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A PROOF OF NOGURA'S CONJECTURE

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Abstract. Answering a question of T. Nogura (1985), we show using the Open Coloring Axiom that the weak diagonal sequence property is preserved by taking products whenever the products themselves are Fréchet. As an application we show, using the same assumption, that the product of two Fréchet groups is Fréchet provided it is sequential. Recall that the product of two Fréchet groups may not be sequential.

Recall that a given topological space X (implicitly assumed to be at least Hausdorff) is *sequential* if every nonclosed subset A of X contains a sequence which converges to a point outside of A. Recall also that a space X is said to be *Fréchet* if every subset A of X which accumulates to some point x in X contains a sequence which converges to x. It is usually in the realm of Fréchet spaces that one considers various ways to obtain a converging sequence out of a given sequence of converging sequences (see, e.g., [1], [2], [3], [4], [5]). For example, the well-known *diagonal-sequence property* states that if $\{x_{nk}\}$ is a double-indexed sequence of members of X such that for some $x \in X$ and all n, $x_{nk} \to_k x$, then for each n we can choose $k(n)$ such that

$$x_{nk(n)} \to_n x.$$

This turns out to be quite a strong property preserved under products and imposing first-countability when X is countable and the topology of X is an analytic subset of 2^X (see [10]). Consider however the *weak diagonal-sequence property* which asserts that we can make the choice $k(n)$ in such a way that some infinite subsequence of $\{x_{nk(n)}\}$ converges to x rather than the sequence itself. This property seems to be considerably less restrictive than the diagonal sequence property as it can be seen, for example, from a result of Nyikos [3] which states that every Fréchet topological group has the weak diagonal sequence property. It turns out that in general the weak diagonal sequence property is not a productive property which motivated T. Nogura [2] to ask if the property is productive under some restriction such as the following.

Question 1 ([2, 3.15])**.** Suppose X and Y are two Fréchet spaces with the weak diagonal sequence property. Suppose further that their product $X \times Y$ is Fréchet. Does $X \times Y$ have the weak diagonal sequence property?

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The purpose of this note is to give a positive answer to Nogura's question assuming the Open Coloring Axiom [8, §8]. The use of OCA is quite natural here, as it is easily seen (see [6]) that the existence of a nontrivial coherent sequence of functions $c_f : \Gamma_f \to 2$ ($f \in \mathbb{N}^{\mathbb{N}}$) gives the negative answer, where for ($f \in \mathbb{N}^{\mathbb{N}}$) we put

$$\Gamma_f = \{(n,k) \in \mathbb{N}^2 : \ k \leq f(n)\}.$$

In fact, the recent article of Simon and Tironi [7], which motivated our work, shows that the existence of some strong counterexample to Nogura's conjecture is actually equivalent to the existence of a nontrivial coherent sequence $c_f : \Gamma_f \to 2$ ($f \in \mathbb{N}^{\mathbb{N}}$). Recall that OCA implies that every coherent sequence $c_f : \Gamma_f \to 2$ ($f \in \mathbb{N}^{\mathbb{N}}$) is trivial i.e., induced modulo-finite by a single map $d : \mathbb{N}^2 \to 2$ (see [8, §8]). What we show here is that a different application of OCA gives us a proof of Nogura's conjecture in its full generality.

Theorem 2 (OCA)**.** *Suppose that X and Y are Fréchet spaces with the weak diagonal sequence property. Suppose further that their product $X \times Y$ is Fréchet. Then $X \times Y$ has the weak diagonal sequence property.*

Proof. Suppose there is a pair of spaces X and Y satisfying the hypothesis of the theorem but their product $X \times Y$ fails to have the weak diagonal sequence property. A Fréchet space without the weak diagonal sequence property is easily seen to contain a copy of the sequential fan S_ω, the topological space on $\mathbb{N}^2 \cup \{\infty\}$ with only ∞ nonisolated while the neighborhood base of ∞ is generated by the complements of the sets of the forms Γ_f ($f \in \mathbb{N}^{\mathbb{N}}$). So, from our assumption, one easily constructs two Fréchet topologies τ_X and τ_Y on $\mathbb{N}^2 \cup \{\infty\}$ with ∞ as the only nonisolated point such that τ_X and τ_Y both have the weak diagonal sequence property while the topology of S_ω is generated by $\tau_X \cup \tau_Y$ as a subbasis. For $n \in \mathbb{N}$, set

$$C_n = \{n\} \times \mathbb{N}.$$

Note that each C_n is a converging sequence in both topologies τ_X and τ_Y. Let

$$\mathcal{A} = \{A \subseteq \mathbb{N}^2 : \ A \to_{\tau_X} \infty \text{ and } A \cap C_n \text{ is finite for all } n\},$$
$$\mathcal{B} = \{B \subseteq \mathbb{N}^2 : \ B \to_{\tau_Y} \infty \text{ and } B \cap C_n \text{ is finite for all } n\}.$$

