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ACCESSIBLE DOMAINS IN THE HEISENBERG GROUP

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Abstract. We study the problem of accessibility of boundary points for domains in the sub–Riemannian setting of the first Heisenberg group. A sufficient condition for accessibility is given. It is a Dini–type continuity condition for the horizontal gradient of the defining function. The sharpness of this condition is shown by examples.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain in the Euclidean space and let $x_0 \in \Omega$ be a fixed basepoint. A boundary point $x \in \partial\Omega$ is said to be accessible from x_0 if there exists a continuous rectifiable curve $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x_0$, $\gamma(1) = x$ and $\gamma(t) \in \Omega$ for all $t \in [0,1)$. The property of accessibility is independent from the choice of the basepoint, but it depends on the regularity of the boundary $\partial\Omega$ in a neighborhood of x. If Ω has Lipschitz boundary, then every boundary point is accessible, but there are many examples of domains with fractal–type boundary, for instance the snowflake domain, whose boundary points are still accessible. For a simply connected domain in the complex plane the question of accessibility is a classical subject in complex function theory. By a theorem of Gehring and Hayman it is equivalent to the finiteness of the length of the image of radii under the Riemann mapping [16]. If the domain is not simply connected, the problem of accessibility is much more complicated and is at present the subject of research in complex dynamics (see [13], [17], [19]).

The notion of accessibility is of a purely metric nature and the definition of accessible domain can be introduced in a general metric space. If the metric space is locally bi–Lipschitz equivalent to some open set of $\mathbb{R}^n$, then the analysis is reduced to the Euclidean case. The problem becomes interesting in metric spaces not of Euclidean type and in this paper we study the question in the sub–Riemannian metric setting of the Heisenberg group. By a result of Semmes [18] there is no bi–Lipschitz embedding of the Heisenberg group into any Euclidean space. This is in contrast with the case of Riemannian manifolds and shows the genuinely non–Euclidean nature of the Heisenberg group as a metric space. Regularity properties of domains in Heisenberg and more general Carnot groups have been studied by Hansen and Hueber in [10], by Capogna and Tang in [5], by Capogna and Garofalo in [3] (see also the survey [4]) and more recently by Morbidelli and the second

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author in [14]. In [14] it is proved that any domain of class $C^{1,1}$ in a group of step two is non–tangentially accessible and thus, a fortiori, a John domain. The John property implies the accessibility of boundary points by means of rectifiable curves; however, this property typically fails for domains whose boundary is less than $C^{1,1}$–regular. It could even be expected that the property of accessibility already holds for domains with C^1–regular boundary. In this paper we show that this is not true: in the metric setting of the Heisenberg group the C^1–regularity of a domain does not ensure accessibility. However, an additional Dini condition on the modulus of continuity of the horizontal gradient of the defining function for the boundary (see (2.3)–(2.4)) is shown to be a sufficient condition.

For the sake of simplicity we have chosen to state and prove our theorems in the first Heisenberg group but the same results can be proved without any modification in higher-dimensional Heisenberg groups. We think that a Dini condition similar to (2.4) is sufficient for accessibility of boundary points in every Carnot group of step two, and moreover the techniques introduced are also likely to be useful in groups of steps greater than two.

The paper is organized as follows. In section 2 we recall some background results on the metric structure of the Heisenberg group and give the precise statement of the main theorem. In section 3 we discuss the accessibility condition for a special model domain. In section 4 we give the full proof of our main result.

2. Preliminaries and main result

We begin with some preliminaries about the metric structure of the first Heisenberg group. For an introduction to the subject we refer to the work of Gromov [9]. The underlying space of the first Heisenberg group $(\mathbb{H}^1, \cdot)$ is $\mathbb{R}^3$ and the group operation is given by

$$x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_2 y_1 - x_1 y_2)) \tag{2.1}$$

for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{H}^1$. Clearly, $x^{-1} = (-x_1, -x_2, -x_3)$. For any $x \in \mathbb{H}^1$ the map $\tau_x : \mathbb{H}^1 \to \mathbb{H}^1$ defined by $\tau_x(y) = x \cdot y$ is a *left translation*.

