CYCLICITY CONDITIONS FOR DIVISION ALGEBRAS
OF PRIME DEGREE

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Abstract. Let $D$ be a division algebra of prime degree $p$. A set of criteria is given for cyclicity of $D$ in terms of subgroups of the multiplicative group $D^*$ of $D$. It is essentially shown that $D$ is cyclic if and only if $D^*$ contains a nonabelian metabelian subgroup.

The multiplicative group of a noncommutative division ring has been investigated in various papers by Hua [4], [5], Scott [8], Herstein [3], Amitsur [1], and Huzurbazar [6]. For some recent results see [7]. In this note we concentrate on the case of division algebras of prime degree.

Let $D$ be a division algebra with center $F$ and degree $p$ (i.e., dim$_F D = p^2$), with $p$ prime. The algebra $D$ is called cyclic if it contains a cyclic extension of $F$ of degree $p$. It is known that division algebras of degree $p = 2, 3$ are cyclic; the existence of noncyclic division algebras of prime degree $p \geq 5$ is unknown. If $K$ is a cyclic extension of degree $p$ of $F$ in $D$, then the Skolem–Noether theorem yields an element $z$ such that the inner automorphism $x \mapsto zxz^{-1}$ restricts to a nontrivial automorphism of $K$ and $z^p \in F^*$; see [2, p. 49]. In particular, $zF^*$ is an element of order $p$ in $D^*/F^*$. Conversely, a theorem of Albert (see [2, p. 87]) asserts that $D$ is cyclic if $D^*/F^*$ contains an element of order $p$. The object of this note is to give further equivalent cyclicity conditions in terms of the groups $D^*$ and $D^*/F^*$.

Our main result is the following:

Theorem. Let $D$ be a central division $F$-algebra of prime degree $p$. The following conditions are equivalent:

(a) $D$ is cyclic;
(b) $D^*$ contains a nonabelian soluble subgroup;
(c) $D^*$ contains a nonabelian metabelian subgroup;
(d) $D^*/F^*$ contains a nontrivial finite subgroup;
(e) $D^*/F^*$ contains an element of order $p$;
(f) $D^*/F^*$ contains a nonabelian soluble subgroup;
(g) $D^*/F^*$ contains a nonabelian metabelian subgroup.

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The main ingredient in the proof of \((b) \Rightarrow (a)\) is the following:

**Lemma.** Using the same notation as in the Theorem, suppose \(T \subset D^*\) is a nonabelian soluble subgroup containing \(F^*\). If \(T/F^*\) is infinite, then there is a cyclic extension \(K/F\) such that \(T\) lies in the normalizer \(N_{D^*}(K^*)\) of \(K^*\).

**Proof of the Lemma.** By [7, Lemma 3], we may find an abelian normal subgroup \(A \subset T\) of finite index. Take \(A\) maximal. We have \(A \subset AF^* \subset T\), and \(AF^* \neq T\) since \(T\) is nonabelian. By maximality of \(A\) we conclude that \(A = AF^*\), hence \(F^* \subset A\). Since \(T/F^*\) is infinite, we have \(F^* \neq A\). The subfield \(K = F(A)\) is then maximal in \(D\). Since \(A\) is normal in \(T\) we have \(T \subset N_{D^*}(K^*)\). On the other hand, \(T \not\subset K^*\) since \(T\) is nonabelian. Let \(z \in T \backslash K^*\). Then the inner automorphism obtained from \(z\) restricts on \(K\) to a nontrivial automorphism of \(K/F\). Therefore, \(K/F\) is cyclic, and this completes the proof of the Lemma. \(\square\)

**Proof of the Theorem.** The implications \((a) \Rightarrow (e) \Rightarrow (d), (g) \Rightarrow (f) \Rightarrow (b),\) and \((c) \Rightarrow (b)\) are clear.

\((a) \Rightarrow (c), (g).\) Let \(K \subset D\) be a maximal subfield such that \(K/F\) is cyclic. Consider the normalizer \(N = N_{D^*}(K^*)\) of \(K^*\) in \(D^*\). We have \(N/K^* \cong \text{Gal}(K/F) \cong \mathbb{Z}/p\mathbb{Z}\), hence \(N\) is a metabelian subgroup of \(D^*\). If \(u_1, \ldots, u_p\) is a normal basis of \(K/F\), we may find \(v \in N\) such that \(vu_1v^{-1} = u_2\), hence \(vu_1 \equiv u_1v \mod F^*\). This shows \(N/F^*\) (hence also \(N\)) is not abelian, so \((c)\) and \((g)\) follow.

\((d) \Rightarrow (a).\) Let \(T/F^* \subset D^*/F^*\) be a nontrivial finite subgroup and denote by \(D^1\) the kernel of the reduced norm map \(\text{Nrd}: D^* \to F^*\). Two cases may occur:

- If \(T \cap D^1 \subset F^*\), then for \(x \in T \setminus F^*\) we have \(x^p/\text{Nrd}(x) \in T \cap D^1 \subset F^*\). Thus, by Albert’s criterion (see [2, p. 87]), \(D\) is cyclic.
- If \(T \cap D^1 \not\subset F^*\), pick an element \(x \in T \cap D^1 \setminus F^*\), so \(F(x) \neq F\). Since \(T/F^*\) is finite we have \(x^n \in F^*\) for some integer \(n \neq 0\). Set \(x^n = a\). Taking the reduced norm of both sides of the last equation we obtain \(1 = a^p\) and hence \(x^{mp} = 1\). Let \(m\) be the smallest integer such that \(x^m = 1\). Then, \(m\) is not divisible by the characteristic of \(F\), and \(F(x)\) may be identified with the field \(F(\mu_m)\) of \(m\)-th roots of unity. It follows that \(F(x)/F\) is Galois, hence \((a)\) is proved.