Thus, $\mathcal{A}$ (respectively, $\mathcal{B}$) is the family of all sequences that converge to ∞ relative to τ_X (respectively, relative to τ_Y) and which are orthogonal to each member of the sequence $\{C_n\}$ of converging sequences. Let

$$\mathcal{X} = \{(A,B) \in \mathcal{A} \times \mathcal{B} : A \cap B = \emptyset\}.$$

We endow $\mathcal{X}$ with the standard separable metric topology induced from $2^{\mathbb{N}^2}$. Consider the following subset of the set $\mathcal{X}^{[2]}$ of all unordered pairs of elements of $\mathcal{X}$:

$$\mathcal{K} = \{\{(A,B),(A',B')\} \in \mathcal{X}^{[2]} : (A \cap B') \cup (A' \cap B) \neq \emptyset\}.$$

Note that $\mathcal{K}$ is an open subset of $\mathcal{X}^{[2]}$, so OCA applies to it. Therefore, to finish the proof, it suffices to show that neither of the following two alternatives given by OCA is possible.

Case 1. There is uncountable $\mathcal{Y} \subseteq \mathcal{X}$ such that $\mathcal{Y}^{[2]} \subseteq \mathcal{K}$. We may assume that $\mathcal{Y}$ actually has size $\aleph_1$ and since under OCA subsets of $\mathbb{N}^{\mathbb{N}}$ of size $\aleph_1$ are bounded

in the ordering of eventual dominance (see [8, §8]), going to a subset of $\mathcal{Y}$, we may assume that there is $f \in \mathbb{N}^{\mathbb{N}}$ such that

$$A \cup B \subseteq \Gamma_f \text{ for all } (A, B) \in \mathcal{Y}.$$

Since Γ_f does not accumulate to ∞ in S_ω and since $\tau_X \cup \tau_Y$ is a subbasis of the sequential fan S_ω, we can find $U \in \tau_X$ and $V \in \tau_Y$ such that $\infty \in U \cap V$ and

$$U \cap V \cap \Gamma_f = \emptyset.$$

For $D \subseteq \mathbb{N}^2$ and $n \in \mathbb{N}$, set

$$D/n = D \backslash (n \times \mathbb{N}) \text{ and } D \restriction n = D \cap (n \times \mathbb{N}).$$

Note that for every $(A, B) \in \mathcal{Y}$ there is n such that $A/n \subseteq U$ and $B/n \subseteq V$. So we can find $n \in \mathbb{N}$ and an uncountable subset $\mathcal{Y}_0$ of $\mathcal{Y}$ such that

(a) $A/n \subseteq U$ and $B/n \subseteq V$ for all $(A, B) \in \mathcal{Y}_0$,
(b) $A \restriction n = A' \restriction n$ and $B \restriction n = B' \restriction n$ for all $(A, B), (A', B') \in \mathcal{Y}_0$.

Pick two distinct elements (A, B) and (A', B') of $\mathcal{Y}_0$. Then the unordered pair $\{(A, B), (A', B')\}$ belongs to $\mathcal{K}$, and therefore,

$$U \cap V \cap \Gamma_f \supseteq (A \cap B') \cup (A' \cap B) \neq \emptyset,$$

a contradiction.

Case 2. There is a decomposition

$$\mathcal{X} = \bigcup_{n=0}^{\infty} \mathcal{X}_n$$

such that $(\mathcal{X}_n)^{[2]} \cap \mathcal{K} = \emptyset$ for all n.

For $n \in \mathbb{N}$, set

$$D_n = \bigcup \{A : \ (A, B) \in \mathcal{X}_n \text{ for some } B\}.$$

Then for every $(A, B) \in \mathcal{A} \times \mathcal{B}$ there is n such that $(A \backslash B, B) \in \mathcal{X}_n$ and, therefore,

$$A \subseteq^* D_n \text{ and } B \cap D_n = \emptyset.$$

Pick an ultrafilter $\mathcal{U}$ on $\mathbb{N}^2$ such that for every $S \in \mathcal{U}$ there are infinitely many n such that $C_n \cap S$ is infinite. Let $\epsilon : \mathbb{N} \to 2$ be such that

$$D_n^{\epsilon(n)} \in \mathcal{U} \text{ for all } n \in \mathbb{N}.$$

(Here $D_n^1 = D_n$ and $D_n^0 = \mathbb{N}^2 \backslash D_n$.) Let $\{n_k\}_{k=0}^{\infty}$ be the strictly increasing sequence of elements of $\mathbb{N}$ such that n_k is the minimal integer $> n_{k-1}$ with the property that

