INDUCED LOCAL ACTIONS ON TAUT AND STEIN MANIFOLDS

ANDREA IANNUZZI

(Communicated by Mohan Ramachandran)

Abstract. Let $G = \langle \mathbb{R}, + \rangle$ act by biholomorphisms on a taut manifold $X$. We show that $X$ can be regarded as a $G$-invariant domain in a complex manifold $X^*$ on which the universal complexification $(\mathbb{C}, +)$ of $G$ acts. If $X$ is also Stein, an analogous result holds for actions of a larger class of real Lie groups containing, e.g., abelian and certain nilpotent ones. In this case the question of Steinness of $X^*$ is discussed.

Introduction

Let $X$ be a complex manifold endowed with an action by biholomorphisms of a connected real Lie group $G$, i.e., $X$ is a complex $G$-manifold. If the Lie algebra of the universal complexification $G^\mathbb{C}$ of $G$ is the complexification of $\text{Lie}(G)$, then one obtains an induced local $G^\mathbb{C}$-action by integrating the $\mathbb{C}$-linear extension of the infinitesimal generator associated to the $G$-action. In many cases this can be understood as the restriction of a global $G^\mathbb{C}$-action, that is, it is possible to realize $X$ as a $G$-invariant domain in a complex $G^\mathbb{C}$-manifold $X^*$ to which we will refer as a globalization of the local $G^\mathbb{C}$-action. For instance, by a result of P. Heinzner ([H]) if $X$ is Stein and $G$ compact, then there exists a Stein globalization $X^*$ with the following universal property: every holomorphic $G$-equivariant map on $X$ to a complex $G^\mathbb{C}$-manifold extends $G^\mathbb{C}$-equivariantly on $X^*$.

Furthermore, for $X$ Stein and $G$ with polar complexification $G^\mathbb{C}$ and cocompact discrete subgroup $\Gamma$ such that $G^\mathbb{C}/\Gamma$ is Stein, equivalent conditions for the existence of a Stein universal globalization are given in [CT]. These can be verified to hold in many concrete situations, however it seems not to be known whether in this setting a globalization always exists. Here we first consider $(\mathbb{R}, +)$-actions on taut manifolds and we prove the following:

Let $X$ be a taut $\mathbb{R}$-manifold. Then there exists a universal globalization $X^*$ of the induced local $\mathbb{C}$-action.

Note that one cannot expect $X^*$ to be taut unless the $\mathbb{R}$-action on $X$ is trivial. If $X$ is also Stein, we show that a similar result holds for $G$ in the above-mentioned class of real Lie groups (Corollary 3). In this case it is natural to ask whether such
a universal globalization is also Stein. For $G = (\mathbb{R}, +)$ it turns out that this is equivalent to a positive answer to the following open question:

Let $Y$ be a complex manifold and assume there exist lower semicontinuous functions $\alpha, \beta : Y \to \mathbb{R}$ such that $\Omega := \{ (\lambda, y) \in \mathbb{C} \times Y : -\beta(y) < \text{Im} \lambda < \alpha(y) \}$ is Stein. Is $Y$ then Stein?

We conclude by pointing out particular cases where this holds true.

**Existence of Globalizations**

For basic facts and results on local actions and their globalizations we refer to [IL] and more generally to [HI, §2-3], from which most notations are inherited. However note that all manifolds are assumed to be Hausdorff (cf. [HI, §3]).

**Theorem 1.** Let $X$ be a taut $\mathbb{R}$-manifold. Then there exists a universal globalization $X^*$ of the induced local $\mathbb{C}$-action.

**Proof.** Note that every leaf $\Sigma$ of Palais’ foliation with respect to the induced local $\mathbb{C}$-action is a non-compact Riemann surface, since its projection $p_{|\Sigma} : \Sigma \to \mathbb{C}$ is not constant. In particular $\Sigma$ is holomorphically separable and [HI] Corollary, p. 438 applies to show univalency of such a local action. Then by [HI, Theorem 2, p. 38] there exists a possibly non-Hausdorff universal globalization $X^*$. The result will follow by showing that $X^*$ is Hausdorff.

For this suppose that there exist elements $x_1$ and $x_2$ in $X^*$ which are not topologically separable. Since $X^* = \mathbb{C} \cdot X$ and $X$ is $\mathbb{R}$-invariant one may assume that $x_1 \in X$ and $x_2 = it \cdot x_0$ with $x_0 \in X$ and $t \in \mathbb{R}^{>0}$. Note that $X$ is Hausdorff, thus $x_2 \notin X$ and consequently the local $\mathbb{C}$-orbit through $x_0$ has necessarily complex dimension one. Then one can choose a local slice $f : \mathbb{B}^{n-1}(1) \to X$ transversal to $\mathbb{C} \cdot x_0$ with $f(0) = x_0$ and a neighborhood $U \subset \mathbb{C}$ of 0 such that $\varphi : U \times \mathbb{B}^{n-1}(1) \to X$ defined by $\varphi(z, s) := z \cdot f(s)$ is a chart of $X$. Here $n$ is the complex dimension of $X$ and $\mathbb{B}^{n-1}(1) := \{ s \in \mathbb{C}^{n-1} : |s| < r \}$ for all $r > 0$.

