PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 11, Pages 3407–3410
S 0002-9939(03)07176-4
Article electronically published on June 19, 2003

# BANACH SPACES HAVING THE RADON-NIKODÝM PROPERTY AND NUMERICAL INDEX 1

MIGUEL MARTÍN

(Communicated by Jonathan M. Borwein)

ABSTRACT. Let $X$ be a Banach space with the Radon-Nikodým property. Then, the following are equivalent.

(i) $X$ has numerical index 1.

(ii) $|x^{**}(x^*)| = 1$ for all $x^* \in \mathrm{ex}(B_{X^*})$ and $x^{**} \in \mathrm{ex}(B_{X^{**}})$.

(iii) $X$ is an almost-CL-space.

(iv) There are a compact Hausdorff space $K$ and a linear isometry $J : X \to C(K)$ such that $|x^{**}(J^*\delta_s)| = 1$ for all $s \in K$ and $x^{**} \in \mathrm{ex}(B_{X^{**}})$.

If $X$ is a real space, the above conditions are equivalent to being semi-nicely embedded in some space $C(K)$.

The numerical index of a Banach space is a constant relating the norm and the numerical radius of operators on the space. Let us present the relevant definitions. For a Banach space $X$, we write $B_X$ for the closed unit ball and $S_X$ for the unit sphere. We denote by $X^*$ the dual space and by $L(X)$ the Banach algebra of all bounded linear operators on $X$. For such an operator $T$, the *numerical radius* of $T$ is

$$v(T) = \sup\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

The *numerical index* of the space $X$ is then given by

$$n(X) = \max\{k \geq 0 \ : k\,\|T\| \leq v(T) \ \ \forall\, T \in L(X)\}.$$

We refer the reader to the books [3, 4] and to the expository paper [13] for general information and background. Recent results can be found in [7, 12, 14, 15].

Let us mention here some facts concerning the numerical index which will be relevant to our discussion. First, one has $v(T^*) = v(T)$ for every $T \in L(X)$, where $T^*$ is the adjoint operator of $T$ (see [3, §9]), and it clearly follows that $n(X^*) \leq n(X)$. The question whether this is actually an equality seems to be open. Second, it is a classical result that $L$- and $M$-spaces have numerical index 1 [6].

The aim of this paper is to characterize Banach spaces with numerical index 1 among those having the Radon-Nikodým property (RNP for short; see [5] for background). To this end, we prove that some general sufficient conditions for a Banach space to have numerical index 1 are also necessary if the space has the RNP.

Received by the editors November 20, 2001.

2000 *Mathematics Subject Classification.* Primary 46B20, 47A12.

This research was partially supported by Spanish MCYT projects no. BFM2000-1467 and BFM2002-00061.

Let us introduce some definitions and comments. A Banach space $X$ is an *almost-CL-space* if $B_X$ is the absolutely closed convex hull of every maximal convex subset of $S_X$. This notion was introduced by Å. Lima [10], generalizing the concept of CL-space given by R. Fullerton [8] in 1960. Real and complex almost-CL-spaces have numerical index 1 (see [13, §4] or [1]), but it is not known whether the reciprocal result is true. As usual, we write $C(K)$ for the Banach space of all continuous functions from the compact Hausdorff space $K$ into the scalar field. Given $s \in K$, $\delta_s$ stands for the functional $f \longmapsto f(s)$ on $C(K)$. Finally, we write $\operatorname{co}(B)$ for the convex-hull of $B$ and, given a convex subset $A$ of $X$, $\operatorname{ex}(A)$ and $\operatorname{dent}(A)$ are, respectively, the set of extreme points and the set of denting points of $A$.

The main result of the paper is the following.

**Theorem 1.** *Let $X$ be a Banach space having the RNP. Then the following are equivalent.*

(*i*) $n(X) = 1$.
(*ii*) $|x^{**}(x^*)| = 1$ *for all* $x^* \in \operatorname{ex}(B_{X^*})$ *and* $x^{**} \in \operatorname{ex}(B_{X^{**}})$.
(*iii*) *$X$ is an almost-CL-space.*
(*iv*) *There are a compact Hausdorff space $K$ and a linear isometry $J : X \to C(K)$ such that* $|x^{**}(J^*\delta_s)| = 1$ *for all* $s \in K$ *and* $x^{**} \in \operatorname{ex}(B_{X^{**}})$.

*Proof.* The implication $(ii) \Rightarrow (i)$ is straightforward. Just use that $v(T) = v(T^*)$ for every $T \in L(X)$. The implication $(iii) \Rightarrow (i)$ has been discussed above.

