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# SEQUENTIAL AND CONTINUUM BIFURCATIONS IN DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT. We examine the bifurcations to positive and sign-changing solutions of degenerate elliptic equations. In the problems we study, which do not represent Fredholm operators, we show that there is a critical parameter value at which an infinity of bifurcations occur from the trivial solution. Moreover, a bifurcation occurs at each point in some unbounded interval in parameter space. We apply our results to non-monotone eigenvalue problems, degenerate semi-linear elliptic equations, boundary value differential-algebraic equations and fully non-linear elliptic equations.

## 1. INTRODUCTION

In this paper we consider the non-linear, degenerate eigenvalue problem

$$Lg(u) = \lambda u, \qquad x \in \Omega := (0,1), \tag{1}$$

$$u = 0, \qquad x \in \partial\Omega, \tag{2}$$

where $Lu := -(a(x)u_x)_x + b(x)u$ and the coefficients $a, b \in C^1(\overline{\Omega})$ satisfy $a > 0$ and $b \geq 0$ on $\overline{\Omega}$. Consequently $L$ is uniformly elliptic, but the non-linear function $g \in C^1(\mathbb{R})$ is assumed to *degenerate* at zero with $g(0) = g'(0) = 0$.

Let us define $\gamma(u) = g(u)/u$ with $\gamma(0) = 0$ and begin with the statement of our assumptions on $g$:

G1. $g$ is an odd, strictly increasing function on $\mathbb{R}$,
G2. $u > 0$ implies $\gamma'(u) > 0$,
G3. $\gamma(u) \to \infty$ as $|u| \to \infty$.

These are all satisfied if, for instance, $g(u) = u|u|^m$, where $m > 0$.

**Definition 1.1.** Let $X, Y$ be Banach spaces, $F : X \times \mathbb{R} \to Y$ be continuous and satisfy $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. Let $\Sigma \subset X \times \mathbb{R}$ denote the set of all non-trivial ($u \neq 0$) solutions of $F(u, \lambda) = 0$. We say that $\lambda_0$ is a *sequential bifurcation point* from the trivial solution for $F(u, \lambda) = 0$ if there is a sequence $(u_n, \lambda_n) \in \Sigma$ such that $(u_n, \lambda_n) \to (0, \lambda_0)$ in $X \times \mathbb{R}$ as $n \to \infty$. If such a sequence $(u_n, \lambda_n)$ lies in some connected set $\mathcal{C} \subset \Sigma$, then $\lambda_0$ is said to be a *continuum bifurcation point.*

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We prove the following for (1)-(2). To each $\lambda > 0$ there is a sequence $u_n(\lambda) \in C^0(\overline{\Omega})$ of solutions of (1)-(2) such that (i) the number of zeros of $u_n(\lambda)$ in $\Omega$ is $n$, (ii) $u_n(\lambda) \to 0$ in $C^0(\overline{\Omega})$ as $n \to \infty$, (iii) $u_n(\lambda) \to 0$ in $C^0(\overline{\Omega})$ as $\lambda \to 0$, (iv) *every* $\lambda > 0$ is a sequential bifurcation point but *not* a continuum bifurcation point and (v) $\lambda = 0$ is a continuum bifurcation point.

We remark that the theory in [1] could be used to obtain *local* versions of some of the results proved here. However, our results are complementary to [1] in that they are global and impose no conditions on the growth of $g^{-1}$ near zero. Furthermore, we establish the existence of an unbounded interval of sequential bifurcation points. For the special case $g(u) = u|u|^m$, we note that a global branch of positive solutions was shown to exist in [2] in a study of flows in porous media.

The remainder of the paper is structured as follows. Section 2 introduces some notation and preliminary results. The main results of the paper appear in Section 3. Finally, in Section 4 we apply our results to non-monotone degenerate eigenvalue problems, degenerate semi-linear elliptic equations, boundary value differential-algebraic equations and fully non-linear elliptic equations.

## 2. Preliminaries

Throughout we write $\overline{U}$ for the closure of $U$ in a given metric space. We denote by $C^k(\overline{\Omega})$ the space of $k$-times differentiable functions on $\overline{\Omega}$, henceforth written simply as $C^k$ when there is no ambiguity. We note here that the imbedding $C^k \hookrightarrow C^r$ is compact if $k > r$. For any $u \in C^0$ with finitely many zeros we shall denote the number of zeros of $u$ in $\Omega$ by $\zeta(u)$.

It is well known that $L : C^2 \to C^0$ together with the Dirichlet boundary condition (2) has positive, simple eigenvalues, henceforth denoted by $\mu_j$ for $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where the principal eigenvalue $\mu_0$ has an associated positive eigenfunction $\phi_0$. Furthermore, $L$ has a continuous inverse $K : C^0 \to C^2$ which induces a compact linear map $K : C^0 \to C^0$.

