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LIMIT SETS AND REGIONS OF DISCONTINUITY OF TEICHMÜLLER MODULAR GROUPS

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ABSTRACT. For a Riemann surface of infinite type, the Teichmüller modular group does not act properly discontinuously on the Teichmüller space, in general. As an analogy to the theory of Kleinian groups, we divide the Teichmüller space into the limit set and the region of discontinuity for the Teichmüller modular group, and observe their properties.

1. INTRODUCTION

The Teichmüller modular group $\mathrm{Mod}(R)$ for a hyperbolic Riemann surface R whose Fuchsian model is of the first kind is the set of homotopy classes of quasiconformal automorphisms of R, and this is a group of biholomorphic automorphisms of the Teichmüller space $T(R)$. If R is of analytically finite type, it is well known that $\mathrm{Mod}(R)$ acts properly discontinuously on $T(R)$. However if R is of infinite type, $\mathrm{Mod}(R)$ does not act properly discontinuously on $T(R)$, in general. In [4], we gave a sufficient condition for the proper discontinuity.

In this paper, we introduce new notions, the limit set and the region of discontinuity for a Teichmüller modular group. Actually, we define these notions for the Teichmüller modular group of a general Riemann surface whose Fuchsian model is not necessarily of the first kind. For this purpose, we consider the reduced Teichmüller modular group $\mathrm{Mod}^{\#}(R)$ acting on a reduced Teichmüller space $T^{\#}(R)$, which will be defined in the next section. The limit set $\Lambda(G)$ for a subgroup G of $\mathrm{Mod}^{\#}(R)$ is the set of points p in $T^{\#}(R)$ such that the orbit of p under G is not discrete, and the region of discontinuity $\Omega(G)$ is the complement of the limit set. This is an analogy to the theory of Kleinian groups acting on the Riemann sphere, and we expect that they satisfy similar properties to that of limit sets and regions of discontinuity for Kleinian groups. We prove some of them. However it seems that the essential natures of limit sets and regions of discontinuity for Teichmüller modular groups are different from the case of Kleinian groups. For example, the orbit of a point in a limit set is not dense in the limit set, in general. Hence we have to devise the proofs.

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We say that a subgroup G of $\mathrm{Mod}^{\#}(R)$ is of the *first kind* if $\Omega(G) = \emptyset$, and otherwise of the *second kind*. We show sufficient conditions for $\mathrm{Mod}^{\#}(R)$ to be of the first kind or of the second kind, and give various examples for each case.

2. Preliminaries

We review theories of Teichmüller spaces and Teichmüller modular groups (cf. [5], [7] and [10]). Throughout this paper, we assume that a Riemann surface R is hyperbolic. In other words, it is represented by $\mathbb{H}/\Gamma$ for some torsion-free Fuchsian group Γ acting on the upper half-plane $\mathbb{H}$. We also assume that R has the non-abelian fundamental group. In other words, the Fuchsian group Γ is non-elementary. We say that R is of *analytically finite type* (g, n) if it is a Riemann surface of genus g from which n punctures are removed.

Fix a Riemann surface R. For pairs (S_i, f_i) of Riemann surfaces S_i and quasiconformal maps f_i of R onto S_i, we say that (S_1, f_1) and (S_2, f_2) are *equivalent* if there exists a conformal map h of S_1 onto S_2 such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every point of the ideal boundary fixed throughout. The *Teichmüller space* $T(R)$ with the base Riemann surface R is the set of all the equivalence classes $[S, f]$ of such pairs (S, f) as above. Further we say that (S_1, f_1) and (S_2, f_2) are *weakly equivalent* if there exists a conformal map h of S_1 onto S_2 such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on R. The *reduced Teichmüller space* $T^{\#}(R)$ with the base Riemann surface R is the set of all the weakly equivalent classes $[S, f]$ of such pairs (S, f) as above.