The differential structure of $\mathbb{H}^1$ is determined by the following so–called *horizontal* vector fields

$$X_1 = \partial_1 + 2x_2\partial_3, \quad \text{and} \quad X_2 = \partial_2 - 2x_1\partial_3, \tag{2.2}$$

which are left invariant for the group law (2.1) and generate by commutators the Lie algebra of the group. The plane distribution spanned by X_1 and X_2 is the *horizontal bundle* of $\mathbb{H}^1$. If Φ is a function of class C^1 (here and in the sequel the C^1–smoothness is always referred to as the differential structure of $\mathbb{R}^3$), then we write $\nabla_0\Phi(x) = (X_1\Phi(x), X_2\Phi(x))$. The vector $\nabla_0\Phi$ is the *horizontal gradient* of the function Φ.

The sub–Riemannian or *Carnot–Carathéodory* metric d on $\mathbb{H}^1$ is defined by using the horizontal vector fields via *admissible curves* as follows. A Lipschitz curve $\gamma : [0, 1] \to \mathbb{R}^3$ is called admissible if

$$\dot\gamma(t) = h_1(t)X_1(\gamma(t)) + h_2(t)X_2(\gamma(t))$$

for a.e. $t \in [0, 1]$, where $h_1, h_2 \in L^\infty(0, 1)$.

Now consider the scalar product on the horizontal bundle which makes X_1, X_2 an orthonormal basis. With respect to this scalar product the sub–Riemannian length of an admissible curve γ is given by

$$\text{length}(\gamma) = \int_0^1 |h(s)|\, ds,$$

where $|h(s)| = \sqrt{h_1^2(s) + h_2^2(s)}$. We can now define the sub–Riemannian length metric $d : \mathbb{H}^1 \times \mathbb{H}^1 \to [0, +\infty)$ by setting

$$d(x, y) = \inf\{\text{length}(\gamma) : \gamma : [0, 1] \to \mathbb{H}^1 \text{ is admissible and } \gamma(0) = x,\ \gamma(1) = y\}.$$

As the vector fields (2.2) satisfy the so–called maximal rank Chow–Hörmander condition (because X_1, X_2 and $[X_1, X_2] = -4\partial_3$ are linearly independent at every point), it follows that admissible connecting curves always exist and so for all $x, y \in \mathbb{H}^1$ we have $d(x, y) < +\infty$. By its definition it is clear that d is a left invariant metric, i.e. $d(z \cdot x, z \cdot y) = d(x, y)$ for all $x, y, z \in \mathbb{H}^1$. This metric generates the same topology as the Euclidean one, however finer properties of the two metrics are quite different. For recent results about Hausdorff measures and dimensions with respect to the metric d we refer to [2].

In this paper we shall deal with the notion of rectifiable curves in terms of the sub–Riemannian metric d. To formulate this precisely let $\gamma : [0, 1] \to \mathbb{H}^1$ be a continuous curve (curves will always be assumed to be continuous) and define its total variation

$$\text{Var}_d(\gamma) = \sup \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})),$$

the sup being taken over all partitions $0 = t_0 < t_1 < ... < t_n = 1$, $n \in \mathbb{N}$. The curve γ is *d–rectifiable* if $\text{Var}_d(\gamma) < +\infty$. The following proposition establishes the link between length and total variation, and states the existence of geodesics in the metric space $(\mathbb{H}^1, d)$.

Proposition 2.1. (i) *A curve $\gamma : [0, 1] \to \mathbb{H}^1$ is d–rectifiable if and only if it is admissible and moreover* $\text{length}(\gamma) = \text{Var}_d(\gamma)$.

(ii) *For all $x, y \in \mathbb{H}^1$ there exists a d–rectifiable curve $\gamma : [0, 1] \to \mathbb{H}^1$ such that $\gamma(0) = x$, $\gamma(1) = y$ and* $\text{length}(\gamma) = d(x, y)$.

According to the first statement above curves that are transversal to the horizontal bundle are never d–rectifiable (no matter how smooth they are). For instance the vertical axis in $\mathbb{R}^3$ is not d–rectifiable. The second statement says that d is a geodesic metric.

Definition 2.2. Let $\Omega \subset \mathbb{H}^1$ be an open domain and let $x_0 \in \Omega$ be a fixed basepoint. A point $x \in \partial\Omega$ is said to be *accessible* if there exists a d–rectifiable curve $\gamma : [0, 1] \to \mathbb{H}^1$ such that $\gamma(0) = x_0$, $\gamma(1) = x$ and $\gamma(t) \in \Omega$ for all $t \in [0, 1)$. The boundary $\partial\Omega$ is said to be *accessible* if all its points are accessible.

