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GLOBAL PROPERTIES OF THE LATTICE OF Π^0_1 CLASSES

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Abstract. Let $\mathcal{E}_\Pi$ be the lattice of Π^0_1 classes of reals. We show there are exactly two possible isomorphism types of end intervals, $[P, 2^\omega]$. Moreover, finiteness is first order definable in $\mathcal{E}_\Pi$.

1. Introduction

The structure of the lattice $\mathcal{E}_\Pi$ of Π^0_1 classes has been investigated in several recent papers, for instance, [3]. A central theme is to compare and contrast the structure with the lattice $\mathcal{E}$ of computable enumerable sets.

In this paper, we solve a number of open problems from the 1999 AMS Summer Conference on Computability Theory. One general problem is to determine which subsets invariant under automorphisms are definable in a given structure. In particular, this is relevant for subsets which are natural in some sense. For $\mathcal{E}_\Pi$, an example is the set of finite classes. We show that this set is definable in $\mathcal{E}_\Pi$, which solves the first part of Problem 6.1 in [2]. The Cantor-Bendixson rank is an important way of classifying Π^0_1 classes. We solve Problem 6.2 of [2] by showing that the family of countable Π^0_1 classes of rank α is definable if and only if $\alpha < \omega$.

Intervals of the lattice $\mathcal{E}_\Pi$ were first studied in [3], where it was shown that, in contrast to the lattice $\mathcal{E}^*$, there are finite initial intervals in the quotient lattice $\mathcal{E}_\Pi / =^*$ which are not Boolean algebras. An important problem here is to characterize all the possible intervals. We show here that there are exactly two possible isomorphism types of *end* intervals, $[P, 2^\omega]$, which answers a question of Herrmann (Problem 6.6 of [2]) and also Problem 9.7 of [8]. As a tool, we prove results on the complexity of possible representations of $\mathcal{E}$ and other structures, which are of interest by themselves. In recent work, Nies has found a Σ_3 sentence separating the two lattices.

2. Preliminaries

2.1. **Some notation.** As in [3] we will be applying some results on effective Boolean algebras and coding due to Nies [10, 11, 12] and also Harrington and Nies [9]. In the first paper, we used the language of c.e. ideals of the computable dense Boolean algebra rather than the language of Π^0_1 classes, to conform to the presentation of

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[9, 10, 11, 12]. Here we will translate some of this background material into the language of Π^0_1 classes, as in [1, 2, 5].

The underlying computable dense Boolean algebra $\mathcal{Q}$ may be thought of as the family of clopen subsets of $\{0,1\}^\omega$. For any finite sequence σ, let $I(\sigma) = \{x : \sigma \prec x\}$. Each clopen set has a unique representation as a finite union of disjoint intervals $I(\sigma_1) \cup \cdots \cup I(\sigma_k)$, where each σ_i has the same length and k is taken to be as small as possible. Then the join ($\vee$) and meet ($\wedge$) operations are clearly computable, as well as the complement operation and the partial ordering relation on $\mathcal{Q}$.

A c.e. Boolean algebra is given by a model $(\mathbb{N}, \preceq, \vee, \wedge)$ such that $\preceq$ is a c.e. relation which is a pre-ordering, $\vee, \wedge$ are total computable binary functions, and the quotient structure $\mathcal{B} = (\mathbb{N}, \preceq, \vee, \wedge)/\approx$ is a Boolean algebra (where $n \approx m \Leftrightarrow n \preceq m \ \&\ m \preceq n$). We can suppose that $0 \in \mathbb{N}$ names the least and $1 \in \mathbb{N}$ the greatest element of $\mathcal{B}$. For Σ^0_k-Boolean algebras, one requires that $\preceq$ be Σ^0_k and that $\wedge, \vee$ be computable in $\emptyset^{(k-1)}$. For a Σ^0_k Boolean algebra $\mathcal{B}$, let

$$\mathcal{I}(\mathcal{B}) \ := \ \text{the lattice of } \Sigma^0_k\text{–ideals of } \mathcal{B}.$$

Clearly, c.e. Boolean algebras correspond to c.e. ideals of $\mathcal{Q}$ and similarly for computable algebras and ideals. At the same time an ideal I of $\mathcal{Q}$ corresponds to a Π^0_1 class P in that $I = \{U \in \mathcal{Q} : P \cap U = \emptyset\}$ and $P = \{0,1\}^\omega - \bigcup I$. We can use this last equation to assign index sets for Π^0_1 classes (as in [4]). Let $\sigma_0, \sigma_1, \ldots$ enumerate $\{0,1\}^{<\omega}$ and let W_e be the eth c.e. subset of ω, as usual. Then the eth Π^0_1 class is given by

$$P_e = \{0,1\}^\omega - \bigcup_{n \in W_e} I(\sigma_n). \tag{1}$$

An ideal I in a Boolean algebra $\mathcal{B}$ is said to be *principal* if there is some b such that $I = \{a : a \leq b\}$. The ideal I corresponding to a Π^0_1 class P as above is principal if and only if P is clopen. Thus, we will refer to a non-clopen Π^0_1 class P as *nonprincipal*. For any Π^0_1 class P, let $S(P)$ be the lattice of Π^0_1 classes Q such that $P \subset Q$.