The problem of finding continuous solutions of (1)-(2) with $g(u(\cdot)) \in C^2$ is therefore equivalent to

$$F(u, \lambda) := g(u) - \lambda K u = 0, \qquad u \in C^0, \tag{3}$$

where $g : C^0 \to C^0$ is the $C^1$ Nemytskii operator for $g$ defined by $(g(u))(x) = g(u(x))$. Our approach to solving (3) will be based on the regularized problem

$$F(u, \lambda; \varepsilon) := g(u) + (\varepsilon I - \lambda K)u = 0, \qquad \varepsilon \geq 0. \tag{4}$$

We define some solution sets. Throughout $E := C^0 \times \mathbb{R}$ is endowed with the norm $\|(u, \lambda)\|_E = \|u\| + |\lambda|$, where $\|\cdot\|$ denotes the sup-norm on $C^0$. The symbol $\langle \cdot, \cdot \rangle$ denotes the usual $L^2$ inner product. For $\varepsilon \geq 0$, $\Sigma(\varepsilon) \subset E$ will denote the set of non-trivial solutions $(u, \lambda)$ of $F(u, \lambda; \varepsilon) = 0$ in $E$. For $j \in \mathbb{N}_0$ we write $\Sigma_j(\varepsilon)$ for the subset of $\Sigma(\varepsilon)$ consisting of functions with $j$ zeros in $\Omega$. By $\Sigma_j^+(\varepsilon)$ ($\Sigma_j^-(\varepsilon)$) we denote the subset of $\Sigma_j(\varepsilon)$ of functions $u$ such that $g(u)_x(0) > 0$ ($g(u)_x(0) < 0$). For notational convenience we will simply write $\Sigma$ instead of $\Sigma(0)$ and $\Sigma_j^\pm$ for $\Sigma_j^\pm(0)$. We note here that since $g$ is odd, $(u, \lambda) \in \Sigma(\varepsilon)$ if and only if $(-u, \lambda) \in \Sigma(\varepsilon)$. Consequently $\Sigma_j^-(\varepsilon) = -\Sigma_j^+(\varepsilon)$.

*Remark* 1. The map $F : C^0 \times \mathbb{R} \to C^0$ is $C^1$ and has partial Fréchet derivative $d_u F(u, \lambda)[h] = g'(u)h - \lambda K h$ which is not a Fredholm mapping at $u = 0$ since $g'(0) = 0$. Consequently, one cannot use reduction methods based on the implicit function theorem to study bifurcations of (3) from the trivial solution. See also [3, 4, 15]. Moreover, $d_uF(0,\lambda)-\lambda K$ which, for $\lambda\neq 0$, has point spectrum and zero in the essential spectrum, but when $\lambda=0$, the spectrum consists only of zero.

**Lemma 2.1.** *Fix $\varepsilon\geq 0$. If $(u,\lambda)\in\Sigma(\varepsilon)$, then $\lambda>0$; that is, $\Sigma(\varepsilon)\subset C^0\times(0,\infty)$.*

*Proof.* Multiplying the relation $F(u,\lambda;\varepsilon)=0$ by $u$ and integrating over $\Omega$ gives, after setting $v=Ku$,

$$\int_\Omega \varepsilon u^2+ug(u)\ dx=\lambda\int_\Omega uKu\ dx=\lambda\int_\Omega vLv\ dx.$$

Noting that $ug(u)\geq 0$ and $\langle v,Lv\rangle\geq 0$, the result follows. $\square$

**Lemma 2.2.** *For $\varepsilon\in[0,1]$ the following a priori bound applies: to each $\ell>0$ there is an $M(\ell)>0$, independent of $\varepsilon$, such that if $\lambda\in[0,\ell]$, then $\|u\|\leq M(\ell)$ whenever $(u,\lambda)\in\Sigma(\varepsilon)$.*

*Proof.* Suppose that $\varepsilon u+g(u)=\lambda Ku$, where $0\leq\varepsilon\leq 1, 0\leq\lambda\leq\ell$ and let $x_0\in\Omega$ satisfy $\|u\|=|u(x_0)|$. Then

$$||g(u(x_0))|-|-\varepsilon u(x_0)||\leq|g(u(x_0))+\varepsilon u(x_0)|\leq\lambda\|K\||u(x_0)|,$$

where $\|K\|$ denotes the operator norm of $K\in BL(C^0)$. We therefore obtain $\gamma(\|u\|)\leq\lambda\|K\|+\varepsilon\leq\ell\|K\|+1$. Noting that $\gamma|:[0,\infty)\to[0,\infty)$ is surjective (by G3) and non-decreasing (by G2), the result follows on defining $M(\ell)$ to be any positive solution of $\gamma(M)=\ell\|K\|+1$. $\square$

Since $\varepsilon+g'(u)\geq\varepsilon>0$ for all $u\in\mathbb{R}$ and $\varepsilon>0$, the algebraic equation $\varepsilon u+g(u)=v$ has a unique solution $u=G(v;\varepsilon)$, where $G(\cdot;\varepsilon)\in C^1(\mathbb{R})$. When $\varepsilon=0$ we simply have $G(v;0)=g^{-1}(v)$, which is continuous. Moreover, $G:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ is continuous. We shall use this notation throughout and in the following theorem, which is a consequence of global bifurcation theory.