Now we can state our main theorem, which will be proved in section 4.

Theorem 2.3. *Let $\Omega \subset \mathbb{H}^1$ be a domain given by a C^1–smooth defining function $\Phi : \mathbb{H}^1 \to \mathbb{R}$, $\Omega = \{x \in \mathbb{H}^1 : \Phi(x) < 0\}$ and $\partial\Omega = \{x \in \mathbb{H}^1 : \Phi(x) = 0\}$. Assume that $\partial\Omega$ is a regular surface, i.e. $\nabla\Phi(x) \neq 0$ for all $x \in \partial\Omega$. For $x \in \partial\Omega$ let*

$$\omega_x(t) = \sup_{d(x,y) \le t} |\nabla_0\Phi(x) - \nabla_0\Phi(y)|. \tag{2.3}$$

If the Dini–type condition

$$\int_0^\delta \frac{\omega_x(t)}{t}dt < +\infty \tag{2.4}$$

holds for some $\delta > 0$, then $x \in \partial\Omega$ is accessible. Moreover, there are domains $\Omega \subset \mathbb{H}^1$ of class C^1 with non-accessible boundary points.

3. Accessibility for a model domain

In this section we study accessibility in a model case which will serve both as a key step for the proof of Theorem 2.3 and as an example showing the sharpness of condition (2.4). Our main technical device is the so-called Box–Ball estimate, which relates sub–Riemannian metric balls to flat Euclidean boxes.

We shall denote by $B(x,r) = \{y \in \mathbb{H}^1 : d(x,y) \le r\}$ the *Carnot–Carathéodory ball* centered at x having radius $r > 0$. For any $x \in \mathbb{H}^1$ define the homogeneous norm $\|x\| = \max\{(x_1^2 + x_2^2)^{1/2}, |x_3|^{1/2}\}$ and for $r > 0$ introduce the "box"

$$\mathrm{Box}(x,r) = \{x \cdot y : \|y\| \le r\}.$$

The statement of the Box–Ball estimate is as follows.

Proposition 3.1. *There exists a constant $\lambda \in (0,1)$ such that*:

(i) $\|y^{-1} \cdot x\| \le d(x,y) \le \lambda^{-1}\|y^{-1} \cdot x\|$ *for all $x, y \in \mathbb{H}^1$; or, equivalently,*
(ii) $\mathrm{Box}(x,\lambda r) \subset B(x,r) \subset \mathrm{Box}(x,r)$ *for all $x \in \mathbb{H}^1$ and $r > 0$.*

The main result of this section is the following:

Proposition 3.2. *For any $\alpha \ge 1$ let $\Omega = \{x \in \mathbb{H}^1 : \sqrt{x_1^2 + x_2^2} < x_3|\log x_3|^\alpha\}$. The boundary point $0 \in \partial\Omega$ is accessible if and only if $\alpha > 1$.*

Proof. We begin by proving that 0 is accessible if $\alpha > 1$. The function $\psi(t) = t|\log t|^\alpha$ is continuous on $[0,1]$ and enjoys the following properties:

$$\psi(0) = 0, \quad \psi \text{ is increasing in } (0,\delta), \quad \int_0^\delta \frac{1}{\psi(t)}dt < +\infty, \tag{3.5}$$

for some $\delta > 0$. The crucial last integral condition fails exactly when $\alpha = 1$.

For all $k \in \mathbb{N}$ define $t_k = 2^{-k}$ and let $p_k = (0,0,t_k) \in \Omega$. Let $0 < r_k < \sqrt{t_k}$ be the unique solution of the equation $\psi(t_k - r_k^2) = 2r_k/\lambda$. The real number $\lambda \in (0,1)$ is a fixed constant given by Proposition 3.1. With such a choice we have

$$\mathrm{Box}((0,0,t), r_k/\lambda) \subset \Omega \quad \text{for all } t \ge t_k. \tag{3.6}$$

Let $[p_k, p_{k-1}]$ be the line segment having as extremals p_k and p_{k-1}. Since the Euclidean vertical size of the boxes centered on the x_3–axis and with radius r_k is $2r_k^2$, we need at most $[N_k] + 1$ such piled boxes to cover this segment, where the number N_k is given by the condition

$$2N_k r_k^2 = t_{k-1} - t_k = t_k. \tag{3.7}$$

We denote these boxes by $\mathrm{Box}_{kj} = \mathrm{Box}(p_{kj}, r_k)$, with p_{kj} suitable points belonging to the x_3–axis and $j = 1, ..., [N_k]+1$. We also assume that $\mathrm{Box}_{k,j+1}$ lies immediately below Box_{kj} in such a way that the top part of $\partial\mathrm{Box}_{k,j+1}$ is the bottom part of $\partial\mathrm{Box}_{kj}$.