Let us call such a chart an adapted chart of $X$ in $x_0$.

Now $it \cdot \varphi(rU \times \mathbb{B}^{n-1}(r))$ are open neighborhoods of $x_2$ for all $0 < r < 1$ and we are assuming that $x_1$ and $x_2$ are not separable. Therefore there exists a sequence $(z_j, s_j)$ convergent to $(0, 0)$ in $U \times \mathbb{B}^{n-1}(1)$ such that $X \ni it \cdot \varphi(z_j, s_j) \to x_1$. Thus for $y_j := \varphi(z_j, s_j)$ one has $X \ni y_j \to x_0$ and $X \ni it \cdot y_j \to x_1$. Now recall that $X$ is orbit-connected (cf. [CTT] Lemma 1.6) and $\mathbb{R}$-invariant in $X^*$. Then by considering an adapted chart of $X$ in $x_1$ one checks that there exists $\epsilon > 0$ such that $S := \{ z \in \mathbb{C} : -\epsilon < \text{Im} z < \epsilon + \epsilon \} \subset \Omega(y_j)$ for all $j > 0$, where by definition $\Omega(x) := \{ z \in \mathbb{C} : z \cdot x \in X \}$ for all $x \in X$.

Define a sequence of holomorphic functions $h_j : S \to X$ by $h_j(z) := z \cdot y_j$, let $a_0, b_0 \in \mathbb{R}^{>0}$ be given by $\Omega(x_0) = \{ z \in \mathbb{C} : -b_0 < \text{Im} z < a_0 \}$ and note that $it \cdot x_0 \notin X$, hence $a_0 \leq t$. Moreover $h_j(0) \to x_0$ while $ia_0 \cdot x_0 \notin X$ and $is \cdot x_0 \in X$, for $s$ smaller than $a_0$ and close to it, imply that $h_j(a_0) \to \infty$. Since $X$ is taut, this gives a contradiction and concludes the proof.

**Remark 2.** Since $X$ is $\mathbb{R}$-invariant and orbit-connected in $X^*$, there exist lower semicontinuous positive functions $a, b : X \to \mathbb{R}^{>0}$ such that

$$\Omega(x) = \{ z \in \mathbb{C} : -b(x) < \text{Im} z < a(x) \}$$
for all \( x \) in \( X \), where \( \Omega(x) := \{ z \in \mathbb{C} : z \cdot x \in X \} \). An analogous argument as in the above proof applies to show that on a taut manifold, \( a \) and \( b \) are continuous (if \( X \) is Stein one knows that \(-a\) and \(-b\) are plurisubharmonic [F]).

Let \( G \) be a real Lie group with polar complexification \( G^C \), i.e., the \( G \)-equivariant map \( G \times \mathfrak{g} \to G^C \) given by \((g, \xi) \to g \exp i \xi\) is a real analytic diffeomorphism. Furthermore assume that \( G \) admits a discrete cocompact subgroup \( \Gamma \) such that \( G^C/\Gamma \) is Stein. For instance all abelian and compact real Lie groups are of this kind or more generally products of the form \( K \times N \), with \( K \) compact and \( N \) simply connected and nilpotent with rational structure constants (see [Ma, GH]). Since \( G^C \) is polar, the Lie algebra of \( G^C \) is the complexification of \( \mathfrak{g} \), the Lie algebra of \( G \). As a consequence if \( G \) acts on a complex manifold one obtains a holomorphic local action of the complexification \( G^C \) by integrating the holomorphic vector fields given by the \( G \)-action. For \( G \) as above one has

**Corollary 3.** Let \( X \) be a taut and Stein \( G \)-manifold. Then there exists a universal globalization \( X^* \) of the induced local \( G^C \)-action.

**Proof.** For \( \eta \in \mathfrak{g} \), consider the \( \mathbb{R} \)-action on \( X \) defined by \( t \cdot x := (\exp t \eta) \cdot x \) and denote by \( X^*_\eta \) the universal globalization of the induced local \( \mathbb{C} \)-action given by the above theorem. Then the corollary is a consequence of [CTT, Corollary 3.7]. \( \square \)

For an action of a compact Lie group \( G \) on a Stein manifold the universal globalization \( X^* \) is automatically Stein ([H]). It would be interesting to know whether this remains true in the case where \( G \) is not compact and \( X^* \) exists. For \( G = \mathbb{R} \) one has

**Proposition 4.** The following statements are equivalent:

i) Let \( X \) be a Stein \( \mathbb{R} \)-manifold with universal globalization \( X^* \). Then \( X^* \) is Stein.

ii) Let \( Y \) be a complex manifold and assume there exist lower semicontinuous functions \( \alpha, \beta : Y \to \mathbb{R} \) such that \( \Omega := \{ (\lambda, y) \in \mathbb{C} \times Y : -\beta(y) < Im \lambda < \alpha(y) \} \) is Stein. Then \( Y \) is Stein.