We say that two quasiconformal automorphisms h_1 and h_2 of R are *equivalent* if $h_2^{-1} \circ h_1$ is homotopic to the identity by a homotopy that keeps every point of the ideal boundary fixed throughout. The *Teichmüller modular group* $\mathrm{Mod}(R)$ is the set of all the equivalent classes $[h]$ of quasiconformal automorphisms h of R. Further we say that two quasiconformal automorphisms h_1 and h_2 of R are *weakly equivalent* if $h_2^{-1} \circ h_1$ is homotopic to the identity on R. The *reduced Teichmüller modular group* $\mathrm{Mod}^{\#}(R)$ is the set of all the weakly equivalent classes $[h]$ of quasiconformal automorphisms h of R. If R is a Riemann surface whose Fuchsian model is of the first kind, then $T^{\#}(R) = T(R)$ and $\mathrm{Mod}^{\#}(R) = \mathrm{Mod}(R)$.

Similar to the case of $T(R)$, the reduced Teichmüller space $T^{\#}(R)$ is equipped with the reduced Teichmüller distance $d_T(\cdot, \cdot)$ defined by

$$d_T([S_1, f_1], [S_2, f_2]) = \frac{1}{2} \inf_{f_1,\ f_2} \log K(f_1 \circ f_2^{-1}),$$

where $K(\cdot)$ is the maximal dilatation of a quasiconformal map and the infimum is taken over all quasiconformal maps f_1 and f_2 determining $[S_1, f_1]$ and $[S_2, f_2]$ respectively. It is known that, for any quasiconformal map f of R onto S, there exists a quasiconformal map that has the smallest maximal dilatation in the homotopy class of f. This is called an extremal quasiconformal map.

The reduced Teichmüller space $T^{\#}(R)$ is a complete metric space with respect to d_T. An element $\omega = [h] \in \mathrm{Mod}^{\#}(R)$ induces an automorphism of $T^{\#}(R)$ by

$$[S, f] \mapsto [S, f \circ h^{-1}].$$

This is an isometric automorphism with respect to d_T and denoted by ω_*. Namely, we have a homomorphism of $\mathrm{Mod}^{\#}(R)$ to the automorphism group $\mathrm{Aut}(T^{\#}(R))$ of $T^{\#}(R)$. With a few exceptional surfaces which do not appear in our present case, the above homomorphism $\mathrm{Mod}^{\#}(R) \to \mathrm{Aut}(T^{\#}(R))$ is faithful. This was first proved in [2]. Another proof was given in [3]. Therefore we can identify ω_* with ω and omit the asterisk hereafter.

For a non-trivial simple closed curve c on R, we denote the simple closed geodesic that is homotopic to c by c_*. Then for a quasiconformal map f of R, an inequality

$$K(f)^{-1}\ell(c_*) \leq \ell(f(c)_*) \leq K(f)\ell(c_*)$$

holds, where $\ell(\cdot)$ is the hyperbolic length of a curve ([12, Lemma 3.1]).

3. The limit set and the region of discontinuity

In this section, as an analogy to the theory of Kleinian groups, we introduce notions of the limit set and the region of discontinuity for a reduced Teichmüller modular group, and investigate these properties. We begin by giving their definitions.

Definition 1. We say that a point p in $T^{\#}(R)$ is a *limit point* for a subgroup G of $\mathrm{Mod}^{\#}(R)$ if there exist a point $q \in T^{\#}(R)$ and a sequence $\{\chi_n\}$ of distinct elements of G such that $\lim_{n\to\infty} d_T(\chi_n(q), p) = 0$. The set of the limit points is called the *limit set* of G, and denoted by $\Lambda(G)$. The complement $T^{\#}(R) - \Lambda(G)$ of the limit set is denoted by $\Omega(G)$, and called the *region of discontinuity* of G. Similarly, for a subgroup G of the ordinary modular group $\mathrm{Mod}(R)$, we can define $\Lambda(G)$ and $\Omega(G)$ in $T(R)$.