Let p_{kj}^+ and p_{kj}^- be the two intersection points of $\partial\mathrm{Box}_{kj}$ with the x_3–axis, p_{kj}^+ belonging to the top part of $\partial\mathrm{Box}_{kj}$ and p_{kj}^- belonging to the bottom one. In this way we have that $p_{kj}^- = p_{k,j+1}^+$. By Proposition 3.1(ii) and (3.6)

$$p_{kj}^+, p_{kj}^- \in \mathrm{Box}(p_{kj}, r_k) \subset B(p_{kj}, r_k/\lambda) \subset \mathrm{Box}(p_{kj}, r_k/\lambda) \subset \Omega,$$

and thus by Proposition 2.1(ii) there exists a d–rectifiable curve $\gamma_{kj} : [0,1] \to \Omega$ such that $\gamma_{jk}(0) = p_{kj}^+$, $\gamma_{jk}(1) = p_{kj}^-$ and

$$\text{length}(\gamma_{kj}) \le 2r_k/\lambda. \tag{3.8}$$

We denote by γ_k the d–rectifiable curve obtained by joining all the curves γ_{kj}, and we denote by $\gamma : [0,1] \to \mathbb{H}^1$ the curve obtained by joining the curves γ_k (and reparameterizing). Clearly, γ is continuous, $\gamma(t) \in \Omega$ for all $t \in [0,1)$ and $\gamma(0) = 0$ (by definition). We show that $\text{length}(\gamma) < +\infty$. Indeed by (3.8) and (3.7) we find (we assume $N_k \ge 1$)

$$\text{length}(\gamma) = \sum_{k=1}^{\infty} \text{length}(\gamma_k) \le \sum_{k=1}^{\infty} 2(1+[N_k])r_k/\lambda \le \frac{4}{\lambda}\sum_{k=1}^{\infty} N_k r_k = \frac{2}{\lambda}\sum_{k=1}^{\infty}\frac{t_k}{r_k},$$

and using the identity $\psi(t_k - r_k^2) = 2r_k/\lambda$ we get

$$\text{length}(\gamma) \le \frac{4}{\lambda^2}\sum_{k=1}^{\infty}\frac{t_k}{\psi(t_k - r_k^2)}.$$

Since $r_k^2 = t_k/(2N_k) \le t_k/2 = t_{k+1}$, we have $t_k - r_k^2 \ge t_{k+1}$ and thus $\psi(t_k - r_k^2) \ge \psi(t_{k+1})$, because ψ is increasing. We finally get

$$\text{length}(\gamma) \le \frac{16}{\lambda^2}\sum_{k=1}^{\infty}\frac{(t_{k+1} - t_{k+2})}{\psi(t_{k+1})} \le \frac{16}{\lambda^2}\int_0^{1/4}\frac{1}{\psi(t)}dt < +\infty.$$

This proves the first statement of the proposition.

Now consider the case $\alpha = 1$ and let $\psi(t) = t|\log t|$. Notice that

$$\int_0^{\delta}\frac{1}{\psi(t)}dt = +\infty$$

for any $\delta > 0$. We show that the boundary point $0 \in \partial\Omega$ is not accessible from the domain $\Omega = \{x \in \mathbb{H}^1 : \sqrt{x_1^2 + x_2^2} < \psi(x_3)\}$.

