GLOBAL PROPERTIES OF THE LATTICE OF $\Pi^0_1$ CLASSES

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Abstract. Let $E_\Pi$ be the lattice of $\Pi^0_1$ classes of reals. We show there are exactly two possible isomorphism types of end intervals, $[P, 2^\omega]$. Moreover, finiteness is first order definable in $E_\Pi$.

1. Introduction

The structure of the lattice $E_\Pi$ of $\Pi^0_1$ classes has been investigated in several recent papers, for instance, [3]. A central theme is to compare and contrast the structure with the lattice $E$ of computable enumerable sets.

In this paper, we solve a number of open problems from the 1999 AMS Summer Conference on Computability Theory. One general problem is to determine which subsets invariant under automorphisms are definable in a given structure. In particular, this is relevant for subsets which are natural in some sense. For $E_\Pi$, an example is the set of finite classes. We show that this set is definable in $E_\Pi$, which solves the first part of Problem 6.1 in [2]. The Cantor-Bendixson rank is an important way of classifying $\Pi^0_1$ classes. We solve Problem 6.2 of [2] by showing that the family of countable $\Pi^0_1$ classes of rank $\alpha$ is definable if and only if $\alpha < \omega$.

Intervals of the lattice $E_\Pi$ were first studied in [3], where it was shown that, in contrast to the lattice $E$, there are finite initial intervals in the quotient lattice $E_\Pi/ =^*$ which are not Boolean algebras. An important problem here is to characterize all the possible intervals. We show here that there are exactly two possible isomorphism types of end intervals, $[P, 2^\omega]$, which answers a question of Herrmann (Problem 6.6 of [2]) and also Problem 9.7 of [8]. As a tool, we prove results on the complexity of possible representations of $E$ and other structures, which are of interest by themselves. In recent work, Nies has found a $\Sigma_3$ sentence separating the two lattices.

2. Preliminaries

2.1. Some notation. As in [3] we will be applying some results on effective Boolean algebras and coding due to Nies [10, 11, 12] and also Harrington and Nies [9]. In the first paper, we used the language of c.e. ideals of the computable dense Boolean algebra rather than the language of $\Pi^0_1$ classes, to conform to the presentation of...
Here we will translate some of this background material into the language of $\Pi_1^0$ classes, as in [12][13][14].

The underlying computable dense Boolean algebra $Q$ may be thought of as the family of clopen subsets of $\{0,1\}^\omega$. For any finite sequence $\sigma$, let $I(\sigma) = \{x : \sigma \preceq x\}$. Each clopen set has a unique representation as a finite union of disjoint intervals $I(\sigma_1) \cup \cdots \cup I(\sigma_k)$, where each $\sigma_i$ has the same length and $k$ is taken to be as small as possible. Then the join $(\lor)$ and meet $(\land)$ operations are clearly computable, as well as the complement operation and the partial ordering relation on $Q$.

A c.e. Boolean algebra is given by a model $(\mathbb{N}, \leq, \lor, \land)$ such that $\leq$ is a c.e. relation which is a pre-ordering, $\lor, \land$ are total computable binary functions, and the quotient structure $B = (\mathbb{N}, \leq, \lor, \land)/ \approx$ is a Boolean algebra (where $n \approx m \iff n \leq m \land m \leq n$). We can suppose that $0 \in \mathbb{N}$ names the least and $1 \in \mathbb{N}$ the greatest element of $B$. For $\Sigma^0_k$-Boolean algebras, one requires that $\leq$ be $\Sigma^0_2$ and that $\lor, \land$ be computable in $\emptyset^{(k-1)}$. For a $\Sigma^0_k$ Boolean algebra $B$, let

$$I(B) := \text{ the lattice of } \Sigma^0_k \text{-ideals of } B.$$  

Clearly, c.e. Boolean algebras correspond to c.e. ideals of $Q$ and similarly for computable algebras and ideals. At the same time an ideal $I$ of $Q$ corresponds to a $\Pi_1^0$ class $P$ in that $I = \{U \in Q : P \cap U = \emptyset\}$ and $P = \{0,1\}^\omega - \bigcup I$. We can use this last equation to assign index sets for $\Pi_1^0$ classes (as in [14]). Let $\sigma_0, \sigma_1, \ldots$ enumerate $\{0,1\}^{<\omega}$ and let $W_e$ be the $e$th c.e. subset of $\omega$, as usual. Then the $e$th $\Pi_1^0$ class is given by

$$P_e = \{0,1\}^\omega - \bigcup_{n \in W_e} I(\sigma_n).$$

An ideal $I$ in a Boolean algebra $B$ is said to be principal if there is some $b$ such that $I = \{a : a \leq b\}$. The ideal $I$ corresponding to a $\Pi_1^0$ class $P$ as above is principal if and only if $P$ is clopen. Thus, we will refer to a non-clopen $\Pi_1^0$ class $P$ as nonprincipal. For any $\Pi_1^0$ class $P$, let $S(P)$ be the lattice of $\Pi_1^0$ classes $Q$ such that $P \subseteq Q$.

