

## THE DERKSEN INVARIANT VS. THE MAKAR-LIMANOV INVARIANT

ANTHONY CRACHIOLA AND STEFAN MAUBACH

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ABSTRACT. In this article it is shown that the Makar-Limanov invariant of a ring (or variety) can be trivial while the Derksen invariant is not, and vice versa.

### 1. INTRODUCTION

The Makar-Limanov invariant was introduced by Makar-Limanov in [5] to prove that the variety in  $\mathbb{C}^4$  given by the equation  $X^2Y + X + Z^2 + T^3$  is not isomorphic to  $\mathbb{C}^3$ . Later on, Derksen gave an alternative proof in [3] by introducing a different invariant. The general idea was that both invariants are kind of dual, in the sense that they can distinguish the same set of rings from the polynomial rings. However, this article gives examples in which this is not the case, thus stating that the invariants are clearly different.

### 2. DEFINITIONS AND NOTATIONS

In this section,  $R$  denotes a commutative finitely generated  $\mathbb{C}$ -algebra and  $\mathbb{N}$  the non-negative integers.

- Definition 2.1.**
- (i) A map  $D : R \rightarrow R$  is called a *derivation* if it satisfies the Leibniz rule:  $D(ab) = aD(b) + D(a)b$  for all  $a, b \in R$ .
  - (ii) A derivation is called *locally nilpotent* if for each  $a \in R$  there exists  $n \in \mathbb{N}$  such that  $D^n(a) = 0$ .
  - (iii) When  $D$  is a locally nilpotent derivation, we denote by  $R^D$  the kernel of the map  $D$ , i.e.  $R^D := \{a \in R \mid D(a) = 0\}$ .
  - (iv)  $LND(R)$  is the set of all locally nilpotent derivations on  $R$ .
  - (v)  $LND^*(R) := LND(R) \setminus \{0\}$  (notice the zero map “0” is actually a derivation).
  - (vi)  $ML(R) := \bigcap_{D \in LND(R)} R^D$ , the Makar-Limanov invariant of  $R$ .<sup>1</sup>
  - (vii)  $HD(R)$  is the  $\mathbb{C}$ -algebra generated by  $\bigcup_{D \in LND^*(R)} R^D$ .<sup>2</sup>

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<sup>1</sup>The original notation introduced by Makar-Limanov himself was  $AK(R)$ , “absolute kernel”, and this notation is sometimes used too.

<sup>2</sup>This invariant is often denoted by “ $D(R)$ ” but since  $D$  is a very common notation for a derivation, the notation “ $HD$ ” (for Harm Derksen) came into fashion.

**Example 2.2.** If  $R = \mathbb{C}[X_1, \dots, X_n]$ , then  $ML(R) = \mathbb{C}$ , and in the case  $n \geq 2$ ,  $HD(R) = R$ . In the case  $n = 1$ ,  $HD(R) = \mathbb{C}$  (a small exception).

**Corollary 2.3.** If  $ML(R) \neq \mathbb{C}$  (i.e.  $ML(R)$  is larger than  $\mathbb{C}$ ), then  $R$  is not a polynomial ring. If  $\dim(R) \geq 2$  and  $HD(R) \neq R$ , then  $R$  is not a polynomial ring.

3. SPECIFIC RINGS AND THEIR INVARIANTS

**3.1. Makar-Limanov invariant trivial, Derksen invariant not.** In this section we will give a ring whose Makar-Limanov invariant is trivial but its Derksen invariant is not.

**Definition 3.1.** Define the ideal  $I := (X, Y) \subset \mathbb{C}[X, Y]$ , and let

$$\begin{aligned} R &:= \mathbb{C}[X^2, X^3, Y^3, Y^4, Y^5, X^{1+i}Y^{1+j} \mid i, j \in \mathbb{N}] \\ &= \mathbb{C}[X^2, X^3, Y^3, Y^4, Y^5, XY, X^2Y, XY^2, XY^3] \end{aligned}$$

(i.e.  $R = \mathbb{C} \oplus \mathbb{C}X^2 \oplus \mathbb{C}XY \oplus I^3$ ).