For any $k \in \mathbb{N}$ define $t_k = 2^{-k}$, $p_k = (0,0,t_k) \in \Omega$ and $r_k = 4\psi(t_k)$. The number of piled boxes centered on the x_3–axis and having radius r_k necessary to cover the line segment $[p_k, p_{k+1}]$ is at least $[N_k]$, where N_k is given by the condition

$$2N_k r_k^2 = t_k - t_{k+1} = t_{k+1} \quad \Leftrightarrow \quad N_k = \frac{t_{k+1}}{2r_k^2}.$$

We again call these boxes Box_{kj}, $j = 1, ..., [N_k]$, and we assume, as above, that $\mathrm{Box}_{k,j+1}$ lies immediately below Box_{kj}. Different from the boxes considered in the first part of the proof such boxes are wider: their union covers in fact the domain Ω in a neighborhood of $0 \in \partial\Omega$. In this way any continuous curve $\gamma : [0,1] \to \mathbb{H}^1$ such that $\gamma(1) = 0$ and $\gamma(t) \in \Omega$ for all $t \in [0,1)$ must travel through all the boxes Box_{kj}, but at most a finite number. Let us denote by γ_{kj} the intersection of γ with Box_{kj}.

As before let p_{kj}^+ and p_{kj}^- be the intersection points of $\partial\mathrm{Box}_{kj}$ with the x_3–axis, and let q_{kj}^+ and q_{kj}^- be two intersection points of $\partial\mathrm{Box}_{kj}$ with γ_{kj}. Our aim is to estimate the length of γ_{kj} by using the points p^+_{kj} and p^-_{kj}. The first step in this direction is the triangle inequality

$$\mathrm{length}(\gamma_{kj}) \geq d(q^-_{kj}, q^+_{kj}) \geq d(p^-_{kj}, p^+_{kj}) - d(p^-_{kj}, q^-_{kj}) - d(q^+_{kj}, p^+_{kj}).$$

Now notice that $d(p^-_{kj}, p^+_{kj}) \geq \|(p^+_{kj})^{-1} \cdot p^-_{kj}\| \geq r_k = 4\psi(t_k)$, $d(p^-_{kj}, q^-_{kj}) \leq \psi(t_k)$ and $d(p^+_{kj}, q^+_{kj}) \leq \psi(t_k)$. The last two statements follow from the fact that γ_{kj} lies inside Ω. After all, we obtain the estimate $\mathrm{length}(\gamma_{kj}) \geq 2\psi(t_k)$ for all $j = 1, ..., [N_k]$.

If γ_k denotes the intersection of γ with the union $\bigcup_{j=1}^{[N_k]} \mathrm{Box}_{kj}$, then

$$\begin{aligned}\mathrm{length}(\gamma_k) &\geq [N_k]\mathrm{length}(\gamma_{kj}) \geq \frac{N_k}{2}\mathrm{length}(\gamma_{kj}) \\ &\geq N_k\psi(t_k) = \frac{t_{k+1}\psi(t_k)}{2r_k^2} = \frac{t_{k+1}}{2^5\psi(t_k)}.\end{aligned}$$

We used the choice of $r_k = 4\psi(t_k)$.

It is now clear that the length of any continuous curve γ contained in Ω and reaching the origin is estimated from below by the sum of the γ_k as follows (for some $k_0 \in \mathbb{N}$ and $\delta > 0$):

$$\begin{aligned}\mathrm{length}(\gamma) &\geq \sum_{k=k_0}^{+\infty} \mathrm{length}(\gamma_k) \geq \frac{1}{2^5} \sum_{k=k_0}^{+\infty} \frac{t_{k+1}}{\psi(t_k)} \\ &\geq \frac{1}{2^5} \sum_{k=k_0}^{+\infty} \frac{(t_k - t_{k+1})}{\psi(t_k)} \geq \frac{1}{2^5} \int_0^\delta \frac{1}{\psi(t)} dt = +\infty.\end{aligned}$$

This shows that curve γ is not d–rectifiable and thus $0 \in \partial\Omega$ is not accessible. □

We conclude this section with two remarks.

Remark 3.3. From the above proof it is clear that the exact formula $\psi(t) = t|\log t|^\alpha$ was not really essential. Indeed, if $\Omega = \{x \in \mathbb{H}^1 : \sqrt{x_1^2 + x_2^2} < \psi(x_3)\}$ is an open set defined by a continuous function $\psi : [0, +\infty) \to \mathbb{R}$ satisfying conditions (3.5), then the boundary point $0 \in \partial\Omega$ is accessible. The second part of the proof shows the sharpness of (3.5).

Remark 3.4. Notice that $\partial\Omega$ given by $\sqrt{x_1^2 + x_2^2} = \psi(x_3) = x_3|\log x_3|^\alpha$, $\alpha > 0$, is in fact a C^1–smooth regular surface. To see this let $\varphi : [0, \delta) \to [0, \epsilon)$ be a local inverse of ψ and observe that φ is a C^1–smooth function with $\varphi'(0) = 0$. In a neighborhood of $0 \in \partial\Omega$ the boundary $\partial\Omega$ is now given by $\Phi(x) = x_3 - \varphi(\sqrt{x_1^2 + x_2^2}) = 0$.