2.2. An effectively dense $\Sigma^0_3$ Boolean algebra. A c.e. Boolean algebra $B$ is called effectively dense [9] if there is a computable $F$ such that $\forall x \ [F(x) \leq x]$ and $\forall x \neq 0 \ [0 < F(x) < x]$. More generally, a $\Sigma^0_k$ Boolean algebra $B$ is effectively dense if the above holds with some $F \leq_T \emptyset^{(k-1)}$. We first summarize the construction from [9] of an effectively dense $\Sigma^0_3$ Boolean algebra from an arbitrary nonprincipal $\Pi_1^0$ class $P$. We will present these results from the point of view of $\Pi_1^0$ classes rather than c.e. ideals.

The following technical lemma shows that we can make the intervals $I(\sigma_n)$ in disjoint.

**Lemma 2.1.** For any $\Pi_1^0$ class $P$, there is a c.e. set $A$ such that $P = 2^{\omega} - \bigcup_{n \in A} I(\sigma_n)$, where for $m \neq n$, $I(\sigma_m) \cap I(\sigma_n) = \emptyset$.

**Proof.** Let $2^{\omega} - P = \bigcup_n I(\tau_n)$, for some computable sequence $\{\tau_n\}$. For each $n$, express the clopen set $I(\tau_n) = \bigcup_{m < n} I(\tau_m)$ as a finite union $\bigcup_{k \in C_n} I(\sigma_k)$ and let $A = \bigcup_{n < \omega} C_n$. \hfill $\square$

The underlying lattice $E_{\Pi}$ of $\Pi_1^0$ classes may be viewed as a $\Pi_1^0$ structure using the representation given by [9]. That is, there are recursive functions $m$ and $j$ such that $P_a \cup P_b = P_{j(a,b)}$ and $P_a \cap P_b = P_{m(a,b)}$, and the relation “$P_a = P_b$” is $\Pi_2^0$. 


Let $P$ be a nonprincipal $\Pi^0_3$ class. We write $Q_1 \cap Q_2 = P$ if $Q_1 \cap Q_2 = P$ and $Q_1 \cup Q_2 = 2^\omega$, and $P \subseteq Q$ if there exists $Q_2$ such that $Q \cap Q_2 = P$. We observe that for any clopen set $V$, $(P \cup V) \cap (P \cup V^c) = P$, so that $P \subseteq P \cup V$. In $[P, 2^\omega]$, we can define the $\Sigma^0_3$ Boolean algebra of complemented elements as
\[ B(P) = \{ X : P \subseteq X \}. \]

This is indeed a $\Sigma^0_3$ Boolean algebra, since $P_a \in B(P) \iff (3b) [P_a \cap P_b = P \& P_a \cup P_b = 2^\omega]$.

In the following, we recast Definition 4.5 from [12] in the language of $\Pi^0_3$ classes.

**Definition 2.2.** $Q$ is a locally principal extension of $P$ if $P \subseteq Q$ and $Q \setminus P$ is open.

This has a first order definition in the lattice $\mathcal{E}_P$, by the following.

**Lemma 2.3.** $Q$ is a locally principal extension of $P$ if and only if $P \subseteq Q$ and, for all clopen sets $V$, if $P \cap V = \emptyset$, then $Q \cap V$ is clopen.

**Proof.** Let $Q$ be a $\Pi^0_3$ class with $P \subseteq Q$. Suppose first that $Q \setminus P$ is open and let $V$ be a clopen set disjoint from $P$. Then $Q \cap V$ is closed, since both $Q$ and $V$ are closed. $Q \cap V$ is also open, since $Q \cap V = (Q \setminus P) \cap V$. On the other hand, suppose that $Q$ satisfies the condition of the lemma. Then for any $x \in Q \setminus P$, choose a clopen set $V$ such that $x \notin V$ and $P \cap V = \emptyset$. It follows that $Q \cap V$ is a clopen subset of $Q \setminus P$ containing $x$. Thus $Q \setminus P$ is an open set. \hfill \Box

We show that our definition is, in fact, the dual of the definition for c.e. ideals of $Q$ from [12]. An ideal $B$ is a locally principal subideal of $A$ if $B \subseteq A$ and $\forall e \in A [0, e] \cap B$ is principal. The immediate dual (with $Q$, $P$ corresponding to $B$, $A$) is $Q \supseteq P$, and for all clopen $W \supseteq P$, $W \cup Q$ is clopen. Now let $V = W^c$, and note that $V^c \cup Q$ clopen iff $V \cap Q$ clopen.

**Lemma 2.4.** For any nonclopen $\Pi^0_3$ class $P$, there exists a locally principal extension $Q$ of $P$ such that $P \not\subseteq Q$.