**Theorem 2.3.** *For each $\varepsilon>0$ and $j\in\mathbb{N}_0$, there are open, connected and unbounded sets $C_j^\pm(\varepsilon)\subset\Sigma_j^\pm(\varepsilon)$ such that $(0,\varepsilon\mu_j)\in\overline{C_j^\pm(\varepsilon)}$. Furthermore, for every $\lambda>\varepsilon\mu_j$ there exist $(\pm u_{j,\varepsilon},\lambda)\in C_j^\pm(\varepsilon)$, so that $(\varepsilon\mu_j,\infty)\subset\Pi\left(C_j^\pm(\varepsilon)\right)$, where $\Pi:E\to\mathbb{R}$ is the natural projection.*

*Proof.* For each fixed $\varepsilon>0$, apply global bifurcation results [13] to $v=\lambda KG(v;\varepsilon)$ and use the nodal properties of solutions to regular elliptic equations to demonstrate the existence of disjoint, unbounded continua $C_j^\pm(\varepsilon)$ with the stated properties. The existence of $(\pm u_{j,\varepsilon},\lambda)$ for $\lambda>\varepsilon\mu_j$ follows from the unboundedness of $C_j^\pm(\varepsilon)$ in $E$, Lemma 2.1 and Lemma 2.2. $\square$

If $u$ is a non-trivial solution of (4) with $\varepsilon>0$, then the zeros of the function $\varepsilon u+g(u)$ are transverse. The following result shows that transversality persists when $\varepsilon=0$.

**Theorem 2.4** (see [9, Theorem 2.2])**.** *Suppose that $f\in C^0(\mathbb{R})$ is strictly increasing and $f(0)=0$. If $u\in C^2(\overline{\Omega})$ is a solution of the initial value problem $Lu=f(u)$ on $\overline{\Omega}$ with $u(\alpha)=u_x(\alpha)=0$ for some $\alpha\in\overline{\Omega}$, then $u\equiv 0$ on $\overline{\Omega}$. Furthermore, $u$ has a finite number of zeros in $\overline{\Omega}$.*

**Corollary 2.5.** *If $(u,\lambda)\in\Sigma$, then $\zeta(u)=\zeta(g(u))<\infty$ and all zeros of $g(u)$ in $\overline{\Omega}$ are transverse. In particular, $\Sigma=\bigcup_{j=0}^\infty\left(\Sigma_j^+\cup\Sigma_j^-\right)$.*

*Proof.* If $(u,\lambda) \in \Sigma$ and $v := g(u)$, then $Lv = \lambda g^{-1}(v)$. The result follows from Lemma 2.1 and Theorem 2.4 with $f(v) = \lambda g^{-1}(v)$. □

## 3. The main results

In this section we prove the main results on the existence of non-trivial solutions of (3) and the nature of bifurcation points.

### 3.1. Existence of non-trivial solutions.

We begin with an existence and uniqueness result for elliptic equations.

**Lemma 3.1.** *Suppose $Au := -(\alpha(x)u_x)_x + \beta(x)u$, where $\alpha$ and $\beta$ satisfy the same assumptions as $a$ and $b$. Let $\lambda > 0$ and $\varepsilon \geq 0$ be fixed. If there exists a positive subsolution $\psi$ of the elliptic problem*

$$Av = \lambda G(v;\varepsilon), \qquad v(0) = v(1) = 0, \tag{5}$$

*then there exists a unique non-trivial, non-negative solution $v$ of* (5)*. Moreover, $v \geq \psi$.*

*Proof.* By assumption G3, $\lim_{v\to\infty} G(v;\varepsilon)/v = 0$ for fixed $\varepsilon \geq 0$. In particular this implies that $\limsup_{v\to\infty} \lambda G(v;\varepsilon)/v < \kappa_0$, where $\kappa_0$ denotes the principal eigenvalue of $A$. It is well known [6, 11] that non-negative solutions of the associated parabolic problem

$$v_t = -Av + \lambda G(v;\varepsilon), \qquad v(0,t) = v(1,t) = 0 \tag{6}$$

(with continuous initial condition $v(x,0) = v_0(x)$) have non-empty omega-limit sets $\omega(v_0)$ contained in the equilibrium set, comprising of solutions of (5). In particular, since $\psi$ is also a subsolution of (6), there exists a solution $v$ of (5) such that $v \geq \psi$. It therefore remains only to establish the uniqueness of $v$.

Suppose $w$ is any non-trivial, non-negative solution of (5). By G1 and the maximum principle, $w > 0$ in $\Omega$. Now, $\int_0^1 vAw - wAv\; dx = 0$ so that

$$\lambda \int_0^1 vG(w;\varepsilon) - wG(v;\varepsilon)\; dx = \int_0^1 \lambda vw \left( \frac{G(w;\varepsilon)}{w} - \frac{G(v;\varepsilon)}{v} \right)\; dx = 0.$$

By G2, $s \mapsto G(s;\varepsilon)/s$ is decreasing for all $s > 0$. Hence if $v$ and $w$ are ordered in $C^0$, then $v = w$ and $v$ is unique. If $v$ and $w$ are not ordered in $C^0$, then, for any $v_0 \geq \max\{v,w\}$, $\omega(v_0)$ must contain a solution $z$ of (5) such that $z \geq \max\{v,w\}$, whence $z \neq v$ and $z \neq w$. Hence $z$ and $v$ are ordered in $C^0$ and the above argument (with $w$ replaced by $z$) yields $z = v$, a contradiction. □

The following result is crucial, showing that non-trivial $j$-zero solutions of the regularized problem (4) cannot accumulate on the trivial branch as $\varepsilon \to 0$, except possibly at the origin.