4. Proof of the main theorem

In this section we prove Theorem 2.3. We have a domain $\Omega \subset \mathbb{H}^1$ of class C^1 given by $\Omega = \{x \in \mathbb{H}^1 : \Phi(x) < 0\}$. The function Φ is of class C^1 in the usual sense and satisfies $\nabla\Phi(x) \neq 0$ for all $x \in \partial\Omega$. A boundary point $x \in \partial\Omega$ is said to be *non-characteristic* if $\nabla_0\Phi(x) \neq 0$, while it is said to be *characteristic* if $X_1\Phi(x) = X_2\Phi(x) = 0$.

Non-characteristic points are always accessible. Assume, for instance, that $x \in \partial\Omega$ and $X_1\Phi(x) < 0$. The curve $\gamma(t) = \exp(tX_1)(x)$ is d–rectifiable and satisfies $\gamma(0) = x$ and $\gamma(t) \in \Omega$ for $t \in (0, \delta)$.

Characteristic points are notoriously difficult to handle. Such difficulties arise in various problems related to hypoelliptic partial differential equations (see [12],

[11] and [8]) or to geometric measure theory (see [7] and [6]). The size of the characteristic set in terms of Hausdorff measures was recently studied in [1]. Notice that in the domains Ω considered in the previous section the point $0 \in \partial\Omega$ was characteristic. Proposition 3.2 shows that accessibility may fail at characteristic points.

If $x \in \partial\Omega$ is characteristic, then $\partial_3\Phi(x) \neq 0$, because $\nabla_0\Phi(x) = 0$ but $\nabla\Phi(x) \neq 0$. Therefore, by the implicit function theorem the boundary $\partial\Omega$ can be represented in parametric form in a neighborhood of x by a function $x_3 = f(x_1, x_2)$. Without loss of generality we can assume that $\Phi(x) = f(x_1, x_2) - x_3$ for some function f of class C^1. Furthermore, both the horizontal gradient and the Heisenberg metric are invariant under group translations and so we can assume without loss of generality that our boundary point is $x = 0$.

Proof of Theorem 2.3. Let $0 \in \partial\Omega$ be a characteristic point of the boundary of $\Omega = \{x \in \mathbb{H}^1 : \Phi(x) < 0\}$. Define

$$\omega(t) = \sup_{d(x,0)\leq t} |\nabla_0\Phi(x)|.$$

We have to prove that if the following Dini condition holds (we assume $\delta = 1$ in (2.4)):

$$\int_0^1 \frac{\omega(t)}{t} dt < +\infty, \tag{4.9}$$

then the boundary point $0 \in \partial\Omega$ is accessible.

We can assume that $\Phi(x) = x_3 - f(x_1, x_2)$ where $f \in C^1(\mathbb{R}^2)$ is a function such that $f(0) = 0$ and $\nabla f(0) = 0$ (because $\nabla_0\Phi(0) = 0$). We write $z = (x_1, x_2)$ and define

$$\bar{\omega}(t) = \sup_{|z|\leq t} |\nabla f(z)| \quad \text{and} \quad \varphi(t) = \int_0^t \bar{\omega}(s)ds.$$

The function $\varphi : [0, +\infty) \to \mathbb{R}$ so defined is continuous, differentiable, $\varphi(0) = \varphi'(0) = 0$, and $t \mapsto \varphi'(t) = \bar{\omega}(t)$ is non-decreasing. Moreover

$$\begin{aligned} |f(z)| &= \Big| \int_0^1 \langle \nabla f(sz), z\rangle ds \Big| \leq |z| \int_0^1 |\nabla f(sz)| ds \\ &\leq |z| \int_0^1 \bar{\omega}(s|z|)ds = \int_0^{|z|} \bar{\omega}(s)ds = \varphi(|z|). \end{aligned} \tag{4.10}$$

We distinguish two cases. *First case*: there exists $\eta > 0$ such that $\bar{\omega}(t) = \varphi'(t) = 0$ for $0 \leq t \leq \eta$. In this case (4.10) yields that $f(z) = 0$ for $|z| < \eta$ and the accessibility of $0 \in \partial\Omega$ easily follows.