**Proof.** By Lemma 2.3 let $2^\omega \setminus P = \bigcup_n U_n$, where $\{ U_n \}_{n < \omega}$ is a computable sequence of disjoint intervals. Now choose a noncomputable c.e. set $B$ and let $Q = 2^\omega \setminus \bigcup_{n \notin B} U_n$. Clearly, $P \subset Q$ and furthermore, $Q \setminus P = \bigcup_{n \notin B} U_n$ is an open set. Suppose by way of contradiction that $R$ is a $\Pi^0_3$ class such that $Q \cup R = 2^\omega$ and $Q \cap R = P$. But this means that $2^\omega \setminus R = Q \setminus P$. Then $e \in B \iff R \cap U_n \neq \emptyset$, which is a co-c.e. condition, contradicting the assumption that $B$ is not computable. \hfill \Box

If $P \subseteq Q$, we define in $B(Q)$ the filter
\[ \mathcal{R}_P(Q) = \{ X : P \subseteq X \text{ & } Q \subseteq X \}. \]

Note that $\{ e : P_e \in \mathcal{R}_P(Q) \}$ is a $\Sigma^0_3$ set. Thus, we define the $\Sigma^0_3$ Boolean algebra
\[ \mathcal{K} = B(Q)/\mathcal{R}_P(Q). \]

Recall that a $\Sigma^0_3$ Boolean algebra $B$ is effectively dense [12] if there is a function $f$ computable in $0'$ such that for any $a \neq 0^B$, $0 <^B f(a) <^B a$. The following result is obtained by dualizing Lemma 3.6 of [3]. (Note that this reverses the ordering of the Boolean algebra, but this process does not affect effective density. In other words, $B$ is effectively dense just if the corresponding Boolean algebra with the reverse ordering is.)
Lemma 2.5. \( K \) is effectively dense.

2.3. A definability lemma. A result in Nies [12 Lemma 6.3]) will be very important for us. We translate the result into the notation of \( \Pi_1^0 \) classes and filters. (Actually, the result in [12] is more general, since it is proven for any effectively dense Boolean algebra \( B \), while we only apply the case \( B = Q \).

A filter \( F \) of \( B(Q) \) is \( k \)-acceptable if \( F \) has a \( \Sigma_k^0 \) index set and \( \mathcal{R}_P(Q) \subseteq F \). For example, \( \mathcal{R}_P(Q) \) itself is \( 3 \)-acceptable.

A class \( C \) of filters of \( B(Q) \) containing \( \mathcal{R}_P(Q) \) is uniformly definable if, for some formula \( \psi(X; P_1, \ldots, P_n, P, Q) \) in the language of lattices with \( 0, 1 \), \( F \in C \) if and only if there are parameters \( P_1, \ldots, P_n \in E \) such that

\[
F = \{ X : Q \sqsubseteq X \land E \models \psi(X; P_1, \ldots, P_n, P, Q) \}.
\]

Lemma 2.6 (Definability Lemma). Let \( P \) be a nonprincipal \( \Pi_1^0 \) class and let \( Q \) be a locally principal extension of \( P \) such that \( P \not\in Q \). Then the class of \( k \)-acceptable filters of \( B(Q) \) is uniformly definable for each odd \( k \geq 3 \).

The result is obtained dualizing the one in [12]. One uses induction over odd \( k \geq 3 \). As an illustration, consider \( k = 3 \). In the language of \( \Pi_1^0 \) classes, one proves that \( F \) is a \( 3 \)-acceptable filter in \( B(Q) \) if and only if there is a parameter \( C \in E \), \( Q \subseteq C \), such that

\[
F = \{ X \in B(Q) : (\exists R \in \mathcal{R}_P(Q) \land R \sqsubseteq X \cup C) \}.
\]

3. Complexity of representations

In this section, we prove two results restricting the possible Turing complexity of representations of the relativized lattice \( E^N \), and of the lattice \( I(B) \) of ideals of an effectively dense \( \Sigma_k^0 \) Boolean algebra.

Suppose \( S \) is a finite signature containing an equality symbol \( \approx \) and constants \( c_0, c_1, \ldots \), and let \( D \) be the set of atomic relations and negations of atomic relations over \( S \) without free variables (typical examples of \( D \) are \( f c_n = f g c_m \) and \( -R c_n c_m \), where \( n, m \in \mathbb{N} \), \( f, g \) are unary function symbols and \( R \) is a binary relation symbol in \( S \)). A representation is a subset \( \mathcal{R} \) of \( D \) such that exactly one of an atomic relation or its negation is in \( \mathcal{R} \), and \( E_{\mathcal{R}} = \{ c_n, c_m : c_n \approx c_m \in \mathcal{R} \} \) is an equivalence relation compatible with \( \mathcal{R} \). In the following we identify \( c_n \) with the number \( n \). A pair \( \langle \mathcal{R}, \alpha \rangle \) is a representation of an \( S \)-structure \( A \) if \( \alpha : \mathbb{N} \to A \) is onto and the canonical \( S \)-structure on equivalence classes of \( E_{\mathcal{R}} \) is isomorphic to \( A \) via \( \alpha \). For \( Y \subseteq \mathbb{N} \), a subset of \( A \) is \( \Sigma_1^1(Y) \) if its preimage under \( \alpha \) is. If \( a = \alpha(n) \), we say that \( n \) is an index for \( a \).

For a countable \( S \)-structure \( A \) and \( Y \subseteq \mathbb{N} \), we write \( A \leq_T Y \) if there is a representation \( \langle \mathcal{R}, \alpha \rangle \) of \( A \) such that \( \mathcal{R} \leq_T Y \). In other words, for a relation symbol \( R \) in \( S \), say binary, and including \( \approx \), we can decide recursively in \( Y \) whether \( R n m \in \mathcal{R} \), and for a function symbol \( f \in S \), say binary, given \( n, m \), we can recursively in \( Y \) determine an index for \( f^A(\alpha(n), \alpha(m)) \).