$(iv) \Rightarrow (i)$. We fix $T \in L(X)$ and we have to prove that $v(T) = \|T\|$. For each $s \in K$, we take $x^{**} \in \operatorname{ex}(B_{X^{**}})$ such that

$$\left|x^{**}\big(T^*(J^*\delta_s)\big)\right| = \|T^*(J^*\delta_s)\|.$$

Since $|x^{**}(J^*\delta_s)| = 1$, we have

$$v(T) = v(T^*) \geq \|T^*(J^*\delta_s)\|$$

for each $s \in K$. It is clear that

$$\|T\| = \|T^*\| = \sup\{\|T^*(J^*\delta_s)\| \ : \ s \in K\}$$

and, so, $v(T) = \|T\|$ and $n(X) = 1$.

$(i) \Rightarrow (ii)$. We use [12, Lemma 1] to get $|x^*(x)| = 1$ for all $x \in \operatorname{dent}(B_X)$ and $x^* \in \operatorname{ex}(B_{X^*})$. Since $X$ has the RNP, it may be concluded that

$$\operatorname{ex}(B_{X^{**}}) \subseteq \overline{\operatorname{dent}(B_X)}^{w^*},$$

so $|x^{**}(x^*)| = 1$ for all $x^* \in \operatorname{ex}(B_{X^*})$ and $x^{**} \in \operatorname{ex}(B_{X^{**}})$.

$(i) \Rightarrow (iii)$. Let $F$ be a maximal convex subset of $S_X$. By using Hahn-Banach and Krein-Milman Theorems, it is easily checked that there exists $x^* \in \operatorname{ex}(B_{X^*})$ such that

$$F = \{x \in B_X \ : \ x^*(x) = 1\}.$$

So, by [12, Lemma 1], $\operatorname{dent}(B_X)$ is contained in the absolutely convex hull of $F$. On the other hand, since $X$ has the RNP, $B_X = \overline{\operatorname{co}}(\operatorname{dent}(B_X))$. Thus, $B_X$ is the closed absolutely convex hull of $F$ and $X$ is an almost-CL-space.

$(i) \Rightarrow (iv)$. For $x^* \in B_{X^*}$, we have

$$x^* \in \operatorname{ex}(B_{X^*}) \quad \Leftrightarrow \quad |x^*(x)| = 1 \text{ for each } x \in \operatorname{dent}(B_X)\,. \tag{1}$$

One implication is [12, Lemma 1]; the other one arises from the fact that $\operatorname{dent}(B_X)$ is norming for $X^*$.

It follows from (1) that $K = \mathrm{ex}\,(B_{X^*})$, with the $w^*$ topology, is a Hausdorff compact space. Let $J$ be the canonical injection from $X$ into $C(K)$. Now, since $J^*\delta_s = s \in \mathrm{ex}\,(B_{X^*})$, we deduce from $(ii)$ that $|x^{**}(J^*\delta_s)| = 1$ for all $s \in K$ and $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$. $\square$

Let us give some remarks on the above theorem.

Condition $(ii)$ of the theorem is called E.P.I.P. in [10] and appeared in [11, §4]. In [10, Corollary 3.6], Å. Lima proved $(ii) \Rightarrow (iii)$ for arbitrary real Banach spaces. Nevertheless, we do not know if Lima's result is valid in the complex case without the RNP assumption.

Observe that, when proving $(iv) \Rightarrow (i)$, we do not need the RNP and that the result still holds if we change $C(K)$ to $C_b(\Omega)$, the Banach space of all bounded continuous functions from the topological Hausdorff space $\Omega$ into the scalar field. Therefore, we have the following by-product of the proof of Theorem 1.

**Corollary 2.** *Let $\Omega$ be a Hausdorff topological space and let $X$ be a Banach space. If there exists a linear isometry $J : X \to C_b(\Omega)$ such that*

$$|x^{**}(J^*\delta_s)| = 1 \qquad (s \in \Omega,\ x^{**} \in \mathrm{ex}\,(B_{X^{**}})),$$

*then $n(X) = 1$.*

The above corollary improves one result recently given by D. Werner [16, Corollary 2.2]. Some definitions are required. Following [16], a Banach space $X$ is said to be *nicely embedded* in $C_b(\Omega)$ if there exists a linear isometry $J : X \to C_b(\Omega)$ such that for all $s \in \Omega$ the following properties are satisfied:

(N1) $\|J^*\delta_s\| = 1$.
(N2) $\mathrm{span}(J^*\delta_s)$ is an $L$-summand in $X^*$.

It is clear that nicely embedded spaces fulfill the conditions in Corollary 2, so they have numerical index 1. This is precisely Corollary 2.2 of [16] (see also the introduction of [14]). But the converse result is false even in the finite-dimensional setting. As a matter of fact, in the case when $X$ is the 3-dimensional $L$-space, $n(X) = 1$ and $X^*$ does not have any non-trivial $L$-summand.