2.2. **An effectively dense Σ^0_3 Boolean algebra.** A c.e. Boolean algebra $\mathcal{B}$ is called *effectively dense* [11] if there is a computable F such that $\forall x\ [F(x) \preceq x]$ and $\forall x \not\approx 0\ \ [0 \prec F(x) \prec x]$. More generally, a Σ^0_k Boolean algebra $\mathcal{B}$ is effectively dense if the above holds with some $F \leq_T \emptyset^{(k-1)}$. We first summarize the construction from [3] of an effectively dense Σ^0_3 Boolean algebra from an arbitrary nonprincipal Π^0_1 class P. We will present these results from the point of view of Π^0_1 classes rather than c.e. ideals.

The following technical lemma shows that we can make the intervals $I(\sigma_n)$ in (1) disjoint.

Lemma 2.1. *For any Π^0_1 class P, there is a c.e. set A such that $P = 2^\omega - \bigcup_{n \in A} I(\sigma_n)$, where for $m \neq n$, $I(\sigma_m) \cap I(\sigma_n) = \emptyset$.*

Proof. Let $2^\omega - P = \bigcup_n I(\tau_n)$, for some computable sequence $\{\tau_n\}$. For each n, express the clopen set $I(\tau_n) - \bigcup_{m<n} I(\tau_m)$ as a finite union $\bigcup_{k \in C_n} I(\sigma_k)$ and let $A = \bigcup_{n<\omega} C_n$. □

The underlying lattice $\mathcal{E}_\Pi$ of Π^0_1 classes may be viewed as a Π^0_2 structure using the representation given by (1). That is, there are recursive functions m and j such that $P_a \cup P_b = P_{j(a,b)}$ and $P_a \cap P_b = P_{m(a,b)}$, and the relation "$P_a = P_b$" is Π^0_2.

Let P be a nonprincipal Π^0_1 class. We write $Q_1 \sqcap Q_2 = P$ if $Q_1 \cap Q_2 = P$ and $Q_1 \cup Q_2 = 2^\omega$, and $P \sqsubset Q$ if there exists Q_2 such that $Q \sqcap Q_2 = P$. We observe that for any clopen set V, $(P \cup V) \sqcap (P \cup V^c) = P$, so that $P \sqsubset P \cup V$. In $[P, 2^\omega]$, we can define the Σ^0_3 Boolean algebra of complemented elements as

$$\mathcal{B}(P) = \{X : P \sqsubset X\}.$$

This is indeed a Σ^0_3 Boolean algebra, since $P_a \in \mathcal{B}(P) \iff (\exists b)[P_a \cap P_b = P \& P_a \cup P_b = 2^\omega]$.

In the following, we recast Definition 4.5 from [12] in the language of Π^0_1 classes.

Definition 2.2. Q is a locally principal extension of P if $P \subset Q$ and $Q - P$ is open.

This has a first order definition in the lattice $\mathcal{E}_\Pi$, by the following.

Lemma 2.3. *Q is a locally principal extension of P if and only if $P \subseteq Q$ and, for all clopen sets V, if $P \cap V = \emptyset$, then $Q \cap V$ is clopen.*

Proof. Let Q be a Π^0_1 class with $P \subset Q$. Suppose first that $Q - P$ is open and let V be a clopen set disjoint from P. Then $Q \cap V$ is closed, since both Q and V are closed. $Q \cap V$ is also open, since $Q \cap V = (Q - P) \cap V$. On the other hand, suppose that Q satisfies the condition of the lemma. Then for any $x \in Q - P$, choose a clopen set V such that $x \notin V$ and $P \cap V = \emptyset$. It follows that $Q \cap V$ is a clopen subset of $Q - P$ containing x. Thus $Q - P$ is an open set. □

We show that our definition is, in fact, the dual of the definition for c.e. ideals of $\mathcal{Q}$ from [12]. An ideal B is a locally principal *subideal* of A if $B \subseteq A$ and $\forall e \in A\ [0, e] \cap B$ is principal . The immediate dual (with Q, P corresponding to B, A) is $Q \supseteq P$, and for all clopen $W \supseteq P$, $W \cup Q$ is clopen. Now let $V = W^c$, and note that $V^c \cup Q$ clopen iff $V \cap Q$ clopen.

Lemma 2.4. *For any nonclopen Π^0_1 class P, there exists a locally principal extension Q of P such that $P \not\sqsubset Q$.*

Proof. By Lemma 2.3, let $2^\omega - P = \bigcup_n U_n$, where $\{U_n\}_{n<\omega}$ is a computable sequence of disjoint intervals. Now choose a noncomputable c.e. set B and let $Q = 2^\omega - \bigcup_{n \in B} U_n$. Clearly, $P \subset Q$ and furthermore, $Q - P = \bigcup_{n \notin B} U_n$ is an open set. Suppose by way of contradiction that R is a Π^0_1 class such that $Q \cup R = 2^\omega$ and $Q \cap R = P$. But this means that $2^\omega - R = Q - P$. Then $e \in B \iff R \cap U_n \neq \emptyset$, which is a co-c.e. condition, contradicting the assumption that B is not computable. □