Fact 3.1. Suppose \( A \leq_T Y \) via \( \langle \mathcal{R}, \alpha \rangle \), and \( U \) is a substructure whose domain is \( \Sigma_1^0(Y) \). Then \( U \leq_T Y \) via a representation \( \langle S, \beta \rangle \) such that, in addition, \( Y \) can decide if an atomic relation holds for \( \beta(n), \alpha(m) \).

Proof. To obtain \( \langle S, \beta \rangle \), choose a function \( f \leq_T Y \) such that \( rg(f) = \alpha^{-1}(U) \). Let \( \beta = \alpha \circ f \). \( \square \)
We prove propositions saying that the natural representations of $\mathcal{E}^X$ and $\mathcal{I}(\mathcal{B})$ are not far from optimal.

**Proposition 3.2.** For each $X \subseteq \mathbb{N}$, $\mathcal{E}^X \not\leq_T X'$.

*Proof.* We use some concepts from Nies [10], which we review first. We need the notion of (uniform) coding of extended standard models of arithmetic (extended SMA). An extended SMA is a structure $(M, U)$, where $M \cong \mathbb{N}$ and $U \subseteq M$. In general, a coding with parameters of a relational structure $C$ of finite signature in a structure $D$ is given by a scheme $S$ of formulas $\varphi_D(x, \bar{p})$ (to code the domain) and $\varphi_R(x_1, \ldots, x_n; \bar{p})$ for each $n$-ary relation symbol $R$ in the language of $C$ (including equality $\approx$) such that, for an appropriate list $\bar{t}$ of parameters in $D$, $\varphi_\approx$ defines an equivalence relation on $\{x : D \models \varphi_S(x, \bar{t})\}$ and the structure defined on equivalence classes by the remaining formulas $\varphi_R$ is isomorphic to $C$.

In [10] we show that (the relativizable structure) $\mathcal{E}$ as a lattice satisfies, for some $k$, a coding condition $Co(k)$, which states that there is a scheme of $\Sigma_k$ formulas with parameters so that, for each $X \subseteq \mathbb{N}$, an extended SMA $(M, U) \cong (\mathbb{N}, \bar{X}^{(k+1)})$ (viewed as a structure with two ternary and one unary relation) can be coded in $\mathcal{E}^X$.

We use an argument as in the proof of the Separation Theorem [10] Thm 2.1] to show the claim. Suppose that $\mathcal{E}^X \leq_T Y$ so that there is a representation $(\mathcal{R}, \alpha)$ of $\mathcal{E}^X$ with $\mathcal{R}$ recursive in $Y$ (later, $Y$ will be $X'$). Then the preimage under $\alpha$ of the successor relation of $M$ is c.e. in $Y^{(k-1)}$. Hence there is a function $f \leq_T Y^{(k-1)}$ such that, for all $n$, $\alpha(f(n)) = n^M$ (i.e., $f(n)$ is an index for $n$ in $M$). Then $U$ (viewed as a subset of $\mathbb{N}$) is c.e. in $Y^{(k-1)}$ via the enumeration procedure which enumerates $n$ into $U$ iff the $\Sigma_k$-formula defining $U$ (with a fixed list of parameters in $\mathcal{E}^X$) holds for $\alpha(f(n))$. Since $U = \bar{X}^{(k+1)}$, for $Y = X'$ this implies $\bar{X}^{(k+1)}$ c.e. in $X^{(k)}$, which is not the case. 

In the following, we use notation from Nies [11].

**Proposition 3.3.** Suppose that the $\Sigma^0_k$-Boolean algebra $\mathcal{B}$ is effectively dense. Then $\mathcal{I}(\mathcal{B}) \not\leq_T \emptyset^{(k)}$.

*Proof.* We prove the claim for $k = 1$, i.e., we show that $\mathcal{I}(\mathcal{B}) \not\leq_T \emptyset'$ for a c.e. effectively dense $\mathcal{B}$. For larger $k$, one relativizes this to $\emptyset^{(k-1)}$.

Choose a c.e. separating ideal $I_0$ (defined in [11], (7))] such that $\mathcal{B}/I_0$ is infinite, and let $K$ be the lattice of $\Sigma^0_3$-ideals of $\mathcal{B}$ which contain $I_0$. We show that

$$\mathcal{I}(\mathcal{B}) \leq_T \emptyset' \Rightarrow K \leq_T \emptyset^{(3)}.$$  

This suffices since, by the proof of [11] Lemma 2.4], there is an interval $[C, D]_K$ isomorphic to $\mathcal{E}^3$, the lattice of $\Sigma^0_3$-sets. By Fact 3.1 if $K \leq_T \emptyset^{(3)}$, then also $[C, D]_K \leq_T \emptyset^{(3)}$, which contradicts Proposition 3.2 for $X = \emptyset^{(2)}$. Thus $\mathcal{I}(\mathcal{B}) \not\leq_T \emptyset'$.