**Proposition 3.2.** *Let $j \in \mathbb{N}_0$ be fixed and $0 \leq \varepsilon_n \to 0$ as $n \to \infty$. If $(u_n,\lambda_n) \in \Sigma_j^+(\varepsilon_n)$ satisfies $(u_n,\lambda_n) \to (0,\lambda)$ in $E$ as $n \to \infty$, then $\lambda = 0$. An analogous result holds for $\Sigma_j^-(\varepsilon_n)$.*

*Proof.* Necessarily $\lambda \geq 0$ by Lemma 2.1, so suppose that $\lambda > 0$. We first consider the case $j = 0$ (positive solutions). Fix $\lambda_* \in (0,\lambda)$ and choose $n_0$ such that $\varepsilon_n < \min\{\mu_0, (\lambda_*/\mu_0)\}$ and $\lambda_n > \lambda_*$ for all $n > n_0$. By the degeneracy of $g$ there is a $U > 0$ (independent of $n$) such that $g(u) + \varepsilon_n u \leq (\lambda_*/\mu_0)u$ for all $u \in [0,U]$ and $n > n_0$. Hence there is a $V > 0$ (independent of $n$) such that $G(v;\varepsilon_n) \geq (\mu_0/\lambda_*)v$

for all $v \in [0,V]$ and $n > n_0$. Let us normalise the principal eigenfunction of $L$, $\phi_0$, so that $\|\phi_0\| = V$. Since $G(\phi_0, \varepsilon_n) \geq (\mu_0/\lambda_*)\phi_0$ it follows that

$$-L\phi_0 + \lambda_n G(\phi_0; \varepsilon_n) \geq -L\phi_0 + \lambda_n(\mu_0/\lambda_*)\phi_0 \geq 0$$

and so $\phi_0$ is a subsolution of

$$Lv = \lambda_n G(v; \varepsilon_n), \qquad v(0) = v(1) = 0. \tag{7}$$

Hence by Lemma 3.1 there exists a unique positive solution $w_n$ of (7) and $w_n \geq \phi_0$. Now, $v_n := \varepsilon_n u_n + g(u_n)$ is also a positive solution of (7) and so by uniqueness $v_n = w_n$. But since $v_n = \lambda_n K u_n$ and $u_n \to 0$ in $C^0$ as $n \to \infty$, it follows that $v_n \to 0$ in $C^2$. In particular, by Hopf's boundary point lemma [12] applied to $\phi_0$, there exists an $n_1 > n_0$ such that $v_n < \phi_0$ in $\Omega$ for all $n > n_1$, a contradiction. This proves the result for $j = 0$. The result for $\Sigma_j^-(\varepsilon_n)$ is a trivial consequence of the symmetry of $g$.

Now suppose that $j \geq 1$. If $\xi_n^i$ $(i = 0, ..., j+1)$ denote the zeros of $u_n$ in $\overline{\Omega}$ in increasing order, let $\delta_n^i = \xi_n^{i+1} - \xi_n^i$ $(i = 0, ..., j)$. Then $v_n := \varepsilon_n u_n + g(u_n)$ (suitably restricted) is a constant sign solution of

$$Lv = \lambda_n G(v; \varepsilon_n), \qquad v(\xi_n^i) = v(\xi_n^{i+1}) = 0. \tag{8}$$

Since $\sum_{i=0}^{j} \delta_n^i \equiv 1$, we can assume for some $i$ that $\delta_n^i$ $(=: \delta_n)$ remains uniformly bounded away from zero. Passing to a subsequence if necessary we may assume that $\delta_n \to \delta_\infty \in (0,1]$ as $n \to \infty$. Now rescale the spatial variable $x$ in (8) according to $x \mapsto (x - \xi_n^{i+1})/\delta_n$ and, without loss of generality by the symmetry of $g$, we obtain a sequence $v_n$ of positive solutions of

$$L_n v \lambda_n G(v; \varepsilon_n), \qquad v(0) = v(1) = 0, \tag{9}$$

with $v_n \to 0$ in $C^2$, where $L_n v := -\delta_n^{-2}(a(x)v_x)_x + b(x)v$. If we denote by $\{\mu_0^n, \phi_0^n\}$ the principal eigenpair of the operator $L_n$, then spectral perturbation results for simple eigenvalues [7] show that $\mu_0^n \to \mu_0^\infty$, the principal eigenvalue of $L_\infty$, and $\phi_0^n \to \phi_0^\infty$ in $C^2$, where $\phi_0^\infty$ is the corresponding principal eigenfunction.

Note that there is a $V > 0$ (independent of $n$) and an $n_2 > n_0$ such that $G(v; \varepsilon_n) \geq (\mu_0^\infty + 1)v/\lambda_*$ for all $v \in [0,V]$ and $n > n_2$. If $\phi_0^n$ is normalised so that $\|\phi_0^n\| = V$, then

$$-L_n\phi_0^n + \lambda_n G(\phi_0^n; \varepsilon_n) \geq -L_n\phi_0^n + \lambda_n(\mu_0^\infty + 1)\phi_0^n/\lambda_* \geq (\mu_0^\infty + 1 - \mu_0^n)\phi_0^n \geq 0, \tag{10}$$

for all $n > n_2$ and so $\phi_0^n$ is a positive subsolution of (9) for all such $n$. An identical argument to the $j = 0$ case then leads to a contradiction as before. □

We can now prove the following existence result for (3).