For a Riemann surface R of analytically finite type, $\Lambda(\mathrm{Mod}(R)) = \Lambda(\mathrm{Mod}^{\#}(R)) = \emptyset$. On the other hand, for a Riemann surface R whose Fuchsian model is of the second kind, we always have $\Omega(\mathrm{Mod}(R)) = \emptyset$, since a slight change of the value of a quasiconformal map on the ideal boundary produces a different element of $\mathrm{Mod}(R)$. This is the reason why we consider the reduced modular group $\mathrm{Mod}^{\#}(R)$, not the ordinary modular group $\mathrm{Mod}(R)$, for Riemann surfaces R of infinite type. In the next section, we exhibit an example of a Riemann surface R which satisfies both $\Lambda(\mathrm{Mod}^{\#}(R)) \neq \emptyset$ and $\Omega(\mathrm{Mod}^{\#}(R)) \neq \emptyset$.

We investigate certain properties of the limit set and the region of discontinuity. There are other equivalent definitions of the limit set.

Lemma 1. *For a subgroup G of $\mathrm{Mod}^{\#}(R)$, let $\Lambda'(G)$ be the set of points $p \in T^{\#}(R)$ such that $\lim_{n\to\infty} d_T(\omega_n(p), q') = 0$ for a point $q' \in T^{\#}(R)$ and a sequence $\{\omega_n\}$ of distinct elements of G, and $\Lambda''(G)$ the set of points $p \in T^{\#}(R)$ such that $\lim_{n\to\infty} d_T(\phi_n(p), p) = 0$ for a sequence $\{\phi_n\}$ of distinct elements of G. Then $\Lambda(G) = \Lambda'(G) = \Lambda''(G)$.*

Proof. For any point $p \in \Lambda(G)$, there exist a point $q \in T^{\#}(R)$ and a sequence $\{\chi_n\}$ of distinct elements of G such that $\lim_{n\to\infty} d_T(\chi_n(q), p) = 0$. Since the action of $\mathrm{Mod}^{\#}(R)$ on $T^{\#}(R)$ is isometric, we have

$$\lim_{n\to\infty} d_T(q, \chi_n^{-1}(p)) = \lim_{n\to\infty} d_T(\chi_n(q), p) = 0,$$

which means that $p \in \Lambda'(G)$.

For any point $p \in \Lambda'(G)$, there exist a point $q' \in T^{\#}(R)$ and a sequence $\{\omega_n\}$ of distinct elements of G which satisfy $\lim_{n\to\infty} d_T(\omega_n(p), q') = 0$. Set $\phi_n = \omega_{n+1}^{-1} \circ \omega_n$. Then $\lim_{n\to\infty} d_T(\phi_n(p), p) = 0$. If $\{\phi_n\}$ contains infinitely many distinct elements, then this means that $p \in \Lambda''(G)$. If $\{\phi_n\}$ consists of finitely many elements, then there exists an element ϕ such that $\phi_n = \phi$ for infinitely many n and $\phi(p) = p$. We see that ϕ is of infinite order. Indeed, suppose that ϕ is of finite order $k \geq 1$. Since $\omega_{n+1} = \omega_n \circ \phi^{-1}$, we have $\omega_{n+k} = \omega_n \circ \phi^{-k} = \omega_n$. This contradicts that all ω_n are distinct. Hence ϕ is of infinite order. Thus $\phi^n(p) = p$ implies that $p \in \Lambda''(G)$.

Obviously $\Lambda''(G) \subset \Lambda(G)$. □

First, we show basic properties of the limit set.