Second case: $\varphi'(t) > 0$ for all $t > 0$. The function φ is increasing and thus invertible. We denote by $\psi = \varphi^{-1}$ the inverse function. For the sake of simplicity we shall use the notation $(x_1, x_2) = z$ and $x_3 = t$. Let

$$D = \{(z,t) \in \mathbb{H}^1 : |z| < \psi(t)\} = \{(z,t) \in \mathbb{H}^1 : t > \varphi(|z|)\}.$$

Because of (4.10) $D \subset \Omega = \{(z,t) \in \mathbb{H}^1 : t > f(z)\}$ and moreover $0 \in \partial D$. If 0 is accessible from D it is also accessible from Ω and the theorem will be proved.

The function $\psi : [0, +\infty) \to \mathbb{R}$ is continuous, increasing and $\psi(0) = 0$. If we show that for some $\delta > 0$

$$\int_0^\delta \frac{1}{\psi(t)} dt < +\infty, \tag{4.11}$$

then all the hypotheses in (3.5) are satisfied, and the boundary point $0 \in \partial D$ is accessible from D. Our goal is to prove that (4.9) implies (4.11). To this aim let us introduce

$$\beta(t) := \sup_{0 \le s \le t,\, \varphi(s) \le t^2} \varphi'(s),$$

and notice that

$$\begin{aligned}\beta(t) &= \sup_{0 \le s \le t,\, \varphi(s) \le t^2} \bar{\omega}(s) = \sup_{0 \le s \le t,\, \varphi(s) \le t^2} \sup_{|z| \le s} |\nabla f(z)| \\ &= \sup_{|z| \le t,\, \varphi(|z|) \le t^2} |\nabla f(z)| \le \sup_{|z| \le t,\, |f(z)| \le t^2} |\nabla f(z)|.\end{aligned}$$

The last inequality is implied by (4.10).

By the definition of horizontal gradient we have

$$\nabla_0 \Phi = (\partial_1 f + 2x_2, \partial_2 f - 2x_1) = \nabla f + 2(x_2, -x_1),$$

and this yields $|\nabla f(z)| \le |\nabla_0 \Phi(z, f(z))| + 2|z|$. Since $|z| \le t$ and $|f(z)| \le t^2$ mean $(z, f(z)) \in \mathrm{Box}(0, t) \subset B(0, t/\lambda)$ we have

$$\beta(t) \le 2t + \sup_{d(x,0) \le t/\lambda} |\nabla_0 \Phi(x)| = 2t + \omega(t/\lambda),$$

and by (4.9) it follows that

$$\int_0^\lambda \frac{\beta(t)}{t} dt < +\infty. \tag{4.12}$$

We claim that (4.12) implies (4.11). Without loss of generality assume $\lambda = 1$, $\varphi(1) = 1$ and let $I = \{t \in [0,1] : \varphi(t) > t^2\}$. Then, using the fact that φ' is non-decreasing, we have for $t \in [0,1]$

$$\beta(t) = \begin{cases} \varphi'(\psi(t^2)) & \text{if } t \in I, \\ \varphi'(t) & \text{if } t \in [0,1] \setminus I. \end{cases}$$

The set I is open and can be written as an at most countable union of disjoint open intervals $I = \bigcup_{k=1}^{+\infty} I_k$, where each interval is of the form $I_k = (t_{1k}, t_{2k})$ with $\varphi(t_{1k}) = t_{1k}^2$ and $\varphi(t_{2k}) = t_{2k}^2$ (by continuity). Using this notation write

$$\int_0^1 \frac{\beta(t)}{t} dt = \int_{[0,1]\setminus I} \frac{\varphi'(t)}{t} dt + \sum_{k=1}^{+\infty} \int_{t_{1k}}^{t_{2k}} \frac{\varphi'(\psi(t^2))}{t} dt.$$

Performing the change of variable $\tau = \psi(t^2)$ in each integral in the sum we find

$$\int_{t_{1k}}^{t_{2k}} \frac{\varphi'(\psi(t^2))}{t} dt = \frac{1}{2} \int_{t_{1k}}^{t_{2k}} \frac{(\varphi'(\tau))^2}{\varphi(\tau)} d\tau.$$