If $P \subseteq Q$, we define in $\mathcal{B}(Q)$ the filter

$$\mathcal{R}_P(Q) = \{X : P \sqsubset X \ \&\ Q \subseteq X\}.$$

Note that $\{e : P_e \in \mathcal{R}_P(Q)\}$ is a Σ^0_3 set. Thus, we define the Σ^0_3 Boolean algebra

$$\mathcal{K} = \mathcal{B}(Q)/\mathcal{R}_P(Q).$$

Recall that a Σ^0_3 Boolean algebra $\mathcal{B}$ is *effectively dense* [12] if there is a function f computable in $\emptyset''$ such that for any $a \neq 0^{\mathcal{B}}$, $0 <^{\mathcal{B}} f(a) <^{\mathcal{B}} a$. The following result is obtained by dualizing Lemma 3.6 of [3]. (Note that this reverses the ordering of the Boolean algebra, but this process does not affect effective density. In other words, $\mathcal{B}$ is effectively dense just if the corresponding Boolean algebra with the reverse ordering is.)

Lemma 2.5. *$\mathcal{K}$ is effectively dense.*

2.3. **A definability lemma.** A result in Nies [12, Lemma 6.3]) will be very important for us. We translate the result into the notation of Π^0_1 classes and filters. (Actually, the result in [12] is more general, since it is proven for any effectively dense Boolean algebra $\mathcal{B}$, while we only apply the case $\mathcal{B} = \mathcal{Q}$.)

A filter F of $\mathcal{B}(Q)$ is *k-acceptable$_P$* if F has a Σ^0_k index set and $\mathcal{R}_P(Q) \subseteq F$. For example, $\mathcal{R}_P(Q)$ itself is 3-acceptable$_P$.

A class $\mathcal{C}$ of filters of $\mathcal{B}(Q)$ containing $\mathcal{R}_P(Q)$ is *uniformly definable* if, for some formula $\psi(X; P_1, \dots, P_n, P, Q)$ in the language of lattices with $0, 1$, $F \in \mathcal{C}$ if and only if there are parameters $P_1, \dots, P_n \in \mathcal{E}_\Pi$ such that

$$F = \{X : Q \sqsubset X \ \&\ \mathcal{E}_\Pi \models \psi(X; P_1, \dots, P_n, P, Q)\}.$$

Lemma 2.6 (Definability Lemma). *Let P be a nonprincipal Π^0_1 class and let Q be a locally principal extension of P such that $P \not\sqsubset Q$. Then the class of k-acceptable$_P$ filters of $\mathcal{B}(Q)$ is uniformly definable for each odd $k \geq 3$.*

The result is obtained dualizing the one in [12]. One uses induction over odd $k \geq 3$. As an illustration, consider $k = 3$. In the language of Π^0_1 classes, one proves that F is a 3-acceptable$_P$ filter in $\mathcal{B}(Q)$ if and only if there is a parameter $C \in \mathcal{E}_\Pi$, $Q \subseteq C$, such that

$$F = \{X \in \mathcal{B}(Q) : (\exists R \in \mathcal{R}_P(Q) \ \&\ R \subseteq X \cup C)\}.$$

3. Complexity of representations

In this section, we prove two results restricting the possible Turing complexity of representations of the relativized lattice $\mathcal{E}^X$, and of the lattice $\mathcal{I}(\mathcal{B})$ of ideals of an effectively dense Σ^0_k Boolean algebra.

Suppose S is a finite signature containing an equality symbol $\approx$ and constants $c_0, c_1, \dots$, and let D be the set of atomic relations and negations of atomic relations over S without free variables (typical elements of D are $fc_n = fgc_m$ and $\neg Rc_nc_m$, where $n, m \in \mathbb{N}$, f, g are unary function symbols and R is a binary relation symbol in S). A *representation* is a subset $\mathcal{R}$ of D such that exactly one of an atomic relation or its negation is in $\mathcal{R}$, and $E_\mathcal{R} = \{c_n, c_m : c_n \approx c_m \in \mathcal{R}\}$ is an equivalence relation compatible with $\mathcal{R}$. In the following we identify c_n with the number n. A pair $\langle \mathcal{R}, \alpha \rangle$ is a *representation* of an S–structure $\mathcal{A}$ if $\alpha : \mathbb{N} \mapsto \mathcal{A}$ is onto and the canonical S–structure on equivalence classes of $E_\mathcal{R}$ is isomorphic to $\mathcal{A}$ via α. For $Y \subseteq \mathbb{N}$, a subset of $\mathcal{A}$ is $\Sigma^0_k(Y)$ if its preimage under α is. If $a = \alpha(n)$, we say that n is an *index* for a.

For a countable S–structure $\mathcal{A}$ and $Y \subseteq \mathbb{N}$, we write $\mathcal{A} \leq_T Y$ if there is a presentation $\langle \mathcal{R}, \alpha \rangle$ of $\mathcal{A}$ such that $\mathcal{R} \leq_T Y$. In other words, for a relation symbol R in S, say binary, and including $\approx$, we can decide recursively in Y whether $Rnm \in \mathcal{R}$, and for a function symbol $f \in S$, say binary, given n, m, we can recursively in Y determine an index for $f^\mathcal{A}(\alpha(n), \alpha(m))$.