To prove (2), assume that there is a representation $(\mathcal{R}, \alpha)$ of the lattice $\mathcal{I}(\mathcal{B})$, with $\mathcal{R}$ Turing below $\emptyset'$. Note that $\mathcal{B}$ is canonically isomorphic to the complemented elements in $\mathcal{I}(\mathcal{B})$, a $\Sigma_1$-definable subset of $\mathcal{I}(\mathcal{B})$. Hence from $\mathcal{R}$, using Fact 3.1, we can derive a presentation $(\mathcal{S}, \mathcal{B})$, for $\mathcal{B}$ such that $\mathcal{S} \leq_T \emptyset'$, which we will use in the following. Let $x, r, s$ range over $\mathcal{B}$.

Given a c.e. ideal $L \subseteq I_0$, let

$$J(L) = \{x \in \mathcal{B} : \exists r \in I_0 \forall s \in I_0 [s \wedge r \equiv 0 \Rightarrow x \wedge s \in L]\}.$$
In Nies [11, Lemma 2.3] it is shown that each \( J \in K \) is of the form \( J(L) \) for some c.e. \( L \). Thus, to obtain the desired representation of \( K \), we represent \( J \in L \) by an index for an \( L \) such that \( J = J(L) \). Note that

\[
J(L) = \{ x \in B : \exists \alpha \in \Lambda_0 \; I_0 \cap [0, \forall \wedge x] \subseteq L \}.
\]

Thus, by Fact 3.1, \( \{ x : x \in J(L) \} \) is \( \Sigma^0_2 \) with respect to \( (S, P) \), and \( \Sigma^0_2 \) can be obtained uniformly in an \( R \)-index for \( L \). Then \( J(L_0) \subseteq J(L_1) \) is \( \Pi^0_1 \). For the lattice operations, given \( L_0, L_1 \) note that \( J = J(L_0) \cup J(L_1) \in K \), so there is \( L \) such that \( J = J(L) \). Since we can determine a \( \Sigma^0_2 \) index with respect to \( S \) for \( J \), and equality of \( \Sigma^0_2 \)-ideals (under the representation \( S \)) of \( B \) is \( \Pi^0_2 \), we can find an index for \( L \) using \( \theta^{(3)} \) as an oracle. For \( J(L_0) \cap J(L_1) \) one argues similarly. \( \square \)

4. Nonisomorphic end intervals of \( \mathcal{E}_H \)

In this section, we apply the results from Sections 2 and 3 to the lattice of \( \Pi^0_1 \) classes to show that there are exactly two distinct types of nontrivial end intervals \( [P, 1] \) of \( \mathcal{E}_H \). It is an easy observation that there are at most two, those where \( P \) is principal and where \( P \) is nonprincipal \( \emptyset \).

**Theorem 4.1.** Let \( P \) be nonprincipal. Then \( [P, 1]_\Theta \) is not isomorphic to \( \mathcal{E}_H \).

**Proof.** Suppose for a contradiction that \( [P, 1]_\Theta \cong \mathcal{E}_H \) via \( \Phi \), but \( P \) is nonprincipal. If the structure \( X \) is coded in \( [P, 1]_\Theta \) with first-order formulas and parameters \( P_1, \ldots, P_m \), we will denote by \( \Phi(X) \) the structure coded in \( \mathcal{E}_H \) with the same formulas and the parameter list \( \Phi(P_1), \ldots, \Phi(P_m) \). (Thus, \( X \) behaves the same way in \( [P, 1]_\Theta \) as \( \Phi(X) \) in \( \mathcal{E}_H \).)

By Lemma 2.5 choose a locally principal extension \( Q \) of \( P \) such that \( P \not\subseteq Q \). Then, by Lemma 2.5, the \( \Sigma^0_2 \) Boolean algebra \( B = B(Q)/R_P(Q) \) is effectively dense. Hence, by Proposition 3.3 for \( k = 3, \mathcal{I}(B) \not\leq_T \emptyset^{(3)} \). Taking complements in \( B, \mathcal{I}(B) \) is isomorphic to \( (B, \mathcal{I}(B)) \) of the lattice \( H \) of \( \Sigma^0_2 \) filters of \( B(Q) \) containing \( R_P(Q) \), so \( H \not\leq_T \emptyset^{(3)} \). Note that \( \Phi(P) = \emptyset \). Let \( \bar{Q} = \Phi(Q) \), a nonprincipal \( \Pi^0_1 \) class. Note that \( \Phi(R_P(Q)) = R_{\Phi(Q)} \), so \( \bar{B} = B(\bar{Q})/R_{\Phi}(\bar{Q}) \) is the isomorphic image of \( B \) under \( \Phi \). For the “\( B \)-side”, we have by the Definability Lemma 2.4 and the remark after that, for each \( F \in H \), there is a \( C \subseteq Q \) such that

\[
F = \{ X \in B(Q) : (\exists R \in R_P(Q))(R \subseteq X \cup C) \}.
\]