**Proof.** Let \( \Omega \) be as in ii) and consider the \( \mathbb{R} \)-action by left multiplication on the first component of \( \mathbb{C} \times Y \). Then [CTT, Lemma 1.5] applies to show that \( \mathbb{C} \times Y \) is the universal globalization of \( \Omega \). Thus if i) holds, then \( \mathbb{C} \times Y \) is Stein and consequently so is \( Y \), implying ii).

Conversely for \( X \) as in i) let \( \mathbb{R} \) act diagonally on \( \mathbb{C} \times X \) and by left multiplication on the first component of \( \mathbb{C} \times X^* \). Then the map \( f : \mathbb{C} \times X \to \mathbb{C} \times X^* \) given by \((\lambda, x) \to (\lambda, \lambda^{-1} \cdot x)\) is easily checked to be an \( \mathbb{R} \)-equivariant open embedding. In particular \( f(\mathbb{C} \times X) \) is a Stein \( \mathbb{R} \)-invariant subdomain of \( \mathbb{C} \times X^* \).

Now let \( a, b : X \to \mathbb{R}^{>0} \) be as in Remark 2, fix \( y \in X^* \) and choose \( x \in X \) and \( t \in \mathbb{R} \) such that \( y = it \cdot x \). One has that

\[
(\lambda, y) = (\lambda, \lambda^{-1} \cdot (\lambda + it) \cdot x)
\]

belongs to \( f(\mathbb{C} \times X) \) if and only if \((\lambda + it) \cdot x \in X\), i.e., \(-b(x) - t < Im \lambda < a(x) - t \).

By defining \( \alpha(y) = a(x) - t \) and \( \beta(y) = b(x) + t \) (which is easily verified not to depend on the choice of \( x \) and \( t \)) for all \( y \in X^* \) one has

\[
f(\mathbb{C} \times X) = \{ (\lambda, y) \in \mathbb{C} \times X^* : -\beta(y) < Im \lambda < \alpha(y) \}\]
and statement i) follows from ii) by letting $\Omega = f(\mathbb{C} \times X)$ in $\mathbb{C} \times X^*$, which concludes the proof.

Remark 5. In the following cases it is easy to check that statement ii) holds:

1) $Y$ is holomorphically convex.

For this, first note that for any open Stein neighborhood $U$ in $Y$ the restrictions of $-\alpha$ and $-\beta$ to $U$ define the Stein domain $\Omega \cap (\mathbb{C} \times U)$ in $\mathbb{C} \times U$. It follows that $-\alpha$ and $-\beta$ are plurisubharmonic (see, e.g., [V]).

Now recall that each fiber $F$ of the Remmert reduction of $Y$ (cf. [GR, p. 221]) is a connected compact subspace. In particular $\alpha$ and $\beta$ are constant on $F$, thus $F \cong \{z\} \times F \subset \Omega$ for any fixed $z$ in $\mathbb{C}$ with $-\beta|_F < \text{Im} \, z < \alpha|_F$ and consequently $F$ is holomorphically separable. By compactness and connectness it follows that $F$ consists of a single point, hence $Y$ is Stein.

2) $Y$ is a domain in a Stein manifold $\hat{Y}$.

Here $\Omega$ can be regarded as an open Stein $\mathbb{R}$-invariant subdomain of $\mathbb{C} \times \hat{Y}$, where $\mathbb{R}$ acts by left multiplication on the first component. Since $\mathbb{C} \times \hat{Y}$ is Stein, then $\Omega$ is locally Stein ([DG]).

Moreover the quotient map $\mathbb{C} \times \hat{Y} \to (\mathbb{C} \times \hat{Y})/\mathbb{Z}$ is locally biholomorphic, therefore $\Omega/\mathbb{Z}$ is locally Stein in $(\mathbb{C} \times \hat{Y})/\mathbb{Z} \cong \mathbb{C}^* \times \hat{Y}$, which is Stein, and consequently so is $\Omega/\mathbb{Z}$. Finally $Y$ is easily checked to be biholomorphic to the categorical quotient of $\Omega/\mathbb{Z}$ with respect to the natural induced $S^1$-action, thus it is Stein ([H § 6.5]).

Remark 6. As already noted in the proof of Theorem 1, a complex $\mathbb{R}$-manifold admits a universal globalization $X^*$ which is possibly non-Hausdorff. Note that the same argument used to prove Proposition 4 applies to show the analogous result in the case where $X^*$ and $Y$ are assumed to be in the category of possibly non-Hausdorff complex manifolds.

References


[F] F. Forstnerič, Actions of ($\mathbb{R}$, $+$) and ($\mathbb{C}$, $+$) on complex manifolds, Math. Z. 223 (1996), 123–153. MR 97f:32011


Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, I-40126 Bologna, Italy

E-mail address: iannuzzi@dm.unibo.it

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use