Fact 3.1. *Suppose $\mathcal{A} \leq_T Y$ via $\langle \mathcal{R}, \alpha \rangle$, and $\mathcal{U}$ is a substructure whose domain is $\Sigma^0_1(Y)$. Then $\mathcal{U} \leq_T Y$ via a representation $\langle \mathcal{S}, \beta \rangle$ such that, in addition, Y can decide if an atomic relation holds for $\beta(n), \alpha(m)$.*

Proof. To obtain $\langle \mathcal{S}, \beta \rangle$, choose a function $f \leq_T Y$ such that $rg(f) = \alpha^{-1}(\mathcal{U})$. Let $\beta = \alpha \circ f$. $\square$

We prove propositions saying that the natural representations of $\mathcal{E}^X$ and $\mathcal{I}(\mathcal{B})$ are not far from optimal.

Proposition 3.2. *For each $X \subseteq \mathbb{N}$, $\mathcal{E}^X \not\leq_T X'$.*

Proof. We use some concepts from Nies [10], which we review first. We need the notion of (uniform) coding of *extended standard models of arithmetic* (extended SMA). An extended SMA is a structure (M, U), where $M \cong \mathbb{N}$ and $U \subseteq M$. In general, a coding with parameters of a relational structure C of finite signature in a structure $\boldsymbol{D}$ is given by a scheme S of formulas $\varphi_D(x, \overline{p})$ (to code the domain) and $\varphi_R(x_1, \ldots, x_n; \overline{p})$ for each n–ary relation symbol R in the language of C (including equality $\approx$) such that, for an appropriate list $\overline{d}$ of parameters in $\boldsymbol{D}$, $\varphi_\approx$ defines an equivalence relation on $\{x : \boldsymbol{D} \models \varphi_S(x, \overline{d})\}$ and the structure defined on equivalence classes by the remaining formulas φ_R is isomorphic to C.

In [10] we show that (the relativizable structure) $\mathcal{E}$ as a lattice satisfies, for some k, a coding condition $Co(k)$, which states that there is a scheme of Σ_k formulas with parameters so that, for each $X \subseteq \mathbb{N}$, an extended SMA $(M, U) \cong (\mathbb{N}, \overline{X^{(k+1)}})$ (viewed as a structure with two ternary and one unary relation) can be coded in $\mathcal{E}^X$.

We use an argument as in the proof of the Separation Theorem [10, Thm 2.1] to show the claim. Suppose that $\mathcal{E}^X \leq_T Y$ so that there is a representation $\langle \mathcal{R}, \alpha \rangle$ of $\mathcal{E}^X$ with $\mathcal{R}$ recursive in Y (later, Y will be X'). Then the preimage under α of the successor relation of M is c.e. in $Y^{(k-1)}$. Hence there is a function $f \leq_T Y^{k-1}$ such that, for all n, $\alpha(f(n)) = n^M$ (i.e., $f(n)$ is an index for n in M). Then U (viewed as a subset of $\mathbb{N}$) is c.e. in $Y^{(k-1)}$ via the enumeration procedure which enumerates n into U iff the Σ_k–formula defining U (with a fixed list of parameters in $\mathcal{E}^X$) holds for $\alpha(f(n))$. Since $U = \overline{X^{(k+1)}}$, for $Y = X'$ this implies $\overline{X^{(k+1)}}$ c.e. in $X^{(k)}$, which is not the case. ☐

In the following, we use notation from Nies [11].

Proposition 3.3. *Suppose that the Σ^0_k-Boolean algebra $\mathcal{B}$ is effectively dense. Then $\mathcal{I}(\mathcal{B}) \not\leq_T \emptyset^{(k)}$.*

Proof. We prove the claim for $k = 1$, i.e., we show that $\mathcal{I}(\mathcal{B}) \not\leq_T \emptyset'$ for a c.e. effectively dense $\mathcal{B}$. For larger k, one relativizes this to $\emptyset^{(k-1)}$.

Choose a c.e. separating ideal I_0 (defined in [11, (7)]) such that $\mathcal{B}/I_0$ is infinite, and let $\boldsymbol{K}$ be the lattice of Σ^0_3–ideals of $\mathcal{B}$ which contain I_0. We show that

$$\mathcal{I}(\mathcal{B}) \leq_T \emptyset' \ \Rightarrow\ \boldsymbol{K} \leq_T \emptyset^{(3)}. \tag{2}$$

This suffices since, by the proof of [11, Lemma 2.4], there is an interval $[C, D]_{\boldsymbol{K}}$ isomorphic to $\mathcal{E}^3$, the lattice of Σ^0_3–sets. By Fact 3.1, if $\boldsymbol{K} \leq_T \emptyset^{(3)}$, then also $[C, D]_{\boldsymbol{K}} \leq_T \emptyset^{(3)}$, which contradicts Proposition 3.2 for $X = \emptyset^{(2)}$. Thus $\mathcal{I}(\mathcal{B}) \not\leq_T \emptyset'$.

To prove (2), assume that there is a representation $\langle \mathcal{R}, \alpha \rangle$ of the lattice $\mathcal{I}(\mathcal{B})$, with $\mathcal{R}$ Turing below $\emptyset'$. Note that $\mathcal{B}$ is canonically isomorphic to the complemented elements in $\mathcal{I}(\mathcal{B})$, a Σ_1–definable subset of $\mathcal{I}(\mathcal{B})$. Hence from $\mathcal{R}$, using Fact 3.1, we can derive a presentation $\langle \mathcal{S}, \beta \rangle$, for $\mathcal{B}$ such that $\mathcal{S} \leq_T \emptyset'$, which we will use in the following. Let x, r, s range over $\mathcal{B}$.