Lemma 2. *For a subgroup G of $\mathrm{Mod}^{\#}(R)$, the limit set $\Lambda(G)$ is G-invariant and closed.*

Proof. We may assume that $\Lambda(G) \neq \emptyset$. Let $p \in \Lambda(G)$ and $\chi \in G$. Since p is a limit point, there exists a sequence $\{\chi_n\}$ of distinct elements of G which satisfies $\lim_{n\to\infty} d_T(\chi_n(p), p) = 0$ by Lemma 1. Since χ is an isometry,

$$\lim_{n\to\infty} d_T(\chi \circ \chi_n(p), \chi(p)) = 0.$$

Setting $\omega_n = \chi \circ \chi_n$, we have $\lim_{n\to\infty} d_T(\omega_n(p), \chi(p)) = 0$. Thus $\chi(p) \in \Lambda(G)$ by Lemma 1, and hence $\chi(\Lambda(G)) \subset \Lambda(G)$. Similarly we have $\chi^{-1}(\Lambda(G)) \subset \Lambda(G)$. Thus $\chi(\Lambda(G)) = \Lambda(G)$, which means that $\Lambda(G)$ is G-invariant.

Let $\{p_n\}$ be a sequence of points in $\Lambda(G)$ that converges to a point $p \in T^{\#}(R)$. For each $p_n \in \Lambda(G)$, we can take a sequence $\{\chi_{n,i}\}_{i=1}^{\infty}$ of distinct elements of G such that $\lim_{i\to\infty} d_T(\chi_{n,i}(p_n), p_n) = 0$. For each n, choose $i(n)$ so that $d_T(\chi_{n,i(n)}(p_n), p_n) \leq 1/n$ and all $\chi_{n,i(n)}$ are distinct. Since χ is an isometry, we have

$$\begin{aligned} d_T(\chi_{n,i(n)}(p), p) &\leq d_T(\chi_{n,i(n)}(p), \chi_{n,i(n)}(p_n)) \\ &\quad + d_T(\chi_{n,i(n)}(p_n), p_n) + d_T(p_n, p) \\ &\leq 2d_T(p, p_n) + 1/n. \end{aligned}$$

Then $\lim_{n\to\infty} d_T(\chi_{n,i(n)}(p), p) = 0$. Hence $p \in \Lambda(G)$, which means that $\Lambda(G)$ is closed. □

We classify the limit points for a subgroup G of $\mathrm{Mod}^{\#}(R)$ into three types, $\Lambda_0(G)$, $\Lambda_\infty^1(G)$ and $\Lambda_\infty^2(G)$, according to their stabilizer.

Definition 2. In a subgroup G of $\mathrm{Mod}^{\#}(R)$, the *stabilizer* of a point $p \in T^{\#}(R)$ is defined by $\mathrm{Stab}_G(p) = \{\chi \in G \mid \chi(p) = p\}$.

We define $\Lambda_0(G)$ as the set of points $p \in \Lambda(G)$ such that there exists a sequence $\{\chi_n\}$ of distinct elements of G that satisfies $\lim_{n\to\infty} d_T(\chi_n(p), p) = 0$ and that $\chi_n(p) \neq p$ for all n, and $\Lambda_\infty(G)$ as the set of points $p \in \Lambda(G)$ such that $\mathrm{Stab}_G(p)$ consists of infinitely many elements. Furthermore we divide $\Lambda_\infty(G)$ into two disjoint subsets, $\Lambda_\infty^1(G)$ and $\Lambda_\infty^2(G)$. The $\Lambda_\infty^1(G)$ is the set of points $p \in \Lambda_\infty(G)$ such that there exists an element in $\mathrm{Stab}_G(p)$ that is of infinite order, and the $\Lambda_\infty^2(G)$ is the set of points $p \in \Lambda_\infty(G)$ such that all elements in $\mathrm{Stab}_G(p)$ are of finite order.

It might be the case that $\Lambda_0(G) \cap \Lambda_\infty(G) \neq \emptyset$.

Lemma 3. *The sets $\Lambda_0(G)$, $\Lambda_\infty^1(G)$ and $\Lambda_\infty^2(G)$ are G-invariant.*

Proof. Let $p \in \Lambda_0(G)$. We can take a sequence $\{\chi_n\}_{n=1}^{\infty}$ of distinct elements of G so that $\lim_{n\to\infty} d_T(\chi_n(p), p) = 0$ and $\chi_n(p) \neq p$ for all n. For any $\chi \in G$, we have $\lim_{n\to\infty} d_T(\chi \circ \chi_n(p), \chi(p)) = 0$ and $\chi \circ \chi_n(p) \neq \chi(p)$, which means that $\chi(p) \in \Lambda_0(G)$. Similarly, we have $\chi^{-1}(p) \in \Lambda_0(G)$. Thus $\Lambda_0(G)$ is G-invariant.