The integration interval does not change. By the mean value theorem $\varphi(\tau) = \tau\varphi'(\bar{\tau})$ for some $\bar{\tau} \in (0, \tau)$. Since φ' is non-decreasing we get $\varphi(\tau) \le \tau\varphi'(\tau)$. As a consequence

$$\frac{(\varphi'(\tau))^2}{\varphi(\tau)} \ge \frac{\varphi'(\tau)}{\tau},$$

and all together we obtain

$$\int_0^1 \frac{\beta(t)}{t}dt \geq \int_{[0,1]\setminus I} \frac{\varphi'(t)}{t}dt + \frac{1}{2}\sum_{k=1}^{+\infty}\int_{t_{1k}}^{t_{2k}} \frac{\varphi'(t)}{t}dt \geq \frac{1}{2}\int_0^1 \frac{\varphi'(t)}{t}dt = \frac{1}{2}\int_0^1 \frac{1}{\psi(s)}ds.$$

This proves (4.11) and thus the accessibility of 0. The main statement of Theorem 2.3 is proved.

As far as the statement concerning the sharpness of condition (2.4) is concerned notice that the open set $\Omega = \{x \in \mathbb{H}^1 : \sqrt{x_1^2 + x_2^2} < x_3|\log x_3|\}$ is of class C^1 (see Remark 3.4), but the boundary point $0 \in \partial\Omega$ is not accessible from Ω by Proposition 3.2. $\square$

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References

1. Z. M. Balogh, *Size of characteristic sets and functions with prescribed gradients*, to appear in J. Reine Angew. Math.
2. Z. M. Balogh, M. Rickly, F. Serra Cassano, *Comparison of Hausdorff measures with respect to the Euclidean and the Heisenberg metric*, to appear in Publicacions Math.
3. L. Capogna, N. Garofalo, *Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot-Carathéodory metrics*, J. Fourier Anal. Appl. **4** (1998), 4-5, 403–432. MR **2000k:**35056
4. L. Capogna, N. Garofalo, D. M. Nhieu, *Examples of uniform and NTA domains in Carnot groups*, Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), 103–121, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000. MR **2002k:**30037
5. L. Capogna, P. Tang, *Uniform domains and quasiconformal mappings on the Heisenberg group*, Manuscripta Math. **86** (1995), no. 3, 267–281. MR **96f:**30019
6. B. Franchi, R. Serapioni, F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), no. 3, 479–531.
7. N. Garofalo, D. M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081–1144. MR **97i:**58032
8. N. Garofalo, D. Vassilev, *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups*, Math. Ann. **318** (2000), no. 3, 453–516. MR **2001j:**35042
9. M. Gromov, *Carnot–Carathéodory spaces seen from within*, in Sub–Riemannian Geometry (Eds. A. Bellaïche, J.–J. Risler), Progress in Math. **144**, Birkhäuser Basel (1996), 79–323. MR **2000f:**53034
10. W. Hansen, H. Hueber, *The Dirichlet problem for sublaplacians on nilpotent Lie groups – Geometric criteria for regularity*, Math. Ann. **276** (1987), 537–547. MR **88g:**31017
11. D. Jerison, *The Dirichlet problem for the Kohn Laplacian on the Heisenberg group. II*, J. Funct. Anal. **43** (1981), no. 2, 224–257. MR **83c:**58081b
12. J. J. Kohn, L. Nirenberg, *Non-coercive boundary value problems*, Comm. Pure Appl. Math. **18** (1965), 443–492. MR **31:**6041
13. G. Levin, F. Przytycki, *External rays to periodic points*, Israel J. Math. **94** (1996), 29–57. MR **97d:**58164
14. R. Monti, D. Morbidelli, *Regular domains in homogeneous groups*, Preprint.
15. A. Nagel, E. M. Stein, S. Wainger, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math. **155** (1985), 103–147. MR **86k:**46049
16. Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften, **299** Springer-Verlag, Berlin (1992). MR **95b:**30008

17. F. Przytycki, *Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps*, Fund. Math. **144** (1994), no. 3, 259–278. MR **95g:**58190
18. S. Semmes, *On the non existence of bilipschitz parameterization and geometric problems about A_∞ weights*, Revista Math. Iberoamericana **12** (1996), 337–410. MR **97e:**30040
19. A. Zdunik, *On biaccessible points in Julia sets of polynomials*, Fund. Math. **163** (2000), no. 3, 277–286. MR **2001f:**37058

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