Notice that  $R$  is finitely generated, Noetherian, and a domain.

**Lemma 3.2.**  $ML(R) = \mathbb{C}$ .

*Proof.* Let  $D_1 := Y^3\partial_X$  and  $D_2 := X^2\partial_Y$ . These are locally nilpotent derivations on  $R$ , as can be easily checked. Then  $R^{D_1} = R \cap \mathbb{C}[X, Y]^{D_1} \subseteq \mathbb{C}[X, Y]^{D_1} = \mathbb{C}[Y]$ . Also  $R^{D_2} = R \cap \mathbb{C}[X, Y]^{D_2} \subseteq \mathbb{C}[X, Y]^{D_2} = \mathbb{C}[X]$ . Thus  $\mathbb{C} \subseteq ML(\mathbb{C}) \subseteq R^{D_1} \cap R^{D_2} \subseteq \mathbb{C}[X, Y]^{D_1} \cap \mathbb{C}[X, Y]^{D_2} = \mathbb{C}[Y] \cap \mathbb{C}[X] = \mathbb{C}$ .  $\square$

In order to calculate  $HD(R)$  we first show that every locally nilpotent derivation on  $R$  actually comes from a locally nilpotent derivation on  $\mathbb{C}[X, Y]$ .

**Lemma 3.3.** (i) *The integral closure of  $R$  in  $\mathbb{C}[X, Y]$  is  $\mathbb{C}[X, Y]$ .*  
 (ii) *The integral closure of  $R$  in  $Q(R)$  (the fraction field of  $R$ ) is  $\mathbb{C}[X, Y]$ .*

*Proof.* (i) is easy, since the integral closure of the smaller ring  $\mathbb{C}[X^2, Y^3]$  in  $\mathbb{C}[X, Y]$  is already  $\mathbb{C}[X, Y]$ .

(ii)  $Q(R) = \mathbb{C}(X, Y)$ . Let  $a \in Q(R)$  be integral over  $R$ . Then surely  $a$  is integral over  $\mathbb{C}[X, Y]$ . But  $\mathbb{C}[X, Y]$  is a UFD and thus integrally closed in its fraction field, i.e.,  $a \in \mathbb{C}[X, Y]$  already. Thus the integral closure of  $R$  in  $Q(R)$  is a subset of  $\mathbb{C}[X, Y]$ . Finally, since  $Q(R) = \mathbb{C}(X, Y)$ , we are done by part (i).  $\square$

Notice that if  $D$  is a derivation (not necessarily locally nilpotent) on a domain  $A$ , then it extends uniquely to a derivation on the fraction field  $Q(A)$  of  $A$ , by just forcing  $D(a^{-1}b) = a^{-2}(aD(b) - D(a)b)$  for all  $a \in A, b \in A \setminus \{0\}$ .

**Theorem 3.4** (Seidenberg). *Let  $A$  be a Noetherian domain containing  $\mathbb{Q}$ ,  $K$  its quotient field and  $\tilde{A}$  the integral closure of  $A$  in  $K$ . Let  $D$  be a derivation on  $A$  and  $\tilde{D}$  its unique extension to  $K$ . Then  $\tilde{D}(\tilde{A}) \subseteq \tilde{A}$ .*

This is quoted literally from [4], prop. 1.2.15, page 17, but it is originally from [10].

**Theorem 3.5** (Vasconcelos). *Let  $A \subseteq B$  be an integral extension where  $B$  is a domain and  $\mathbb{Q} \subset A$ . If  $D$  is a derivation on  $B$  such that  $DA \subseteq A$  and the restriction  $D|_A$  of  $D$  to  $A$  is locally nilpotent, then  $D$  is locally nilpotent.*

This is quoted literally from [4], prop. 1.3.37, page 29, but it is originally from [11].