The fact that $\mathrm{Stab}_G(\chi(p)) = \chi \circ \mathrm{Stab}_G(p) \circ \chi^{-1}$ implies that the sets $\Lambda_\infty^1(G)$ and $\Lambda_\infty^2(G)$ are G-invariant. □

We exhibit examples of limit points in $\Lambda_0(\mathrm{Mod}^{\#}(R))$.

Example 1. Set

$$R = \mathbb{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbb{Z}} \left\{ \frac{m}{n} + (2n+1)\sqrt{-1} \right\}.$$

By Example 2 in [4], we see that the base point $[R, id]$ belongs to $\Lambda_0(\mathrm{Mod}^{\#}(R))$.

Example 2. Set

$$z_n = \begin{cases} n + \frac{\sqrt{-1}}{j(n)+1} & (n \neq 0), \\ 0 & (n = 0) \end{cases}$$

where $j(n)$ is the power of the factor 2 when we decompose $|n|$ to the product of primes. Set $R = \mathbb{C} - \sum_{n=-\infty}^{\infty}\{z_n\}$. By Example 6 in [4], we see that the base point $[R, id]$ belongs to $\Lambda_0(\mathrm{Mod}^{\#}(R))$.

We also have an example of a limit point in $\Lambda^1_\infty(\mathrm{Mod}^{\#}(R))$.

Example 3. Let $\hat{R}$ be a compact Riemann surface of genus $g \geq 2$, and R a normal covering surface of $\hat{R}$ whose covering transformation group is a cyclic group $\langle \phi \rangle$ generated by a conformal automorphism ϕ of R. Then $p_0 = [R, id] \in T^{\#}(R)$ and $[\phi] \in \mathrm{Mod}^{\#}(R)$ satisfy $[\phi](p_0) = p_0$. Hence p_0 belongs to $\Lambda^1_\infty(\mathrm{Mod}^{\#}(R))$.

We see that if the limit set has an isolated point, the isolated point belongs to $\Lambda^2_\infty(G)$. However, we do not know whether the limit set has an isolated point.

Theorem 1. *For a subgroup G of $\mathrm{Mod}^{\#}(R)$, the set $\Lambda(G) - \Lambda^2_\infty(G)$ does not have an isolated point.*

Proof. We will show that, for any point $p \in \Lambda(G) - \Lambda^2_\infty(G)$, there exists a sequence $\{p_n\}$ of distinct elements in $\Lambda(G) - \Lambda^2_\infty(G)$ such that $\lim_{n\to\infty} d_T(p_n, p) = 0$.

If $p \in \Lambda_0(G)$, then we can take a sequence $\{\chi_n\}_{n=1}^{\infty}$ of distinct elements of G so that $\lim_{n\to\infty} d_T(\chi_n(p), p) = 0$ and $\chi_n(p) \neq p$ for all n. It follows from Lemma 3 that $\chi_n(p)$ belongs to $\Lambda_0(G)$ for all n.

If $p = [S, f] \in \Lambda^1_\infty(G)$, then there exists an element $\chi = [h] \in \mathrm{Stab}_G(p)$ such that all χ^k are distinct for $k \in \mathbb{N}$. Since $\chi(p) = p$, the quasiconformal automorphism $f \circ h^{-1} \circ f^{-1}$ of S is homotopic to a conformal map ψ. Note that ψ does not have a fixed point in S. Indeed, if ψ has a fixed point x in S, then x is fixed by infinitely many elements $\psi^k \in \mathrm{Aut}(S)$. However it is known that the action of $\mathrm{Aut}(S)$ is properly discontinuous if S has the non-abelian fundamental group ([11, Theorem X. 48]). Thus we have a contradiction. We consider the quotient $S/\langle\psi\rangle$ by the cyclic group $\langle\psi\rangle$, and denote it by $\widehat{S}$. The Riemann surface $\widehat{S}$ is also of hyperbolic type.