Given a c.e. ideal $L \subseteq I_0$, let

$$J(L) = \{x \in \mathcal{B} : \exists r \in I_0 \forall s \in I_0\ [s \wedge r \equiv 0 \Rightarrow x \wedge s \in L]\}.$$

In Nies [11, Lemma 2.3] it is shown that each $J \in \boldsymbol{K}$ is of the form $J(L)$ for some c.e. L. Thus, to obtain the desired representation of $\boldsymbol{K}$, we represent $J \in \mathcal{L}$ by an index for an L such that $J = J(L)$. Note that

$$J(L) = \{x \in \mathcal{B} : \exists r \in I_0 \; I_0 \cap [0, \overline{r} \wedge x] \subseteq L\}.$$

Thus, by Fact 3.1, "$\{x : x \in J(L)\}$" is Σ^0_2 with respect to $\langle \mathcal{S}, \beta \rangle$, and a Σ^0_2–index can be obtained uniformly in an $\mathcal{R}$–index for L. Then "$J(L_0) \subseteq J(L_1)$" is Π^0_3. For the lattice operations, given L_0, L_1 note that $J = J(L_0) \vee J(L_1) \in \boldsymbol{K}$, so there is L such that $J = J(L)$. Since we can determine a Σ^0_2 index with respect to $\mathcal{S}$ for J, and equality of Σ^0_2–ideals (under the representation $\mathcal{S}$) of $\mathcal{B}$ is Π^0_3, we can find an index for L using $\emptyset^{(3)}$ as an oracle. For $J(L_0) \cap J(L_1)$ one argues similarly. □

4. Nonisomorphic end intervals of $\mathcal{E}_\Pi$

In this section, we apply the results from Sections 2 and 3 to the lattice of Π^0_1 classes to show that there are exactly two distinct types of nontrivial end intervals $[P, 1]$ of $\mathcal{E}_\Pi$. It is an easy observation that there are at most two, those where P is principal and where P is nonprincipal [6].

Theorem 4.1. *Let P be nonprincipal. Then $[P, 1]_{\mathcal{E}_\Pi}$ is not isomorphic to $\mathcal{E}_\Pi$.*

Proof. Suppose for a contradiction that $[P, 1]_{\mathcal{E}_\Pi} \cong \mathcal{E}_\Pi$ via Φ, but P is nonprincipal. If the structure $\boldsymbol{X}$ is coded in $[P, 1]_{\mathcal{E}_\Pi}$ with first–order formulas and parameters $P_1, \dots, P_m$, we will denote by $\Phi(\boldsymbol{X})$ the structure coded in $\mathcal{E}_\Pi$ with the same formulas and the parameter list $\Phi(P_1), \dots, \Phi(P_m)$. (Thus, $\boldsymbol{X}$ behaves the same way in $[P, 1]_{\mathcal{E}_\Pi}$ as $\Phi(\boldsymbol{X})$ in $\mathcal{E}_\Pi$.)

By Lemma 2.4, choose a locally principal extension Q of P such that $P \not\subseteq Q$. Then, by Lemma 2.5, the Σ^0_3–Boolean algebra $\mathcal{B} = \mathcal{B}(Q)/\mathcal{R}_P(Q))$ is effectively dense. Hence, by Proposition 3.3 for $k = 3$, $\mathcal{I}(\mathcal{B}) \not\leq_T \emptyset^{(3)}$. Taking complements in $\mathcal{B}$, $\mathcal{I}(\mathcal{B})$ is isomorphic to the lattice $\boldsymbol{H}$ of Σ^0_3 filters of $\mathcal{B}(Q)$ containing $\mathcal{R}_P(Q)$, so $\boldsymbol{H} \not\leq_T \emptyset^{(3)}$. Note that $\Phi(P) = \emptyset$. Let $\widetilde{Q} = \Phi(Q)$, a nonprincipal Π^0_1 class. Note that $\Phi(\mathcal{R}_P(Q)) = \mathcal{R}_\emptyset(\widetilde{Q})$, so $\widetilde{\mathcal{B}} = \mathcal{B}(\widetilde{Q})/_{\mathcal{R}_\emptyset(\widetilde{Q})}$ is the isomorphic image of $\mathcal{B}$ under Φ. For the "$\mathcal{B}$–side", we have by the Definability Lemma 2.6 and the remark after that, for each $F \in \boldsymbol{H}$, there is a $C \supseteq Q$ such that

$$F = \{X \in \mathcal{B}(Q) : (\exists R \in \mathcal{R}_P(Q))(R \subseteq X \cup C)\}. \tag{3}$$