$$E_{n_k} = C_{n_k} \cap \bigcap_{i=0}^{n_{k-1}} D_i^{\epsilon(i)}$$

is infinite; $n_{-1} = 0$. Applying the fact that τ_X has the weak diagonal sequence property to the sequence $\{E_{n_k}\}_{k=0}^{\infty}$ of τ_X−converging sequences to ∞, we find $A \in \mathcal{A}$ such that

$$A \subseteq \bigcup_{k=0}^{\infty} E_{n_k}$$

and such that $A \cap E_{n_k}$ has at most one point for all k. Applying the fact that τ_Y has the weak diagonal sequence property to the same sequence $\{E_{n_k}\}_{k=0}^{\infty}$ of τ_Y-converging sequences to ∞ we find $B \in \mathcal{B}$ such that

$$B \subseteq \bigcup_{k=0}^{\infty} E_{n_k}$$

and such that $B \cap E_{n_k}$ has at most one point for all k. Let n be such that $(A \setminus B, B) \in \mathcal{X}_n$, or in other words

$$A \subseteq^* D_n \text{ and } B \cap D_n = \emptyset.$$

Pick j such that $n_{j-1} \geq n$. Then

$$A \cup B \subseteq^* \bigcup_{k=j}^{\infty} E_{n_k} \subseteq D_n^{\epsilon(n)},$$

a contradiction. This finishes the proof of Theorem 2.

Remark 3. If one assumes that in the hypothesis of Theorem 2 the spaces X and Y are countable and their topologies τ_X and τ_Y are analytic subsets of 2^X and 2^Y, respectively, then the conclusion of Theorem 2 is true with no extra set-theoretic assumption. This follows from a characterization of bisequentiality of analytic topologies given in [11]:

Theorem 4. *An analytic topological space is bisequential if and only if it is Fréchet and has the weak diagonal sequence property.*

Recall that a topological space X is *bisequential* if it has the property that if an ultrafilter $\mathcal{U}$ on X converges to a point x in X, then $\mathcal{U}$ contains a sequence of sets that converge to the point x. This is yet another productive convergence property which strengthens the Fréchet property as well as the weak diagonal sequence property (see, e.g., [4]).

Recall now that the product $G \times H$ of two Fréchet groups G and H may fail to be Fréchet (see [9]). In fact the product $G \times H$ given in [9] is not even countably tight. However, finding such a product $G \times H$ which is countably tight or sequential but not Fréchet is a subtle matter. For example, finding such a product would solve yet another well-known open problem in the area (due to V.I. Malyhin) which asks whether Fréchet property implies metrizability in the realm of countable topological groups (see [4]). While OCA is not sufficient for solving Malyhin's problem it might be relevant to the problem about products of Fréchet groups:

Problem 5. Find an assumption under which it is true that the product of two Fréchet groups is Fréchet if and only if it is countably tight.

The following corollary of the proof of Theorem 2 shows that sequential product or two Fréchet groups is always Fréchet, thus answering Question 6.13(i) of [4].

Theorem 6 (OCA)**.** *The product of two Fréchet groups is Fréchet if and only if it is sequential.*

Proof. Clearly, we may assume that the given two Fréchet groups G and H are countable. Suppose that the product group $K = G \times H$ is sequential but not Fréchet. It is well known and easily seen that every countable sequential space which is not Fréchet contains a closed copy of the *Arens space* [1], the topological space on $\mathbb{N}^2 \cup \{\infty\}$ with ∞ and $(n, 0)$ $(n \in \mathbb{N})$ as nonisolated points, the neighborhood

base of ∞ is generated by the complements of the sets of the forms Γ_f $(f \in \mathbb{N}^{\mathbb{N}})$ and $\{n\} \times \mathbb{N}$ $(n \in \mathbb{N})$, and the neighborhood base of a point of the form $(n, 0)$ is generated by tails of the sequence (n, k) $(k \in \mathbb{N})$. Note that if $\{x_{nk}\}$ is a double-indexed sequence which together with e_K forms a (natural) closed copy of the Arens space in K, then $\{x_{nk}x_{n0}^{-1}\}$ together with e_K forms a (natural) closed copy of the sequential fan S_ω in K. We have already mentioned the result of Nyikos [3] that Fréchet groups have the weak diagonal sequence property. Now observe that the proof of Theorem 2 shows that the product of two countable Fréchet spaces with the weak diagonal sequence property does not contain a closed copy of S_ω. This finishes the proof.

Remark 7. (i) According to a result of [5] the use of an additional axiom to prove Theorem 6 is in some sense necessary.

(ii) Again if one assumes that in the hypothesis of Theorem 6 the groups G and H are countable and their topologies τ_G and τ_H are analytic subsets of 2^G and 2^H, respectively, then the conclusion of Theorem 6 is true with no extra set-theoretic assumption. This follows from the following metrization theorem from [11] which gives the positive answer to the effective version of Malyhin's problem:

Theorem 8. *A countable Fréchet topological group G is metrizable if and only if its topology τ_G is an analytic subset of 2^G.*

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