In the real case, Corollary 2 can be written in a more suitable form by using notation similar to Werner's result. Following Å. Lima [9, 10], a closed subspace $Y$ of a Banach space $X$ is said to be a *semi $L$-summand* if for every $x \in X$ there exists a unique $y \in Y$ such that $\|x - y\| = d(x, Y)$, and moreover this $y$ satisfies $\|x\| = \|y\| + \|x - y\|$. A Banach space $X$ is said to be *semi-nicely embedded* in $C_b(\Omega)$ if there exists a linear isometry $J : X \to C_b(\Omega)$ such that for all $s \in \Omega$ the following properties are satisfied:

(N1) $\|J^*\delta_s\| = 1$.
(N2′) $\mathrm{span}(J^*\delta_s)$ is a semi $L$-summand in $X^*$.

Given a real Banach space $X$ and a point $x \in S_X$, [10, Theorem 3.1] says that $\mathrm{span}(x)$ is a semi-$L$-summand if and only if $|x^*(x)| = 1$ for all $x^* \in \mathrm{ex}\,(B_{X^*})$. Therefore, condition $(iv)$ in Theorem 1 is equivalent to the property of being semi-nicely embedded in some $C(K)$. Therefore, for real spaces, Theorem 1 reads as follows.

**Corollary 3.** *Let $X$ be a real Banach space having the RNP. Then $n(X) = 1$ if and only if $X$ is semi-nicely embedded in some $C(K)$.*

It is now easy to give examples of Banach spaces semi-nicely embedded in some $C(K)$ which are not nicely embedded in any $C_b(\Omega)$. For instance, since the real space $l_1$ has the RNP and $n(l_1) = 1$, it is semi-nicely embedded in $C(\Delta)$, where $\Delta$ is the Cantor set. But $l_\infty$ does not have any non-trivial $L$-summand (see [2, Theorem 6.15], for instance).

## References


[1] M. D. Acosta, *CL-spaces and numerical radius attaining operators*, Extracta Math. **5** (1990), 138–140.

[2] E. Behrends, *M-structure and the Banach-Stone Theorem*, Lecture Notes in Math. **736**, Springer-Verlag, Berlin, 1979. MR **81b:**46002

[3] F. F. Bonsall and J. Duncan, "Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras", London Math. Soc. Lecture Note Ser. **2**, Cambridge University Press, 1971. MR **44:**5779

[4] F. F. Bonsall and J. Duncan, "Numerical Ranges II", London Math. Soc. Lecture Note Ser. **10**, Cambridge University Press, 1973. MR **56:**1063

[5] J. Diestel and J. J. Uhl, "Vector Measures", Math. Surveys **15**, Amer. Math. Soc., Providence, RI, 1977. MR **56:**12216

[6] J. Duncan, C. M. McGregor, J. D. Pryce, and A. J. White, *The numerical index of a normed space*, J. London Math. Soc. **2** (1970), 481–488. MR **41:**8967

[7] C. Finet, M. Martín, and R. Payá, *Numerical index and renorming*, Proc. Amer. Math. Soc. **131** (2003), 871–877.

[8] R. E. Fullerton, *Geometrical characterizations of certain function spaces.* In: Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), pp. 227–236. Pergamon, Oxford, 1961. MR **24:**A2834

[9] Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. **227** (1977), 1–62. MR **55:**3752

[10] Å. Lima, *Intersection properties of balls in spaces of compact operators*, Ann. Inst. Fourier, Grenoble **28** (1978), 35–65. MR **80g:**47048

[11] J. Lindenstrauss, *Extension of compact operators*, Memoirs Amer. Math. Soc. **48**, Providence, RI, 1964. MR **31:**3828

[12] G. López, M. Martín, and R. Payá, *Real Banach spaces with numerical index 1*, Bull. London Math. Soc. **31** (1999), 207–212. MR **99k:**46024

[13] M. Martín, *A survey on the numerical index of a Banach space*, Extracta Math. **15** (2000), 265–276. MR **2002b:**46027

[14] M. Martín and R. Payá, *Numerical index of vector-valued function spaces*, Studia Math. **142** (2000), 269–280. MR **2001i:**46017

[15] M. Martín and A. R. Villena, *Numerical index and Daugavet property for $L_\infty(\mu, X)$*, to appear in Proc. Edinburgh Math. Soc.

[16] D. Werner, *The Daugavet equation for operators on function spaces*, J. Funct. Anal. **143** (1997), 117–128. MR **98c:**47025



Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

*E-mail address*: `mmartins@ugr.es`