So this situation is copied to the $\widetilde{\mathcal{B}}$–side by Φ. There, $\mathcal{R}_\emptyset(\widetilde{Q})$ is the family of clopen sets containing $\widetilde{Q}$. For $C \in \mathcal{E}_\Pi$, $G(C)$ is a filter, where

$$G(C) = \{X \in \mathcal{B}(\widetilde{Q}) : (\exists V)(\widetilde{Q} \subseteq V \subseteq X \cup C)\},$$

and V ranges over the clopen sets. If

$$\boldsymbol{G} = \{G(C) : C \in \mathcal{E}_\Pi\},$$

then $\boldsymbol{G}$ is the isomorphic image under Φ of $\boldsymbol{H}$, and $\boldsymbol{G}$ is a lattice with the standard operations $\vee, \wedge$ on the filters of $\mathcal{B}(Q)$. To conclude the proof we show $\boldsymbol{G} \leq_T \emptyset^{(3)}$. The relation "$P_e \in G(P_c)$" is Σ^0_2 uniformly in C, since "$Q \subseteq V$" is Σ^0_1, and "$V \subseteq X \cup C$" is Π^0_1, being equivalent to "$\overline{V} \cup X \cup C = 2^\omega$". (It is here where the difference between the principal and nonprincipal end intervals becomes apparent, since the set in (3) corresponding to $G(C)$ is merely Σ_3^0.) It follows that the relation "$G(P_c) = G(P_d)$" is Π_3^0. Since Φ is an isomorphism, $\boldsymbol{G}$ is a lattice with the usual operations on filters. To show that these operations are recursive in $\emptyset^{(3)}$, first note that $G(P_c) \cap G(P_d) = G(P_c \cap P_d)$. For the supremum, we have

$$G(P_c) \vee G(P_d) = \{X \cap Y : X \in G(P_c) \ \& \ Y \in G(P_d)\},$$

and this equals $G(P_e)$ for some e. In fact, such an e can be obtained with oracle $\emptyset^{(3)}$, because e satisfies

$$(\forall i)[P_i \in G(P_e) \iff (\exists a, b)(P_a \in G(P_c) \ \& \ P_b \in G(P_d) \ \& \ P_i = P_a \cap P_b)].$$

□

5. Some definable subsets of $\mathcal{E}_\Pi$

In this section, we will demonstrate the definability in $\mathcal{E}_\Pi$ of various sets of Π_1^0 classes, including the finite classes and the minimal classes. Recall that the Cantor-Bendixson derivative $D(P)$ of a closed set P contains exactly the limit points of P. Then $\{P : card(D^n(P)) \geq k\}$ is a Σ_{2n+3}^0 filter for each finite n and k by Theorem 45 of [4]. We will show that this family is in fact definable in $\mathcal{E}_\Pi$.

For a Π_1^0 class P, let $\mathcal{L}(P)$ be the initial segment $[0, P]$ in $\mathcal{E}_\Pi$. In general, $\mathcal{L}(P)$ may not be a Boolean algebra. Hence, we also consider the subfamily $\mathcal{CL}(P)$ of relative clopen subclasses. That is, $\mathcal{CL}(P) = \{P \cap V : V \in \mathcal{Q}\}$. Then $\mathcal{CL}(P)$ is always a Boolean algebra and has a Δ_2^0 representation using as indices Gödel numbers for clopen sets. Recall that P is *thin* if $\mathcal{L}(P)$ is a Boolean algebra; the corresponding ideal I in $\mathcal{Q}$ is said to be *hh-simple* in analogy to $\mathcal{E}$. Then P is thin if and only if $\mathcal{L}(P) = \mathcal{CL}(P)$. It is shown in [4] that the set of indices for thin classes is a Π_4^0 set.

Recall that the *derivative* $\boldsymbol{B}^*$ of a boolean algebra $\boldsymbol{B}$ is $\boldsymbol{B}/_U$, where U is the ideal generated by the atoms of $\boldsymbol{B}$; equivalently, the derivative is the quotient of $\boldsymbol{B}$ modulo the filter generated by the co-atoms. Note that $\boldsymbol{B}^* = \{0\}$ iff $\boldsymbol{B}$ is finite. We say P is *minimal* if $\mathcal{L}(P)^*$ is the trivial Boolean algebra $\{0, 1\}$ and P is *quasi-minimal* if $\mathcal{L}^*(P)$ is finite; the corresponding ideal I in $\mathcal{Q}$ is *maximal* (*quasi-maximal*).

Certainly, any family definable in $\mathcal{L}(P)$ will have an arithmetical index set. As was done in [9] for $\mathcal{E}$, we will obtain a partial converse here.

A closed set is nowhere dense if it does not include any nontrivial clopen set. Note that thin classes and countable classes are nowhere dense.

Theorem 5.1. *Suppose the Π_1^0 class P is nowhere dense. Then for each odd $k \geq 3$, the class of Σ_k^0 filters of $\mathcal{CL}(P)$ is uniformly definable in $\mathcal{E}_\Pi$, via a formula $\varphi(X; P_1, \ldots, P_n, P)$ which does not depend on P.*

The proof of Theorem 5.1 is given below. We first show how to derive the definability of subsets of $\mathcal{E}_\Pi$ from this. If F is a filter of $\mathcal{CL}(P)$, let $\mathcal{A}(F)$ be the filter of $\mathcal{CL}(P)$ generated by the co-atoms of $\mathcal{CL}(P)/F$, so that $\mathcal{CL}(P)/\mathcal{A}(F)$ is the derivative of $\mathcal{CL}(P)/F$. Let $\boldsymbol{B}^{(k)}$ be the kth derivative of $\boldsymbol{B}$. It follows from Theorem 4.7 of [1] that for any Π_1^0 class P, $\mathcal{CL}(P)^{(k)}$ is effectively isomorphic to $\mathcal{CL}(D^k(P))$.