First, suppose that $\widehat{S}$ is not of analytically finite type (0,3). We take a sequence of quasiconformal maps $\{\widehat{g_n}\}$ of $\widehat{S}$ which are not homotopic to a conformal map on $\widehat{S}$ for all n and satisfy $\lim_{n\to\infty} K(\widehat{g_n}) = 1$. In particular, a lift $\widetilde{g_n}$ of $\widehat{g_n}$ to $\mathbb{H}$ is not the restriction of a conformal map on the limit set $\Lambda(\widehat{\Gamma})$ of the Fuchsian model $\widehat{\Gamma}$ of $\widehat{S}$. Here we note the following lemma.

Lemma 4 ([9, Lemma 2.22])**.** *For a normal subgroup Γ of a non-elementary Fuchsian group $\widehat{\Gamma}$, we have $\Lambda(\Gamma) = \Lambda(\widehat{\Gamma})$ if $\Gamma \neq \{id\}$. Here $\Lambda(\cdot)$ means the limit set of a Fuchsian group.*

Since S is a normal covering surface of $\widehat{S}$, Lemma 4 says that the lift $\widetilde{g_n}$ is not the restriction of a conformal map on $\Lambda(\Gamma)$, either. Here Γ is the Fuchsian model of S. Then a lift g_n of $\widehat{g_n}$ to S is not homotopic to a conformal map on S, and $p_n = [g_n(S), g_n \circ f] \in T^{\#}(R)$ is different from p for each n. We also have

$$\lim_{n\to\infty} d_T(p, p_n) \leq \lim_{n\to\infty} \log K(g_n) = 0.$$

Since $d_T(\chi^k(p_n), p_n) = \log K(g_n \circ \psi^k \circ {g_n}^{-1})$ and since $g_n \circ \psi^k \circ {g_n}^{-1}$ $(k = 1, 2, \cdots)$ are distinct conformal automorphisms of $g_n(S)$, we see that p_n belongs to $\Lambda^1_\infty(G)$. Thus $\{p_n\}$ is a desired sequence.

If $\widehat{S}$ is of analytically finite type (0,3), then we consider $\langle\psi^2\rangle$ instead of $\langle\psi\rangle$. Then $S/\langle\psi^2\rangle$ is not of analytically finite type (0,3), and the same proof can be applied. □

Corollary 1. *For a subgroup G of $\mathrm{Mod}^{\#}(R)$ such that $\Lambda(G) - \Lambda^2_\infty(G)$ is not empty, the limit set $\Lambda(G)$ is an uncountable set.*

Proof. By Theorem 1, the closure $\overline{\Lambda(G) - \Lambda^2_\infty(G)}$ of $\Lambda(G) - \Lambda^2_\infty(G)$ is a perfect closed set. In a complete metric space, every non-empty perfect closed set is an uncountable set (Cantor; cf. [6, p. 156]). Then $\overline{\Lambda(G) - \Lambda^2_\infty(G)}$ is an uncountable set. Since $\overline{\Lambda(G) - \Lambda^2_\infty(G)} \subset \Lambda(G)$, the limit set $\Lambda(G)$ is also an uncountable set. □

Remark 1. The Riemann surfaces R in Examples 1, 2 and 3 satisfy $\Lambda(\mathrm{Mod}^{\#}(R)) - \Lambda^2_\infty(\mathrm{Mod}^{\#}(R)) \neq \emptyset$.

In the theory of Kleinian groups, it is known that the limit set of a non-elementary Kleinian group Γ coincides with the closure of the loxodromic fixed points of Γ, and it also coincides with the closure of the set of limit points that are not fixed by any elements of Γ. On the analogy of this fact, we propose the following problems.