**Lemma 3.6.** *If  $D$  is a locally nilpotent derivation on  $R$ , then it extends uniquely to a locally nilpotent derivation on  $\mathbb{C}[X, Y] \longrightarrow \mathbb{C}[X, Y]$ .*

*Proof.* The integral closure of  $R$  in  $Q(R)$  is  $\mathbb{C}[X, Y]$  (by 3.3). So by the above theorem of Seidenberg  $D$  extends uniquely to  $\mathbb{C}[X, Y]$ . By the above theorem of Vasconcelos we see that  $D$  is locally nilpotent. □

**Lemma 3.7.** *If  $f \in \mathbb{C}[X, Y]$  is a coordinate,  $p(T) \in \mathbb{C}[T]$  and  $p(f) \in R$ , then  $XY$  does not appear as a monomial in  $p(f)$ .*

*Proof.*  $f = f_0 + f_1 = f_0 + aX + bY + cX^2 + dXY + eY^2 + g$  for some  $g \in I^3$  and  $a \neq 0$  or  $b \neq 0$ . Now  $p(f) = q(f_1)$  for some  $q(T) \in \mathbb{C}[T]$ .

$$\begin{aligned} q(f_1) &= \lambda_0 + \lambda_1 f_1 + \lambda_2 f_1^2 + \dots + \lambda_n f_1^n \\ &= \lambda_0 + \lambda_1(aX + bY + cX^2 + dXY + eY^2) \\ &\quad + \lambda_2(aX + bY + cX^2 + dXY + eY^2)^2 + g' \quad g' \in I^3 \end{aligned}$$

and since  $a \neq 0$  or  $b \neq 0$  and  $q(f) \in R$ , we must have  $\lambda_1 = 0$ . Thus

$$\begin{aligned} q(f) &= \lambda_0 + \lambda_2(aX + bY + cX^2 + dXY + eY^2)^2 + g' \quad g' \in I^3 \\ &= \lambda_0 + \lambda_2(a^2X^2 + 2abXY + b^2Y^2) + g'' \quad g'' \in I^3, \end{aligned}$$

but since the monomial  $Y^2$  appears in no element of  $R$  and  $q(f) \in R$ , we must have  $\lambda_2 b^2 = 0$  which implies  $2\lambda_2 ab = 0$ , which is the coefficient of  $XY$ . □

**Lemma 3.8.** *Let  $D \in LND(R)$ . Suppose there exists  $g \in R^D$  such that the coefficient of  $XY$  of  $g$  is non-zero ( $XY$  appears in  $g$ ). Then  $D = 0$ .*

*Proof.* We know by Lemma 3.6 that  $D$  can be extended as a locally nilpotent derivation to  $\mathbb{C}[X, Y]$ . Suppose  $D \neq 0$ . Thus  $\mathbb{C}[X, Y]^D = \mathbb{C}[f]$  for some coordinate  $f$  by Rentschler's theorem [9]. Hence  $g = p(f) \in R^D$ . But now by Lemma 3.7, the coefficient of  $XY$  must be zero, a contradiction. Hence our assumption that  $D$  was non-zero was wrong, thus  $D = 0$ . □

**Lemma 3.9.**  $HD(R) \neq R$ .

*Proof.* If we show that  $XY \notin HD(R)$ , then we are done. Suppose  $g_1, \dots, g_n \in R$  are elements of kernels of non-zero locally nilpotent derivations such that  $XY = p(g_1, \dots, g_n)$  for some  $p \in \mathbb{C}[T_1, \dots, T_n]$ . Then since  $g_i \in R$  we have that  $g_i = c_i + a_i X^2 + b_i XY + h_i$  for some  $a_i, b_i, c_i \in \mathbb{C}, h_i \in (X^3, X^2Y, XY^2, Y^3)$ . We may assume that  $c_i = 0$ . Furthermore, by Lemma 3.8  $b_i = 0$ . Let  $p'$  be the part of  $p$  which is linear. Now  $XY = p'(a_1 X^2, \dots, a_n X^2) + h'$  for some  $h' \in (X^3, X^2Y, XY^2, Y^3)$ . This gives a contradiction. □

**3.2. Derksen invariant trivial, Makar-Limanov invariant not.** In this section we will give a class of rings with trivial Derksen invariant but non-trivial Makar-Limanov invariant. Let  $A$  be a commutative domain over  $\mathbb{C}$  with transcendence degree 1 such that  $A$  is not isomorphic to  $\mathbb{C}[X]$ . For example, take  $A$  to be the coordinate ring of a curve which is not isomorphic to the line. We will examine the ring  $R := A[X_1, \dots, X_n]$  for  $n \geq 2$ .