**Theorem 3.3.** *Let $\lambda > 0$ and $j \in \mathbb{N}_0$ be given. Then there exist $(\pm u_j, \lambda) \in \Sigma_j^\pm$; that is, $\Pi\left(\Sigma_j^\pm\right) = (0, \infty)$.*

*Proof.* Let $\varepsilon_n \to 0$ be any positive sequence. From Lemma 2.2 and Proposition 3.2 with $\lambda_n \equiv \lambda$, there is a sequence $u_n$ of $C^2$ solutions of (4) which is $C^0$-bounded and bounded away from zero in $C^0$. Since $Ku_n$ is therefore $C^2$-bounded we may pass to a subsequence if necessary and assume that there is a $z \in C^1$ such that $Ku_n \to z$ in $C^1$. Hence, it follows that $\varepsilon_n u_n + g(u_n) \to \lambda z$ in $C^1$, from where $\varepsilon_n u_n \to 0$ in $C^0$, so that $g(u_n) \to \lambda z$ in $C^0$. Consequently, $u_n \to g^{-1}(\lambda z) =: u$ in $C^0$. Therefore,

$$\begin{aligned} \|g(u) - \lambda K u\| &= \|(g(u) - g(u_n)) + (g(u_n) - \lambda K u_n) + (\lambda K u_n - \lambda K u)\| \\ &\leq \|g(u) - g(u_n)\| + \varepsilon_n \|u_n\| + \lambda \|K\| \|u_n - u\| \to 0. \end{aligned}$$

Hence $u$ is a solution of (3). Since $z$ is a $C^1$-limit of functions with exactly $j$ transverse zeros we have $\zeta(z) = j$, whence $\zeta(u) = \zeta(g(u)) = \zeta(\lambda z) = j$. □

3.2. **Sequential and continuum bifurcations.** We may now establish the existence of an unbounded interval of sequential bifurcation points.

**Theorem 3.4.** *For each $\lambda > 0$ there exists a sequence $(u_j, \lambda) \in \Sigma$ such that $\zeta(u_j) = j$ and $u_j \to 0$ in $C^0$ as $j \to \infty$. In particular, every $\lambda \geq 0$ is a sequential bifurcation point for* (3).

*Proof.* Clearly, for each fixed $\lambda > 0$ there are infinitely many solutions of (3), $u_j$, parameterised by the number of zeros $j \in \mathbb{N}_0$. Recall that the corresponding zeros of $g(u_j)$ are transverse. We claim that $\lim_{j\to\infty} u_j = 0$ in $C^0$. Using the bound $\|u_j\| \leq M(\lambda)$ from Lemma 2.2, we may assume (on passing to a subsequence) that there is a $z \in C^1$ such that $Ku_j \to z$ in $C^1$, so that $g(u_j) \to \lambda z$ in $C^1$ and therefore $u_j \to g^{-1}(\lambda z)$ in $C^0$. If $u := g^{-1}(\lambda z)$, then $u$ is a solution of (3). Since $\zeta(g(u_j)) = j$, $g(u)$ cannot have finitely many zeros in $\Omega$. Hence by Theorem 2.4 $g(u) = 0$, from where $z = 0$. Hence $g(u_j) \to 0$ in $C^1$ and therefore $u_j \to 0$ in $C^0$.

In turn, this implies that $\lambda = 0$ is a sequential bifurcation point, simply by setting $\lambda_n = 1/n$ and choosing any $(\overline{u}_n, \lambda_n) \in \Sigma$ with $\|\overline{u}_n\| \leq 1/n$. □

Next we examine the question of which $\lambda \geq 0$ are continuum bifurcation points.

**Lemma 3.5.** *If $\mathcal{C} \subset \Sigma$ is connected and $(u, \lambda), (u', \lambda') \in \mathcal{C}$, then $\zeta(u) = \zeta(u')$.*

*Proof.* Let $(u, \lambda) \in \mathcal{C}$ and suppose that $(u_n, \lambda_n) \in \mathcal{C}$ satisfies $(u_n, \lambda_n) \to (u, \lambda)$ as $n \to \infty$. Using $g(u_n) \equiv \lambda K u_n$ we find that $g(u_n) \to g(u)$ in $C^1$ and because $g(u)$ has finitely many transverse zeros, $\zeta(u_n)\zeta(g(u_n)) = \zeta(g(u)) = \zeta(u)$ for all $n$ sufficiently large. This shows that $\zeta(\cdot)$ is an integer-valued continuous function on $\mathcal{C}$ and is therefore constant on $\mathcal{C}$. □

**Corollary 3.6.** *For all $\lambda > 0$, $\lambda$ is not a continuum bifurcation point.*

*Proof.* If $\lambda > 0$ is a continuum bifurcation point, then there exists a connected set $\mathcal{C} \subset \Sigma$ and a sequence $(u_n, \lambda_n) \in \mathcal{C}$ such that $(u_n, \lambda_n) \to (0, \lambda)$ in $E$. By Lemma 3.5 there exists a $j \in \mathbb{N}_0$ such that $(u_n, \lambda_n) \in \Sigma_j$ for all $n$. Passing to a subsequence if necessary, we may assume without loss of generality that $(u_n, \lambda_n) \in \Sigma_j^+$ for all $n$. By Proposition 3.2 with $\varepsilon_n \equiv 0$ it follows that $\lambda = 0$, a contradiction. □

**Theorem 3.7.** *$\lambda = 0$ is a continuum bifurcation point for* (3).

*Proof.* For each $\lambda > 0$ there is a unique $(u^+, \lambda) \in \Sigma_0^+$ by Theorem 3.3 and Lemma 3.1. We prove that the map $\lambda \mapsto u^+(\lambda)$ (with $u^+(0) = 0$) from $[0, \infty) \to C^0$ is continuous.