Proposition 5.2. *If F is a filter of $\mathcal{CL}(P)$ which is definable in $(\mathcal{E}_\Pi, P)$, then so is $\mathcal{A}(F)$. The formula defining $\mathcal{A}(F)$ only depends on the one defining F, not on the particular choice of P.*

Proof. Suppose that F is a Σ^0_k filter, $k \geq 3$. Then $\mathcal{A}(F)$ is Σ^0_{k+2}. Using Theorem 5.1, we may define $\mathcal{A}(F)$ as the least Σ^0_{k+2} filter of $\mathcal{CL}(P)$ which contains all the elements of F and all $B \subset P$ such that $B/_F$ is a co-atom in $\mathcal{CL}(P)/_F$. □

In the following theorem, the case $n = 1$ and $\boldsymbol{B} = \{0\}$ gives a first–order definition of finiteness.

Theorem 5.3. *Let $n > 0$ and let $\boldsymbol{B}$ be a finite Boolean algebra or $\boldsymbol{B} = \{0\}$. Then*

$$\{P : \mathcal{CL}(P)^{(n)} \ \cong \boldsymbol{B}\} \text{ is definable in } \mathcal{E}_\Pi$$

without parameters.

Proof. Let $F^P_0 = \{1\}$ and for each n, let $F^P_{n+1} = \mathcal{A}(F_n)$. It follows from Proposition 5.2 that there is a formula φ_n (independent of P) which defines F^P_n in $\mathcal{CL}(P)$. Hence, we can express that the quotient algebra of $\mathcal{CL}(P)$ modulo F^P_n is isomorphic to $\boldsymbol{B}$. □

Corollary 5.4. *The following families of Π^0_1 classes are definable in $\mathcal{E}_\Pi$ without parameters:*

(a) *For any fixed n and k, $\{P : card(D^n(P)) \leq k\}$.*
(b) *The minimal classes.*

Proof. (a) This follows from Theorem 5.3 and the fact that $\mathcal{CL}(D^n(P))$ is isomorphic to $\mathcal{CL}(P)^{(n)}$.

(b) P is minimal if and only if, for all $Q \in \mathcal{L}(P)$, either Q is finite or $P - Q$ is finite. □

This corollary takes care of countable classes of finite rank, but we can also consider uncountable classes of finite rank. (Recall that the *rank* of a class P is the least α such that $D^{\alpha+1}(P) = D^\alpha(P)$.)

Proposition 5.5. *For each ordinal α, the family of Π^0_1 classes of rank α is definable in $\mathcal{E}_\Pi$ if and only if α is finite.*

Proof. For infinite α, the family cannot be definable since $\{e : card(D^\alpha(P_e)) = \emptyset\}$ is $\Sigma^0_{2\alpha+1}$ complete and thus not arithmetical, by Theorem 45 of [4]. For finite α, P has rank α if and only if $F^P_{\alpha+1} = F^P_\alpha$, where F^P_α is defined as in the proof of Theorem 5.3. □

In the following we refer to Tarski's classification of the completions T of the theory of Boolean algebras, in the form presented in Chang and Keisler [7, Section 5.5.]. They assign invariants $m(\boldsymbol{B}), n(\boldsymbol{B}) \in \omega+1$ to Boolean algebras and prove that two Boolean algebras are elementarily equivalent iff they have the same invariants. Thus, if T is a completion of the theory of Boolean algebras, we can also write $m(T), n(T)$ for $m(\boldsymbol{B}), n(\boldsymbol{B})$, where $\boldsymbol{B}$ is some model of T.

Theorem 5.6. *For any completion T of the theory of Boolean algebras, except possibly the one with invariants $m(T) = \infty$ and $n(T) = 0$, the family of Π^0_1 classes such that $\mathcal{CL}(P) \models T$ is definable in $\mathcal{E}_\Pi$ without parameters.*

Note that the theorem is nontrivial since some completions are not finitely axiomatizable.

Proof. To define the invariants for a Boolean algebra $\boldsymbol{B}$, one introduces definable ideals $\boldsymbol{I}_k$: let $\boldsymbol{I}_0 = \{0\}$, and let $\boldsymbol{I}_{k+1}$ be the preimage in $\boldsymbol{B}$ of the ideal of $\boldsymbol{B}/I_k$ generated by the atomic and the atomless elements. Consider $\boldsymbol{B} = \mathcal{CL}(P)$. There are formulas $\psi(P)$ in the language of $\mathcal{E}_\Pi$ expressing that $\boldsymbol{B}/I_k$ has n atoms, or, using Theorem 5.1 for ideals, there are infinitely many atoms (that is, the ideal generated by the atoms is nonprincipal). Thus, we can express that $\mathcal{CL}(P)$ satisfies the required invariants. □

Proof of Theorem 5.1. We use the Definability Lemma 2.6. Given a locally principal extension Q of a thin Π^0_1 class P and the effectively dense Σ^0_3 Boolean algebra $\mathcal{K} = \mathcal{B}(Q)/\mathcal{R}_P(Q)$ as above, consider the Boolean homomorphism $\Phi : \mathcal{CL}(P) \to \mathcal{K}$ defined by

$$\Phi(P \cap V) = (Q \cup V)/\mathcal{R}_P(Q).$$

We first show that this map is well defined. Suppose that $P \cap V = P$, then $P \cap V^c = \emptyset$, so that $Q \cap V^c$ is clopen by Lemma 2.3, and thus $Q \cup V = (Q \cap V^c) \cup V$ is also clopen; hence, $P \sqsubset Q \cup V$.