**Lemma 3.10.**  $HD(R) = R$ .

*Proof.* The kernels of the partial derivatives generate  $R$ . □

Of course, the same observation shows that a polynomial ring with at least two variables over any algebra always has trivial Derksen invariant. Lemma 3.10 and Example 2.2 are special cases. To show that  $ML(R) \neq \mathbb{C}$ , we will use the following

**Theorem 3.11** (Makar-Limanov). *Suppose  $A$  is a commutative domain over  $\mathbb{C}$  with transcendence degree 1. Then  $ML(A[X_1, \dots, X_n]) = ML(A)$  for each  $n \geq 1$ .*

This theorem provides an alternate proof of the Abhyankar-Eakin-Heinzer cancellation theorem for curves [1], in the characteristic zero case. For a proof of Theorem 3.11, see [6] or [7]. We also need the following standard facts. If  $S$  is a commutative domain over  $\mathbb{C}$  and  $D \in LND(S)$ , then  $S^D$  is algebraically closed. Moreover, if  $D(s) = 1$  for some  $s \in S$ , then  $S = S^D[s]$ . See, for example, [4] for proofs.

**Lemma 3.12.**  $ML(R) = A$ .

*Proof.* By Theorem 3.11,  $ML(R) = ML(A)$ . Suppose  $ML(A) \neq A$ , so that there exists a non-zero locally nilpotent derivation  $D$  on  $A$ . Suppose  $A^D \neq \mathbb{C}$ . Then  $\text{trdeg}_{\mathbb{C}}(A^D) = 1 = \text{trdeg}_{\mathbb{C}}(A)$ , and so every element of  $A$  is algebraically dependent over  $A^D$ . Since  $A^D$  is algebraically closed,  $A^D = A$ , contradicting the choice of  $D$ . Hence  $A^D = \mathbb{C}$ . Let  $a \in A$  such that  $D(a) \in A^D$  (always possible with a locally nilpotent derivation), say  $D(a) = \lambda$ . Let  $s = \lambda^{-1}a$ , so that  $D(s) = 1$ . Then  $A = \mathbb{C}[s]$ . This contradicts our original assumption on  $A$ .  $\square$

Notice that  $\text{trdeg}_{\mathbb{C}}(R) > 2$  for the class of rings  $R$  in this section. This is a necessary condition for an example of our type. Using the fact that the kernel of a locally nilpotent derivation is algebraically closed, one can show that if  $\text{trdeg}_{\mathbb{C}}(R) = 1$  or  $2$  and  $HD(R) = R$ , then  $ML(R) = \mathbb{C}$ .

**3.3. Derksen and Makar-Limanov invariants both non-trivial.** Finally, in this section we will give a class of rings whose Derksen and Makar-Limanov invariants are both non-trivial and are moreover the same. Let  $R$  be the ring over  $\mathbb{C}$  given by the equation  $x^n y = P(z)$ , where  $n > 1$  and  $\deg(P) > 1$ . Danielewski used surfaces of this type to give a negative answer to the generalized Zariski cancellation question [2]. We make use of the following

**Theorem 3.13.**  $LND(R) = x^n \mathbb{C}[x] \frac{\partial}{\partial z}$ , where  $R$  is viewed as a subring of  $\mathbb{C}[x, x^{-1}, z]$ .

For the proof, see [8]. In particular, the kernel of every locally nilpotent derivation on  $R$  is  $\mathbb{C}[x]$ . As a result, we have

**Lemma 3.14.**  $HD(R) = ML(R) = \mathbb{C}[x]$ .

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DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202  
E-mail address: [crach@math.wayne.edu](mailto:crach@math.wayne.edu)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIJMEGEN, TOERNOOIVELDT, 6525 ED  
NIJMEGEN, THE NETHERLANDS  
E-mail address: [stefanm@sci.kun.nl](mailto:stefanm@sci.kun.nl)