THE DIOPHANTINE EQUATION $2x^2 + 1 = 3^n$

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Abstract. Let $p$ be a rational prime and $D$ a positive rational integer coprime with $p$. Denote by $N(D, 1, p)$ the number of solutions $(x, n)$ of the equation $Dx^2 + 1 = p^n$ in rational integers $x \geq 1$ and $n \geq 1$. In a paper of Le, he claimed that $N(D, 1, p) \leq 2$ without giving a proof. Furthermore, the statement $N(D, 1, p) \leq 2$ has been used by Le, Bugeaud and Shorey in their papers to derive results on certain Diophantine equations. In this paper we point out that the statement $N(D, 1, p) \leq 2$ is incorrect by proving that $N(2, 1, 3) = 3$.

1. Introduction

Let $D_1$ and $D_2$ be coprime positive rational integers, and let $p$ be a rational prime coprime with $D_1 D_2$. Denote by $N(D_1, D_2, p)$ the number of solutions $(x, n)$ of the following equation:

$$D_1 x^2 + D_2 = p^n \quad \text{in rational integers} \quad x \geq 1, n \geq 1.$$

In paper [5], Le claimed that $N(D_1, 1, p) \leq 2$ and the proof could be found in [3] and [4]. Le used $N(D_1, 1, p) \leq 2$ and related results to deduce the main result of [5]. In the proof of Theorem 2 of [2], Bugeaud and Shorey used Le’s result $N(D_1, 1, p) \leq 2$ to claim that $N(2, 1, 3) = 2$, by giving the solutions $(x, n) = (1, 1)$ and $(2, 2)$. (See also the remarks on page 59 of [2].)

By looking at papers [3] and [4], we cannot find a proof for the statement $N(D_1, 1, p) \leq 2$ which was claimed by Le [5]. Unfortunately, it is not difficult to verify that $N(2, 1, 3) \geq 3$ by considering $(x, n) = (1, 1)$, $(2, 2)$ and $(11, 5)$. In this paper we point out that the statement $N(D_1, 1, p) \leq 2$ is incorrect by proving that $N(2, 1, 3) = 3$.

2. $N(2, 1, 3) = 3$

To determine positive rational integral solutions $(x, n)$ of $2x^2 + 1 = 3^n$ we apply unique factorization in the imaginary quadratic field $\mathbb{Q}(\sqrt{-2})$ to reduce the problem to a question about a Fibonacci-type integer sequence. Then, by Proposition 2.1, a result of Beukers [11], we prove that the Diophantine equation $2x^2 + 1 = 3^n$ has exactly three positive rational integral solutions, namely $(x, n) = (1, 1), (2, 2)$ and $(11, 5)$.
The following proposition is part of Lemma 7 of [1]:

**Proposition 2.1.** Let \( \theta = 1 + \sqrt{-2} \) and \( \alpha = \sqrt{-2} \). Then all rational integral solutions \( n > 0 \) of the equations \( \alpha \theta^n - \bar{\alpha} \bar{\theta}^n = \alpha - \bar{\alpha} \) or \( -(\alpha - \bar{\alpha}) \) are \( n = 1, 2 \) and \( 5 \), where \( \theta \) and \( \bar{\alpha} \) denote the algebraic conjugates of \( \theta \) and \( \alpha \), respectively.

Let \( \mathbb{Z} \) be the set of rational integers and \( R \) denote the ring of algebraic integers in the quadratic field \( \mathbb{Q}(\sqrt{-2}) \). Then \( R = \{ a + b\sqrt{-2} \mid a, b \in \mathbb{Z} \} \). It is known that \( R \) is a unique factorization domain. Let \( \theta = 1 + \sqrt{-2} \) and \( \bar{\theta} = 1 - \sqrt{-2} \). Then \( \theta \bar{\theta} = 3 \) and \( \theta^2 = 2 \theta - 3 \). The equation \( 2x^2 + 1 = 3^n \) factors in \( R \) as

\[
(1 + x\sqrt{-2})(1 - x\sqrt{-2}) = \theta^n \bar{\theta}^n, \quad \text{if} \quad x \in \mathbb{Z}.
\]

Note that \( \theta \) and \( \bar{\theta} \) are irreducible in \( R \). If \( \theta \bar{\theta} \mid 1 + x\sqrt{-2} \), then there exist rational integers \( i, j, 1 \leq i \leq n, 1 \leq j \leq n \) such that \( 1 + x\sqrt{-2} = u\theta^i \bar{\theta}^j \), where \( u \) is either \( 1 \) or \( -1 \). Suppose \( i \leq j \). Then

\[
1 + x\sqrt{-2} = u\theta^i \bar{\theta}^j
= u(\theta \bar{\theta})^j \bar{\theta}^{-i}
= u3^j (2 - \theta)^j \bar{\theta}^{-i}
= u3^j (2 - \theta)^j - \binom{j - i}{1} 2^{j - i - 1} \theta + \binom{j - i}{2} 2^{j - i - 2} \theta^2 + \cdots
+ (-1)^j (\theta^{j - i} \bar{\theta}^{-i})
= u3^j (A + B\sqrt{-2}),
\]