Furthermore, note that with the canonical representations, Φ is Δ^0_3.

Claim 5.7. *Suppose that for all clopen V, $P \sqsubset Q \cup V$ implies that $P \subset V$. Then Φ is an embedding.*

Proof. Suppose that $P \cap V$ is in the kernel of Φ. Then $P \sqsubset Q \cup V^c$, so that $P \subset V^c$, and hence $P \cap V = 0$. □

Next we show such a Q exists.

Claim 5.8. *Suppose that P is nowhere dense. Then there exists a locally principal extension Q of P such that, for all clopen V, $P \sqsubset Q \cup V$ implies that $P \subset V$.*

Proof. First note that, if $P \nsubseteq V$, then $P \cup V$ is nonprincipal. To see this, suppose that $P \cup V = U$ is clopen, so that $P - V = U - V$ is a nonempty clopen subset of P. Since P is nowhere dense, $U - V = \emptyset$, so that $P \subseteq V$, contradiction.

Recall that $2^\omega - P = \bigcup_n U_n$, where the clopen sets U_n are disjoint. We will define Q to be $2^\omega - \bigcup_{n \in A} U_n$ for a certain c.e. set A. We build A by extending the construction in the proof of Lemma 2.4 above. Let P_e be the eth Π^0_1 class and express P_e as the intersection of a uniformly computable, decreasing sequence of clopen sets $P_{e,s}$. It suffices to meet the following requirements for each clopen V and each e:

$$R_{V,e} : (Q \cup V) \sqcap P_e = P \;\Rightarrow\; P \subset V.$$

Construction of A: In the beginning all requirements are declared unsatisfied. At stage s of the construction, we may select a candidate U_n for some requirement, and we may take action on some requirements, as follows.

1. If $R_{V,e}$ is the highest priority requirement without a candidate and $V^c \cap U_s \neq 0$, then declare U_s to be its candidate.

2. For each unsatisfied requirement $R_{V,e}$ which has a candidate U_n, if now $U \cap P_{e,s} = \emptyset$, then put n into A and declare the requirement satisfied.

We observe that no U_n can be a candidate for more than one requirement.

Verification: Clearly, Q is a locally principal extension of P. Suppose by way of contradiction that the requirement $R = R_{V,e}$ is not met. Then $(Q \cup V) \sqcap P_e = P$ and $P \cup V$ is nonprincipal. By the first assumption, $(Q \cup V) \cup P_e = 2^\omega$ and $(Q \cup V) \cap P_e = P$. Then for each n, we have $U_n \subset Q \cup V \cup P_e$ and we have $U_n \cap (Q \cup V) \cap P_e = \emptyset$. By the second assumption, there are infinitely many n such that $V^c \cap U_n \neq \emptyset$. That is, suppose by way of contradiction that only finitely many of the U_n meet V^c, say $\{U_i : i \in F\}$ for some finite set F. Then it is easily seen that $P \cup V = V \cup (2^\omega - \bigcup_{i \in F} U_i)$. Thus, R eventually receives a candidate U_n. Now there are two cases.

Case I: First, suppose that $U_n \cap P_e = \emptyset$. Then for some s, $U_n \cap P_{e,s} = \emptyset$, so that $n \in A$ and $U_n \cap Q = \emptyset$ by the construction. But we chose n such that $U_n \cap V^c \neq \emptyset$, that is, $U_n \nsubseteq V$, which contradicts $U_n \subset Q \cup V \cup P_e$.

Case II: Suppose that $U_n \cap P_e \neq \emptyset$. Then by the construction $n \notin A$. Thus, $U_n \subset Q$, which contradicts $U_n \cap (Q \cup V) \cap P_e = \emptyset$. □

Now, given a nowhere dense Π^0_1 class P and odd $k \geq 3$, then $\varphi(X; P_1, \ldots, P_n, P)$ for Theorem 5.1 is obtained as follows. Suppose $P \subseteq Q$ is as in Claim 5.8. Since Φ is 1-1 and is Δ^0_3, if $\mathcal{H}$ ranges through the k-acceptable$_P$ filters of $\mathcal{B}(Q)$, then $\Phi^{-1}(\mathcal{H}/_{\mathcal{R}_P(Q)})$ ranges through the Σ^0_k filters of $\mathcal{CL}(P)$. Let $\varphi(X; P_1, \ldots, P_n, Q)$ be

$$\exists Q[P, Q \text{ as in Claim 5.8 } \& \ \exists V[X = P \cap V \ \& \ \psi(Q \cup V; P_1, \ldots, P_n, P, Q)]]$$

where $\psi(X; P_1, \ldots, P_n, P, Q)$ is the formula from the Definability Lemma 2.6. □

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