Fix $\lambda \geq 0$ and let $\lambda_n > 0$ be any sequence satisfying $\lambda_n \to \lambda$ as $n \to \infty$. Let $u_n^+ := u^+(\lambda_n)$. Suppose that $u^+(\cdot)$ is not continuous at $\lambda$; then there is a $\delta > 0$ such that $\|u_n^+ - u^+(\lambda)\| \geq \delta$ for all $n$. By Lemma 2.2, $u_n^+$ is bounded in $C^0$. From $u_n^+ = \lambda_n K g^{-1}(u_n^+)$ and the compactness of $K$, there exists a convergent subsequence, say $u_{n_j}^+ \to u^*$ in $C^0$. Hence $u^*$ is a solution of $Lu^* = \lambda g^{-1}(u^*)$. By Proposition 3.2, if $\lambda > 0$, then $u^* = u^+(\lambda)$, while if $\lambda = 0$, then $u^* = 0$. Either way this contradicts the above $\delta$-bound. □

We now utilise a theorem from topological analysis to obtain connectedness results for the sets of non-trivial sign-changing solutions.

**Definition 3.8.** Suppose that $(Z,d)$ is a complete metric space and that $\{S_n\}_{n=0}^{\infty}$ is a family of connected subsets of $Z$. For $S \subset Z$ define $d(z,S) := \inf_{s\in S} d(s,z)$,

$$S_{\text{inf}} := \left\{ z \in Z : \lim_{n\to\infty} d(z,S_n) = 0 \right\},$$

$$S_{\text{sup}} := \left\{ z \in Z : \liminf_{n\to\infty}\ d(z,S_n) = 0 \right\}.$$

**Theorem 3.9** (see [17])**.** *Suppose that $\bigcup_{n=0}^{\infty} S_n$ is relatively compact in $Z$. If $S_{\text{inf}} \neq \emptyset$, then $S_{\text{sup}}$ is a non-empty, closed and connected subset of $Z$.*

**Theorem 3.10.** *Let $j \in \mathbb{N}_0$ be given. There exist unbounded, closed and connected sets $\mathcal{C}_j^{\pm} \subset \Sigma_j^{\pm} \cup \{(0,0)\}$ such that $(0,0) \in \mathcal{C}_j^{\pm}$. In particular, $\Pi\left(\mathcal{C}_j^{\pm}\right) = [0,\infty)$.*

*Proof.* Let $\varepsilon_n \to 0$ be any positive sequence. For fixed $\nu > 0$ let $S_n^{+,j}(\nu)$ be the maximal connected component of $C_j^+(\varepsilon_n) \cap (C^0 \times [0,\nu])$ which contains $(u,\lambda) = (0, \epsilon_n \mu_j)$ in its closure, where $C_j(\varepsilon)$ is defined in Theorem 2.3. Note that by Theorem 2.3, $S_n^{+,j}(\nu)$ contains non-trivial elements of the form $(u,\lambda)$ for all $\lambda \in [\varepsilon_n \mu_j, \nu]$, provided $n$ is sufficiently large and $(0, \varepsilon_n \mu_j) \in \overline{S_n^{+,j}(\nu)}$. By the compactness of $[0,\nu]$ and of the operator $K : C^0 \to C^0)$ it follows that $\bigcup_{n=0}^{\infty} S_n^{+,j}(\nu)$ is relatively compact in $E$. Clearly $(0,0) \in S_{\text{inf}}^{+,j}(\nu)$ and so $S_{\text{inf}}^{+,j}(\nu)$ is non-empty. Hence by Theorem 3.9 $S_{\text{sup}}^{+,j}(\nu)$ is non-empty, closed and connected in $E$.

Now, by the construction of solutions in Theorem 3.3 it follows that

$$\{(u_j,\lambda) \in \Sigma_j^+ \ :\ \lambda \in (0,\nu]\} \cup \{(0,0)\} \subset S_{\text{inf}}^{+,j}(\nu) \subset S_{\text{sup}}^{+,j}(\nu).$$

Moreover, if $(u,\lambda) \in S_{\text{sup}}^{+,j}(\nu)$ there exists a sequence $(u_n,\lambda_n) \in S_n^{+,j}(\nu)$ such that $(u_n,\lambda_n) \to (u,\lambda)$ in $E$. Then,

$$\begin{aligned} \|g(u) - \lambda K u\| \quad &\leq \quad \|g(u) - g(u_n)\| + |\lambda_n - \lambda| \|K u_n\| \\ &\qquad + \lambda \|K(u_n - u)\| + \varepsilon_n \|u_n\| \to 0, \end{aligned}$$

so that $(u,\lambda)$ is a solution of (3). By Proposition 3.2 and Theorem 2.4 either $(u,\lambda) = (0,0)$ or $(u,\lambda) \in \Sigma_j^+$ for some $j \in \mathbb{N}_0$.