So this situation is copied to the \( \bar{B} \)-side by \( \Phi \). There, \( R_{\Phi}(\bar{Q}) \) is the family of clopen sets containing \( \bar{Q} \). For \( C \in \mathcal{E}_H \), \( G(C) \) is a filter, where

\[
G(C) = \{ X \in B(\bar{Q}) : (\exists V)(\bar{Q} \subseteq V \subseteq X \cup C) \},
\]

and \( V \) ranges over the clopen sets. If

\[
G = \{ G(C) : C \in \mathcal{E}_H \},
\]

then \( G \) is the isomorphic image under \( \Phi \) of \( H \), and \( G \) is a lattice with the standard operations \( \lor, \land \) on the filters of \( B(Q) \). To conclude the proof we show \( G \not\leq_T \emptyset^{(3)} \). The relation “\( P \in G(P) \)” is \( \Sigma^0_2 \) uniformly in \( C \), since “\( Q \subseteq V \)” is \( \Sigma^0_2 \), and “\( V \subseteq X \cup C \)” is \( \Pi^0_1 \), being equivalent to “\( \forall X \cup C = 2^\omega \)” (It is here where the
difference between the principal and nonprincipal end intervals becomes apparent, since the set in \( \mathbb{E} \) corresponding to \( G(C) \) is merely \( \mathcal{F}_1 \). It follows that the relation “\( G(P_c) = G(P_d) \)” is \( \Pi^0_1 \). Since \( \Phi \) is an isomorphism, \( G \) is a lattice with the usual operations on filters. To show that these operations are recursive in \( \Phi^{(3)} \), first note that \( G(P_c) \cap G(P_d) = G(P_c \cap P_d) \). For the supremum, we have
\[
G(P_c) \cup G(P_d) = \{X \cap Y : X \in G(P_c) \& Y \in G(P_d)\},
\]
and this equals \( G(P_e) \) for some \( e \). In fact, such an \( e \) can be obtained with oracle \( \Phi^{(3)} \), because \( e \) satisfies
\[
(\forall i)(P_i \in G(P_e) \iff (\exists a, b)(P_a \in G(P_i) \& P_b \in G(P_d) \& P_i = P_a \cap P_b)).
\]

\[\square\]

5. Some definable subsets of \( \mathcal{E}_k \)

In this section, we will demonstrate the definability in \( \mathcal{E}_k \) of various sets of \( \Pi^0_1 \) classes, including the finite classes and the minimal classes. Recall that the Cantor-Bendixson derivative \( D(P) \) of a closed set \( P \) contains exactly the limit points of \( P \). Then \( \{P : \text{card}(D^n(P)) \geq k\} \) is a \( \Sigma^0_{k+1} \) filter for each finite \( n \) and \( k \) by Theorem 45 of [3]. We will show that this family is in fact definable in \( \mathcal{E}_k \).

For a \( \Pi^0_1 \) class \( P \), let \( \mathcal{L}(P) \) be the initial segment \([0, P]\) in \( \mathcal{E}_k \). In general, \( \mathcal{L}(P) \) may not be a Boolean algebra. Hence, we also consider the subfamily \( \mathcal{C} \mathcal{L}(P) \) of relative clopen subclasses. That is, \( \mathcal{C} \mathcal{L}(P) = \{P \cap V : V \in \mathcal{Q}\} \). Then \( \mathcal{C} \mathcal{L}(P) \) is always a Boolean algebra and has a \( \Delta^0_2 \) representation using as indices Gödel numbers for clopen sets. Recall that \( P \) is thin if \( \mathcal{L}(P) \) is a Boolean algebra; the corresponding ideal \( I \) in \( \mathcal{Q} \) is said to be hh-simple in analogy to \( \mathcal{E} \). Then \( P \) is thin if and only if \( \mathcal{L}(P) = \mathcal{C} \mathcal{L}(P) \). It is shown in [4] that the set of indices for thin classes is a \( \Pi^0_1 \) set.

Recall that the derivative \( B^* \) of a boolean algebra \( B \) is \( B/U \), where \( U \) is the ideal generated by the atoms of \( B \); equivalently, the derivative is the quotient of \( B \) modulo the filter generated by the co-atoms. Note that \( B^* = \{0\} \) iff \( B \) is finite. We say \( P \) is minimal if \( \mathcal{L}(P)^* \) is the trivial Boolean algebra \( \{0, 1\} \) and \( P \) is quasi-minimal if \( \mathcal{L}^*(P) \) is finite; the corresponding ideal \( I \) in \( \mathcal{Q} \) is maximal (quasi-maximal).

Certainly, any family definable in \( \mathcal{L}(P) \) will have an arithmetical index set. As was done in [7] for \( \mathcal{E} \), we will obtain a partial converse here.

A closed set is nowhere dense if it does not include any nontrivial clopen set. Note that thin classes and countable classes are nowhere dense.

**Theorem 5.1.** Suppose the \( \Pi^0_1 \) class \( P \) is nowhere dense. Then for each odd \( k \geq 3 \), the class of \( \theta^k \) filters of \( \mathcal{C} \mathcal{L}(P) \) is uniformly definable in \( \mathcal{E}_k \), via a formula \( \varphi(X; P_1, \ldots, P_n, P) \) which does not depend on \( P \).

The proof of Theorem 5.1 is given below. We first show how to derive the definability of subsets of \( \mathcal{E}_k \) from this. If \( F \) is a filter of \( \mathcal{C} \mathcal{L}(P) \), let \( \mathcal{A}(F) \) be the filter of \( \mathcal{C} \mathcal{L}(P) \) generated by the co-atoms of \( \mathcal{C} \mathcal{L}(P)/F \), so that \( \mathcal{C} \mathcal{L}(P)/\mathcal{A}(F) \) is the derivative of \( \mathcal{C} \mathcal{L}(P)/F \). Let \( B^{(k)} \) be the \( k \)th derivative of \( B \). It follows from Theorem 4.7 of [1] that for any \( \Pi^0_1 \) class \( P \), \( \mathcal{C} \mathcal{L}(P)^{(k)} \) is effectively isomorphic to \( \mathcal{C} \mathcal{L}(D^k(P)) \).
Proposition 5.2. If $F$ is a filter of $\mathcal{CL}(P)$ which is definable in $(\mathcal{E}_\Pi, P)$, then so is $A(F)$. The formula defining $A(F)$ only depends on the one defining $F$, not on the particular choice of $P$.

