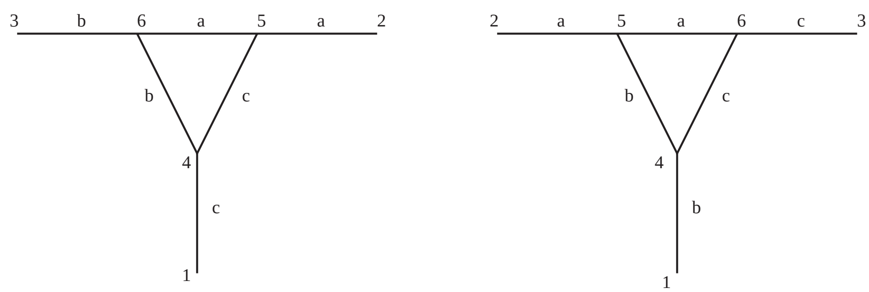

FIGURE 3. Associated interior domains for isospectral graphs

4. Proof of Theorem 1.1 and Theorem 1.2

We use the machinery of section 2 and the constructions of section 3 to prove the main results.

Proof of Theorem 1.1. We call two weighted graphs isomorphic if there is a vertex bijection which respects the edge structure and weighting.

We note that the graphs dual to those in Figure 1 are not isomorphic. To see this, note that an isomorphism must carry the interior of one graph to the interior of the other. It is then easy to see that the central triangles must be mapped to each other. There is no such map that can be extended to the remainder of the graph which preserves the edge weighting.

We check that $\mathcal{G}_1 = \mathcal{G}_{1abc}$ and $\mathcal{G}_2 = \mathcal{G}_{2abc}$ are isospectral.

From Figure 3 we can use (2.8) to write down the corresponding Laplace operators on the interiors of the respective domains:

$$\Delta_1 = -2\begin{pmatrix} (a+c) & 0 & 0 & \frac{-c}{2} & 0 & 0 \\ 0 & (a+b) & 0 & 0 & \frac{-a}{2} & 0 \\ 0 & 0 & (b+c) & 0 & 0 & \frac{-b}{2} \\ \frac{-c}{2} & 0 & 0 & (b+c) & \frac{-c}{2} & \frac{-b}{2} \\ 0 & \frac{-a}{2} & 0 & \frac{-c}{2} & (a+c) & \frac{-a}{2} \\ 0 & 0 & \frac{-b}{2} & \frac{-b}{2} & \frac{-a}{2} & (a+b) \end{pmatrix}, \tag{4.1}$$

$$\Delta_2 = -2\begin{pmatrix} (a+b) & 0 & 0 & \frac{-b}{2} & 0 & 0 \\ 0 & (a+c) & 0 & 0 & \frac{-a}{2} & 0 \\ 0 & 0 & (b+c) & 0 & 0 & \frac{-c}{2} \\ \frac{-b}{2} & 0 & 0 & (b+c) & \frac{-b}{2} & \frac{-c}{2} \\ 0 & \frac{-a}{2} & 0 & \frac{-b}{2} & (a+b) & \frac{-a}{2} \\ 0 & 0 & \frac{-c}{2} & \frac{-c}{2} & \frac{-a}{2} & (a+c) \end{pmatrix}. \tag{4.2}$$

By inspection, the corresponding characteristic polynomials are equal, thus proving that the domains are isospectral.

Proof of Theorem 1.2. From (4.2) and (4.1) we can compute the heat content asymptotics. Setting $W_V(x) = 1$ for all $x \in \mathcal{G}_{iabc},\ i = 1, 2$, we have

$$\begin{aligned} q_{0,1} = q_{0,2} &= 6, \\ q_{1,1} = q_{1,2} &= -4(a+b+c), \\ q_{2,1} = q_{2,2} &= 3(a^2+b^2+c^2) + 2(ab+ac+bc), \\ q_{3,1} = q_{3,2} &= -2(a^3+b^3+b^2c+bc^2+c^3+a^2(b+c)+a(b^2-bc+c^2)) \end{aligned}$$

and

$$q_{4,1} - q_{4,2} = \frac{8}{4!}(c-b)(a-c)(a-b)(a+b+c).$$

This proves Theorem 1.2

By Proposition 2.7, we have an immediate corollary:

Corollary 4.1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be as above. Then*

$$\mathrm{mspec}(\mathcal{G}_1) \neq \mathrm{mspec}(\mathcal{G}_2).$$

Final remarks

There is a discrete analog of the method of eigenfunction transplantation which can be used to see that the graphs above are isospectral. We have studied several of the examples constructed in [BCDS]. We have found that results analogous to those we present here hold for the examples that we checked.

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Department of Mathematics, New College of Florida, Sarasota, Florida 34243
E-mail address: `ptm@virtu.sar.usf.edu`

The Courant Institute of Mathematical Sciences, New York, New York 10276-0907
E-mail address: `rjm243@nyu.edu`