Clearly, $S_{\text{sup}}^{+,j}(\nu) \subset S_{\text{sup}}^{+,j}(\nu')$ if $\nu < \nu'$ and it follows that $\mathcal{C}_j^+ : \bigcup_{\nu>0} S_{\text{sup}}^{+,j}(\nu)$ has the stated properties. The result for $\mathcal{C}_j^-$ follows similarly. □

**Example 1.** Consider a semi-linear, degenerate elliptic equation $\Delta\varphi(v) + \lambda f(v) = 0$ with Dirichlet boundary conditions on an annulus $R_1 < |y| < R_2$ in $\mathbb{R}^n$ [8]. Suppose that $\varphi$ and $f$ are strictly increasing, odd functions satisfying $\varphi(0) = f(0) = 0$. Setting $u = f(v)$ one obtains $\Delta g(u) + \lambda u = 0$, where $g(u) := \varphi(f^{-1}(u))$. Suppose that $\varphi$ and $f$ are such that $g$ satisfies G1-G3. Now, radially symmetric solutions satisfy $(r^{n-1} g(u)_r)_r + \lambda r^{n-1} u = 0$, where $r = |y|$. Setting $x = r^n/n$ then yields the equivalent problem $-(a(x) g(u)_x)_x = \lambda u$ for $x \in (R_1^n/n, R_2^n/n)$, where $a(x) := (nx)^{2(1-1/n)}$, to which the results of this section apply. Such a situation occurs when $\varphi(v) = v|v|^{m-1}$ and $f(v) = v|v|^{p-1}$ for $m > p > 0$.

## 4. Applications

4.1. **Non-monotone eigenvalue problems.** Here we apply our main results to problems where $g$ is only *locally* monotonic near zero. We still obtain infinitely many solution sets in $E$ parameterised by zeros together with an unbounded interval of sequential (but not continuum) bifurcation points.

**Lemma 4.1.** *Let $\delta > 0$ and suppose that $g : [0,\delta] \to [0,\infty)$ is a strictly increasing $C^1$ function which is $C^2$ on $(0,\delta]$ with $g(0) = g'(0) = 0$ and $g''(\delta) > 0$. If $\gamma(u) = g(u)/u$ satisfies $\gamma'(u) > 0$ on $(0,\delta]$, then there exists an odd, strictly increasing $C^1$ extension $\overline{g} : \mathbb{R} \to \mathbb{R}$ such that $g|_{[0,\delta]} = \overline{g}|_{[0,\delta]}$. Moreover, if $\overline{\gamma}(u) := \overline{g}(u)/u$, then $\overline{\gamma}'(u) > 0$ for all $u > 0$ and $\overline{\gamma}(u) \to \infty$ as $|u| \to \infty$.*

*Proof.* Since $u^2\gamma'(u) = ug'(u) - g(u)$ we have $g'(\delta) > 0$. Now define $\overline{g}$ to be the odd extension of the function

$$\begin{cases} g(u) & : \quad 0 \le u \le \delta, \\ g(\delta) + (u-\delta)g'(\delta) + \frac{1}{2}(u-\delta)^2 g''(\delta) & : \quad u \ge \delta, \end{cases}$$

and then for $|u| \ge \delta$ we have $u^2\overline{\gamma}'(u) = \delta^2\gamma'(\delta) + \frac{1}{2}g''(\delta)(u^2 - \delta^2) > 0$. □

We can now deduce the following result when $g$ is only *locally* monotonic.

**Theorem 4.2.** *For some $\delta > 0$ suppose that $g : [-\delta,\delta] \to \mathbb{R}$ is a strictly increasing, odd, $C^1$ function which is $C^2$ on $[-\delta,\delta]\backslash\{0\}$ and $g(0) = g'(0) = 0, g''(\delta) > 0$. If $\gamma'(u) \ge 0$ on $(0,\delta]$, then there exist closed, connected sets $\mathcal{C}_j^{\pm} \subset \Sigma_j^{\pm} \cup \{(0,0)\}$ such that $(0,0) \in \mathcal{C}_j^{\pm}$. At least one, but possibly both, of the following is true:*

(1) *$\mathcal{C}_j^{\pm}$ is unbounded,*
(2) *there exists a $(u,\lambda) \in \mathcal{C}_j^{\pm}$ such that $\|u\| = \delta$.*

*Furthermore, for each $\lambda > 0$ there exists a sequence $u_j \in \Sigma$ such that $\zeta(u_j) \to \infty$ and $u_j \to 0$ in $C^0$ as $j \to \infty$. In particular, every $\lambda \ge 0$ is a sequential bifurcation point and $\lambda = 0$ is a continuum bifurcation point for* (3)*.*

*Proof.* Use Lemma 4.1 to replace (3) by $\overline{g}(u) = \lambda K u$ to which Theorems 3.10 and 3.4 apply. The result follows from the fact that solutions of $\overline{g}(u) = \lambda K u$ with $\|u\| \le \delta$ also satisfy (3). □

4.2. **Degenerate diffusion equations.** Consider a quasi-linear parabolic equations of the form

$$v_t - (a(x)D(v)_x)_x + b(x)D(v) = \lambda f(v), \tag{11}$$

supplied with Dirichlet boundary conditions and given initial data. Such equations arise naturally in many branches of the physical and biological sciences [5, 14]. Upon setting $u = f(v)$ and defining $g(u) = D(F(u))$ (see below) one may use Theorem 4.2 to obtain information on the existence of equilibrium solutions of (11) whenever $f$ and $D$ are monotonic near zero. We omit the trivial proof.