Proof. Suppose that $F$ is a $\Sigma^0_k$ filter, $k \geq 3$. Then $A(F)$ is $\Sigma^0_{k+2}$. Using Theorem 5.1, we may define $A(F)$ as the least $\Sigma^0_{k+2}$ filter of $\mathcal{CL}(P)$ which contains all the elements of $F$ and all $B \subset P$ such that $B/F$ is a co-atom in $\mathcal{CL}(P)/F$.

In the following theorem, the case $n = 1$ and $B = \{0\}$ gives a first-order definition of finiteness.

Theorem 5.3. Let $n > 0$ and let $B$ be a finite Boolean algebra or $B = \{0\}$. Then

$$\{P : \mathcal{CL}(P)^{(n)} \cong B\}$$

is definable in $\mathcal{E}_\Pi$ without parameters.

Proof. Let $F^P_0 = \{1\}$ and for each $n$, let $F^P_{n+1} = A(F_n)$. It follows from Proposition 5.2 that there is a formula $\varphi_n$ (independent of $P$) which defines $F^P_n$ in $\mathcal{CL}(P)$. Hence, we can express that the quotient algebra of $\mathcal{CL}(P)$ modulo $F^P_n$ is isomorphic to $B$. \hfill \Box

Corollary 5.4. The following families of $\Pi^0_1$ classes are definable in $\mathcal{E}_\Pi$ without parameters:

(a) For any fixed $n$ and $k$, $\{P : \text{card}(D^n(P)) \leq k\}$.

(b) The minimal classes.

Proof. (a) This follows from Theorem 5.3 and the fact that $\mathcal{CL}(D^n(P))$ is isomorphic to $\mathcal{CL}(P)^{(n)}$.

(b) $P$ is minimal if and only if, for all $Q \in \mathcal{L}(P)$, either $Q$ is finite or $P - Q$ is finite. \hfill \Box

This corollary takes care of countable classes of finite rank, but we can also consider uncountable classes of finite rank. (Recall that the rank of a class $P$ is the least $\alpha$ such that $D^{\alpha+1}(P) = D^\alpha(P)$.)

Proposition 5.5. For each ordinal $\alpha$, the family of $\Pi^0_1$ classes of rank $\alpha$ is definable in $\mathcal{E}_\Pi$ if and only if $\alpha$ is finite.

Proof. For infinite $\alpha$, the family cannot be definable since $\{e : \text{card}(D^\alpha(P_e)) = \emptyset\}$ is $\Sigma^0_{2n+1}$ complete and thus not arithmetical, by Theorem 45 of [4]. For finite $\alpha$, $P$ has rank $\alpha$ if and only if $F^P_{\alpha+1} = F^P_{\alpha}$, where $F^P_{\alpha}$ is defined as in the proof of Theorem 5.3. \hfill \Box

In the following we refer to Tarski's classification of the completions $T$ of the theory of Boolean algebras, in the form presented in Chang and Keisler [7, Section 5.5.]. They assign invariants $m(B), n(B) \in \omega+1$ to Boolean algebras and prove that two Boolean algebras are elementarily equivalent iff they have the same invariants. Thus, if $T$ is a completion of the theory of Boolean algebras, we can also write $m(T), n(T)$ for $m(B), n(B)$, where $B$ is some model of $T$. 


**Theorem 5.6.** For any completion $T$ of the theory of Boolean algebras, except possibly the one with invariants $m(T) = \infty$ and $n(T) = 0$, the family of $\Pi_0^1$ classes such that $CL(P) \models T$ is definable in $E_H$ without parameters.

Note that the theorem is nontrivial since some completions are not finitely axiomatizable.

**Proof.** To define the invariants for a Boolean algebra $B$, one introduces definable ideals $I_k$: let $I_0 = \{0\}$, and let $I_{k+1}$ be the preimage in $B$ of the ideal of $B/I_k$ generated by the atomic and the atomless elements. Consider $B = CL(P)$. There are formulas $\psi(P)$ in the language of $E_H$ expressing that $B/I_k$ has $n$ atoms, or, using Theorem 5.1 for ideals, there are infinitely many atoms (that is, the ideal generated by the atoms is nonprincipal). Thus, we can express that $CL(P)$ satisfies the required invariants.

**Proof of Theorem 5.1.** We use the Definability Lemma 2.6. Given a locally principal extension $Q$ of a thin $\Pi_0^1$ class $P$ and the effectively dense $\Sigma_0^1$ Boolean algebra $K = B(Q)/R_P(Q)$ as above, consider the Boolean homomorphism $\Phi : CL(P) \to K$ defined by

$$\Phi(P \cap V) = (Q \cup V)/R_P(Q).$$

We first show that this map is well defined. Suppose that $P \cap V = P$, then $P \cap V^c = \emptyset$, so that $Q \cap V^c$ is clopen by Lemma 2.3, and thus $Q \cup V = (Q \cap V^c) \cup V$ is also clopen; hence, $P \subset Q \cup V$.