where \( A \) and \( B \) are rational integers. Since \( \{1, \sqrt{-2}\} \) is an integral basis of \( R \), the equality \( 1 + x\sqrt{-2} = u3^j (A + B\sqrt{-2}) \) is impossible. Suppose \( j < i \). Then by the same argument, we also reach a contradiction. We conclude that \( 3 = \theta \bar{\theta} \) does not divide \( 1 + x\sqrt{-2} \). Similarly, we also know that \( 3 = \theta \bar{\theta} \) does not divide \( 1 - x\sqrt{-2} \). Hence we have \( \theta^n = u(1 + x\sqrt{-2}) \) or \( \theta^n = u(1 - x\sqrt{-2}) \), where \( u \) is either \( 1 \) or \( -1 \). Equivalently, we have \( \sqrt{-2} \theta^n = u(\sqrt{-2} - 2x) \) or \( \sqrt{-2} \theta^n = u(\sqrt{-2} + 2x) \). From these equations we find that \( \sqrt{-2} \theta^n = a + \theta \) or \( a - \theta \) for some rational integer \( a \). Conversely, for some rational integer \( m > 0 \), if \( \sqrt{-2} \theta^m = a + \theta \) or \( a - \theta \) for \( a \in \mathbb{Z} \), then \( (a \pm 1)^2 + 2 = 2 \times 3^m \). This means that either \( \{\frac{a + 1}{2}, m\} \) or \( \{\frac{a - 1}{2}, m\} \) is a solution of \( 2x^2 + 1 = 3^n \). To summarize, we have proved that the equation \( 2x^2 + 1 = 3^n \) has a positive rational integral solution \((x, n) \) for \( n = m \) if and only if \( \sqrt{-2} \theta^m = a + \theta \) or \( a - \theta \) for \( a \in \mathbb{Z} \).

The problem now is to determine exactly those powers \( n \) such that \( \sqrt{-2} \theta^n \) can be expressed either in the form \( a + \theta \) or \( a - \theta \) for \( a \in \mathbb{Z} \). Since \( \{1, \theta\} \) is also an integral basis of \( R \), \( \sqrt{-2} \theta^n \) can be expressed as \( \sqrt{-2} \theta^n = a_n + b_n \theta \), for \( a_n, b_n \in \mathbb{Z} \). By \( \theta^2 = 2 \theta - 3 \), we have

\[
a_{n+1} + b_{n+1} \theta = \sqrt{-2} \theta^{n+1}
= (\sqrt{-2} \theta^n) \theta
= (a_n + b_n \theta) \theta
= -3b_n + (a_n + 2b_n) \theta,
\]

which implies that \( b_{n+2} = 2b_{n+1} - 3b_n \). Thus the sequence of rational integers \( b_n \) is completely determined by this binary linear recurrence and the initial values \( b_1 = 1 \)
and \( b_2 = -1 \). The sequence \( \{b_n\}_{n=1}^{\infty} \) begins:

\[
1, -1, -5, -7, 1, 23, 43, 17, -95, \ldots.
\]

Since \( b_1 = b_5 = 1 \) and \( b_2 = -1 \), we are provided with the three solutions to the equation \( 2x^2 + 1 = 3^n \), namely, \((x, n) = (1, 1), (2, 2), \) and \((11, 5)\). Now, the problem is to prove that there are no further occurrences of \( 1 \) or \(-1 \) in the sequence \( \{b_n\}_{n=1}^{\infty} \).

**Proposition 2.2.** Let the sequence of rational integers \( b_n \) be defined by the equations:

\[
b_1 = 1, b_2 = -1 \text{ and } b_{n+2} = 2b_{n+1} - 3b_n.
\]

Then \( b_n = 1 \) or \(-1 \) only for \( n = 1, 2 \) and 5.

**Proof.** To apply Proposition 2.1, we define \( \alpha = \sqrt{-2} \). Then \( \alpha \theta = \sqrt{-2}(1 + \sqrt{-2}) = b_2 - b_1 \theta \). Suppose, for all rational integers \( k, \) \( 1 \leq k \leq n \), that \( \alpha \theta^k = b_{k+1} - b_k \theta \).

Then we have

\[
\alpha \theta^{n+1} = (\alpha \theta^n) \theta
= (b_{n+1} - b_n \theta)(2 - \theta)
= 2b_{n+1} - 2b_n \theta - b_{n+1} \theta + b_n \theta^2
= 2b_{n+1} - 2b_n \theta - b_{n+1} \theta + b_n (2 \theta - 3)
= (2b_{n+1} - 3b_n) - b_{n+1} \theta
= b_{n+2} - b_{n+1} \theta.
\]

By induction, we prove that \( \alpha \theta^n = b_{n+1} - b_n \theta \) for \( n > 0 \).

From \( \alpha \theta^n = b_{n+1} - b_n \theta \), it follows that \( \alpha \theta^n - \bar{\alpha} \theta^n = b_n (\theta - \bar{\theta}) = b_n (\alpha - \bar{\alpha}) \), which implies that

\[
b_n = \frac{\alpha \theta^n - \bar{\alpha} \theta^n}{\alpha - \bar{\alpha}}.
\]

By Proposition 2.1, \( b_n = 1 \) or \(-1 \) only for \( n = 1, 2 \) and 5. \[ \square \]

To summarize, we have proved the following:

**Theorem 2.3.** The Diophantine equation \( 2x^2 + 1 = 3^n \) has exactly three positive rational integral solutions, namely \((x, n) = (1, 1), (2, 2) \) and \((11, 5)\).