**Theorem 4.3.** *Suppose that $D, f \in C^1(\mathbb{R})$ are odd, strictly increasing functions such that $D(0) = D'(0) = f(0) = 0$ and $f'(0) > 0$. Let $F$ denote the local $C^1$ inverse of $f$ near $0$. If there exists a $\delta^* > 0$ such that $D \in C^2(0,\delta^*]$ and $uF'(u)D'(F(u)) - D(F(u)) > 0$ on $(0,\delta^*]$, then the conclusions of Theorem 4.2 hold for equilibrium solutions of* (11) *for each $\delta \le \delta^*$ for which $(D(F))''(\delta) > 0$. In particular, the latter conditions hold for all sufficiently small $\delta > 0$ whenever $D, f \in C^3(\mathbb{R})$, $D''(0) = 0$ and $D'''(0) > 0$.*

**Example 2.** Theorem 4.3 applies to a degenerate form of the Chafée-Infante problem (see [6])

$$v_t - (v|v|^m)_{xx} = \lambda v(1 - v^2), \qquad m > 0.$$

**Example 3.** Consider the *slow diffusion* problem

$$u_t - (a(x)[\exp(-1/u)]_x)_x = \lambda u$$

with Dirichlet boundary conditions, where $g(u) := [\exp(-1/u)]$ denotes the odd extension of $\exp(-1/u)$ for $u > 0$. Theorem 4.3 applies to the associated steady-state problem. Note however, that the global results of Section 3 do not apply even though $g$ is globally monotonic due to the failure of the coercivity condition G3. Due to the *flat* nature of $g$ at $u = 0$, the results of [1] do not apply to this equation.

4.3. **Boundary value differential-algebraic equations.** We can also use the above results to find steady-states of parabolic systems

$$\begin{aligned} u_t + Lu &= \lambda F(u,v), \qquad u(0,t) = u(1,t) = 0, \\ v_t &= G(u,v), \end{aligned}$$

or equivalently, the boundary value differential-algebraic equation (DAE)

$$Lu = \lambda F(u,v),\ G(u,v) = 0, \qquad u(0) = u(1) = 0. \tag{12}$$

Problems of this nature are considered in [10], motivated by interactions between diffusive and non-diffusive species. We have the following theorem regarding solutions of (12).

**Theorem 4.4.** *Suppose that $F$ and $G$ are $C^r$ functions with $r \geq 4$ such that $F(0,0) = G(0,0) = 0$, $G_v(0,0) = G_{vv}(0,0) = 0$, $F(-u,-v) = -F(u,v)$ and $G(u,-v) = -G(-u,v)$. If $G_uF_vG_{vvv} < 0$ at $(0,0)$, then $\lambda = 0$ is a continuum bifurcation point to a branch of positive solutions of* (12)*. There are countably many sets of non-trivial solutions $\mathcal{C}_j \subset C^2(\overline{\Omega}) \times C^0(\overline{\Omega}) \times \mathbb{R}$ such that $\mathcal{C}_j \cup \{(0,0,0)\}$ is connected, and if $(u,v,\lambda) \in \mathcal{C}_j$, then $u$ and $v$ have $j$ zeros in $\Omega$. Every $\lambda \in (0,\infty)$ is a sequential bifurcation point, but no element of $(0,\infty)$ is a continuum bifurcation point.*

*Proof.* Apply the implicit function theorem to $G(u,v) = 0$ and solve this constraint as $u = U(v)$, where $U(0) = U'(0) = U''(0) = 0$ and $U'''(0) = -G_{vvv}(0,0)/G_u(0,0) \neq 0$. Then (12) is reduced to $LU(v) = \lambda F(U(v),v)$, so now set $w = F(U(v),v)$. This can be solved by the inverse function theorem for $v = V(w)$ such that $V(0) = 0, V'(0) = 1/F_v(0,0)$ and $V''(0) = -F_{vv}(0,0)/F_v(0,0)^3$. Now, (12) is locally equivalent to $LU(V(w)) = \lambda w$, so we set $g(w) = U(V(w))$.

Now, the hypotheses on $F$ and $G$ ensure that $U$ and $V$ are odd functions, so that $g(w)$ is also odd; now set $\gamma(w) = g(w)/w$. Differentiating, we see that $g(w) = \xi w^3 + o(w^3)$ where $\xi = -G_{vvv}G_uF_v/(G_u^2F_v^4) > 0$ and where each of these derivatives is evaluated at $(u,v) = (0,0)$. Hence there is a $\delta > 0$ such that $g(w) > 0, \gamma'(w) > 0$ on $(0,\delta]$ and $g''(\delta) > 0$. One can now apply Theorem 4.2 to $Lg(w) = \lambda w$. □

**Example 4.** The hypotheses of Theorem 4.4 are satisfied by the steady-state problem for the reaction-diffusion system

$$\begin{aligned} u_t - u_{xx} &= \lambda \sin v, \quad u(0,t) = u(1,t) = 0, \\ v_t &= u + u^2v - v^3. \end{aligned}$$

*Remark* 2. Fully non-linear elliptic equations of the form

$$Lu = f(u, Lu), \qquad u(0) = u(1) = 0, \tag{13}$$

can be written as a boundary value DAE by setting $v = Lu$, $F(u,v) = v$ and $G(u,v) = f(u,v) - v$. Problems of this type are studied, for instance, in [16]. A solution of (12) when $\lambda = 1$ provides a solution of (13) and these can be obtained using Theorem 4.4 with suitable restrictions on $f$.

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