Furthermore, note that with the canonical representations, $\Phi$ is $\Delta^0_3$.

**Claim 5.7.** Suppose that for all clopen $V$, $P \subset Q \cup V$ implies that $P \subset V$. Then $\Phi$ is an embedding.

**Proof.** Suppose that $P \cap V$ is in the kernel of $\Phi$. Then $P \subset Q \cup V^c$, so that $P \subset V^c$.

Next we show such a $Q$ exists.

**Claim 5.8.** Suppose that $P$ is nowhere dense. Then there exists a locally principal extension $Q$ of $P$ such that, for all clopen $V$, $P \subset Q \cup V$ implies that $P \subset V$.

**Proof.** First note that, if $P \not\subset V$, then $P \cup V$ is nonprincipal. To see this, suppose that $P \cup V = U$ is clopen, so that $P - V = U - V$ is a nonempty clopen subset of $P$. Since $P$ is nowhere dense, $U - V = \emptyset$, so that $P \not\subset V$, contradiction.

Recall that $2^\omega - P = \bigcup_n U_n$, where the clopen sets $U_n$ are disjoint. We will define $Q$ to be $2^\omega - \bigcup_{n \in A} U_n$ for a certain c.e. set $A$. We build $A$ by extending the construction in the proof of Lemma 2.4 above. Let $P_e$ be the $e$th $\Pi_0^1$ class and express $P_e$ as the intersection of a uniformly computable, decreasing sequence of clopen sets $P_{e,s}$. It suffices to meet the following requirements for each clopen $V$ and each $e$:

$$R_{V,e} : (Q \cup V) \cap P_e = P \Rightarrow P \subset V.$$

**Construction of $A$:** In the beginning all requirements are declared unsatisfied. At stage $s$ of the construction, we may select a candidate $U_n$ for some requirement, and we may take action on some requirements, as follows.

1. If $R_{V,e}$ is the highest priority requirement without a candidate and $V^c \cap U_n \neq \emptyset$, then declare $U_n$ to be its candidate.
2. For each unsatisfied requirement $R_{V,c}$ which has a candidate $U_n$, if now $U \cap P_{e,s} = \emptyset$, then put $n$ into $A$ and declare the requirement satisfied.

We observe that no $U_n$ can be a candidate for more than one requirement.

Verification: Clearly, $Q$ is a locally principal extension of $P$. Suppose by way of contradiction that the requirement $R = R_{V,e}$ is not met. Then $(Q \cup V) \cap P_e = \emptyset$ and $P \cup V$ is nonprincipal. By the first assumption, $(Q \cup V) \cap P_e = 2^\omega$ and $(Q \cup V) \cap P_e = \emptyset$. Then for each $n$, we have $U_n \subset Q \cup V \cup P_e$ and we have $U_n \cap (Q \cup V) \cap P_e = \emptyset$. By the second assumption, there are infinitely many $n$ such that $V^c \cap U_n \neq \emptyset$. That is, suppose by way of contradiction that only finitely many of the $U_n$ meet $V^c$, say $\{U_i : i \in F\}$ for some finite set $F$. Then it is easily seen that $P \cup V = V \cup (2^\omega - \bigcup_{i \in F} U_i)$. Thus, $R$ eventually receives a candidate $U_n$. Now there are two cases.

Case I: First, suppose that $U_n \cap P_e = \emptyset$. Then for some $s$, $U_n \cap P_{e,s} = \emptyset$, so that $n \in A$ and $U_n \cap Q = \emptyset$ by the construction. But we chose $n$ such that $U_n \cap V^c \neq \emptyset$, that is, $U_n \nsubseteq V$, which contradicts $U_n \subset Q \cup V \cup P_e$.

Case II: Suppose that $U_n \cap P_e \neq \emptyset$. Then by the construction $n \notin A$. Thus, $U_n \subset Q$, which contradicts $U_n \cap (Q \cup V) \cap P_e = \emptyset$. \qed

Now, given a nowhere dense $\Pi^0_2$ class $P$ and odd $k \geq 3$, then $\varphi(X; P_1, \ldots, P_n, P)$ for Theorem 5.1 is obtained as follows. Suppose $P \subseteq Q$ is as in Claim 5.8. Since $\Phi$ is 1-1 and is $\Delta^0_k$, if $\mathcal{H}$ ranges through the $k$-acceptable filters of $B(Q)$, then $\Phi^{-1}(\mathcal{H}/_{\mathcal{R}_F(Q)})$ ranges through the $\Sigma^0_k$ filters of $\mathcal{CL}(P)$. Let $\varphi(X; P_1, \ldots, P_n, P, Q)$ be
\[
\exists Q[P, Q \text{ as in Claim 5.8 } \& \forall V[X = P \cap V \& \psi(Q \cup V; P_1, \ldots, P_n, P, Q)]
\]
where $\psi(X; P_1, \ldots, P_n, P, Q)$ is the formula from the Definability Lemma 2.6 \qed

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