\((b) \Rightarrow (a).\) Let \(S \subset D^*\) be a nonabelian soluble subgroup and set \(T = SF^*\). Then \(T\) is also nonabelian soluble. If \(T/F^*\) is infinite, then the Lemma yields \((a)\). If \(T/F^*\) is finite, then we have \((d)\), hence also \((a)\) since \((d) \Rightarrow (a)\) has been proved above.

The proof of the Theorem is thus complete. \(\square\)

We conclude with a few observations on the case where \(F\) contains all the roots of unity of order prime to the characteristic. This hypothesis imposes severe restrictions on subgroups of \(D^*/F^*:\)

**Corollary.** If \(F\) contains all the roots of unity of order prime to the characteristic, then every nontrivial finite subgroup of \(D^*/F^*\) (if any) is cyclic of order \(p\) or elementary abelian of order \(p^2\).

**Proof.** Let \(T/F^*\) be a nontrivial finite subgroup of \(D^*/F^*\). In view of the hypothesis on \(F\), the proof of \((d) \Rightarrow (a)\) above shows that \(T \cap D^1 \subset F^*\), and that \(x^p\text{Nrd}(x)^{-1} \in\)
$F^*$ for all $x \in T$. Since the derived group $T'$ is in $D^1$, it follows that $T/F^*$ is abelian and $p$-torsion, hence it is elementary abelian of exponent $p$. If $x_1, \ldots, x_r \in T$ have distinct images in $T/F^*$, it is easily seen that $x_1, \ldots, x_r$ are linearly independent over $F$ (see [9, p. 130]). Therefore, the order of $T/F^*$ is at most $p^2$.

In contrast with the Theorem, the group $D^1$ of reduced norm 1 elements (or, equivalently, the derived group $D'$ since $D' = D^1$ by [2, Theorem 4, p. 164]) does not contain any nonabelian soluble subgroup when $F$ satisfies the hypothesis in the Corollary above.

**Proposition.** Using the same notation as in the Theorem, if $F$ contains all the roots of unity of order prime to the characteristic, then the group $D^1$ does not contain any nonabelian soluble subgroup.

**Proof.** Suppose $S \subset D^1$ is a nonabelian soluble subgroup and let $T = SF^*$. As a first step in the proof, we show that there is a maximal subfield $K \subset D$ such that $K/F$ is cyclic and $S \subset T \subset N_{D^1}(K^*)$. This readily follows from the Lemma if $T/F^*$ is infinite.

If $T/F^*$ is cyclic, then $T$ (hence also $S$) is abelian, a contradiction.

If $T/F^*$ is elementary abelian of order $p^2$, let $x \in T \setminus F^*$ and let $y \in T$ be an element which does not commute with $x$. We have $yx^{-1} \in xF^*$ since $T/F^*$ is abelian, hence the inner automorphism obtained from $y$ induces a nontrivial automorphism of $F(x)$. Therefore, $F(x)/F$ is a cyclic extension of $F$. Since $T/F^*$ is generated by the images of $x$ and $y$, it follows that $T$ lies in the normalizer of $F(x)^*$.

The Corollary shows that the two cases above exhaust all the possibilities for $T/F^*$ when it is finite. Therefore, we may always find a maximal subfield $K \subset D$ with $K/F$ cyclic and $S \subset T \subset N_{D^1}(K^*)$. Pick $z \in D^*$ such that the inner automorphism obtained from $z$ restricts to a nontrivial automorphism of $K/F$ and let $z^p = a \in F^*$. Then

$$N_{D^1}(K^*) = K^* \cup K^*z \cup \cdots \cup K^*z^{p-1}.$$  

Since $S$ is not abelian, we must have $S \not\subset K^*$; hence $kz^i \in S$ for some $k \in K^*$ and some integer $i$ with $1 \leq i \leq p - 1$. Since $S \subset D^1$, it follows that $N_{D^1}(kz^i) = 1$, hence

$$N_{K/F}(k) = \begin{cases} a^{-i} & \text{if } p \text{ is odd}, \\ -a^{-1} & \text{if } p = 2. \end{cases}$$

Therefore, in both cases $a$ is a norm from the extension $K/F$: if $p$ is odd this is because $i$ is prime to $p$, and if $p = 2$ this is because $-1$ is a square in $F$ by hypothesis. By [2, p. 73], it follows that $D$ is not a division algebra, a contradiction.

**Example.** The Corollary and the Proposition do not hold without hypothesis on $F$, as the following example shows. Let $D = (-3, -1)\mathbb{Q}$ be the quaternion algebra with basis $1, i, j, ij$ such that $i^2 = -3$, $j^2 = -1$, and $ji = -ij$ over the field $\mathbb{Q}$ of rational numbers. Then $\omega = (-1 + i)/2 \in D$ has order 3 and reduced norm 1.

Now, $\omega$ and $j$ generate a subgroup of $D^1$ which is a generalized quaternion group of order 12 (and their images in $D^*/F^*$ generate a subgroup isomorphic to $S_3$, the symmetric group on three elements).
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