Problem 1. For a subgroup G of $\mathrm{Mod}^{\#}(R)$, the set $\Lambda_0(G)$ is dense in $\Lambda(G) - \Lambda^2_\infty(G)$. The closure of $\Lambda_\infty(G)$ coincides with $\Lambda(G)$.

Next, we consider the proper discontinuity of G on the region of discontinuity.

Definition 3. We say that a subgroup $G \subset \mathrm{Mod}^{\#}(R)$ acts on a subregion $\Omega \subset T^{\#}(R)$ *properly discontinuously* if for any $p \in \Omega$, there exists a constant $r > 0$ such that the set $\{\chi \in G \mid \chi(B(p,r)) \cap B(p,r) \neq \emptyset\}$ consists of only finitely many elements. Here $B(p,r)$ is an open ball centered at p with radius r.

Proposition 1. *Let G be a subgroup of $\mathrm{Mod}^{\#}(R)$. For any point p in $T^{\#}(R) - \Lambda_0(G)$, there exists a constant $r > 0$ such that $\chi(B(p,r)) \cap B(p,r) = \emptyset$ for any $\chi \in G - \mathrm{Stab}_G(p)$.*

Proof. Suppose that, for any $n \in \mathbb{N}$, there exists an element $\chi_n \in G - \mathrm{Stab}_G(p)$ such that $\chi_n(B(p,1/n)) \cap B(p,1/n) \neq \emptyset$. We take a point $q_n \in B(p,1/n)$ so that $\chi_n(q_n) \in B(p,1/n)$. Since χ_n is an isometry, we have

$$d_T(p, \chi_n(p)) \leq d_T(p, \chi_n(q_n)) + d_T(\chi_n(q_n), \chi_n(p)) \leq 2/n.$$

Hence $\lim_{n\to\infty} d_T(p, \chi_n(p)) = 0$. If $\{\chi_n\}$ contains infinitely many distinct elements, then $p \in \Lambda_0(G)$ since $\chi_n(p) \neq p$ for all n. This contradicts $p \notin \Lambda_0(G)$. If $\{\chi_n\}$ consists of finitely many elements, then $\chi_n(p) = p$ for a sufficiently large n. This contradicts $\chi_n \notin \mathrm{Stab}_G(p)$. □

If $p \in \Omega(G)$, then $\mathrm{Stab}_G(p)$ consists of only finitely many elements. Thus we have the following corollary.

Corollary 2. *Let G be a subgroup of $\mathrm{Mod}^{\#}(R)$. Then G acts on $\Omega(G)$ properly discontinuously.*

Remark 2. In general, for a group consisting of isometric transformations acting on a complete metric space, the limit set and the region of discontinuity can be defined as in Definition 1, and they satisfy the same properties as in Lemmas 1, 2, 3 and Proposition 1.

In the last of this section, we propose problems on properties of the limit sets and the regions of discontinuity.

Problem 2. For a subgroup G of $\mathrm{Mod}^{\#}(R)$ such that $\Omega(G)$ is not empty, the limit set $\Lambda(G)$ is nowhere dense in $T^{\#}(R)$.

Problem 3. For a subgroup G of $\mathrm{Mod}^{\#}(R)$ such that $\Omega(G)$ is not empty, the region of discontinuity $\Omega(G)$ is connected.

4. Teichmüller modular group of the second kind

In this section, we consider sufficient conditions for $\mathrm{Mod}^{\#}(R)$ to have a non-empty region of discontinuity. The conditions are given in terms of hyperbolic geometry on R.

Definition 4. For a subgroup G of $\mathrm{Mod}^{\#}(R)$, we say that G is of the *first kind* if $\Omega(G) = \emptyset$, and otherwise of the *second kind*.

Definition 5. For a constant $M > 0$, we define R_M to be the subset of points $p \in R$ such that there exists a non-trivial simple closed curve passing through p whose hyperbolic length is less than M. The set R_ϵ is called the *ϵ-thin part* of R if $\epsilon > 0$ is smaller than the Margulis constant. Further, a connected component of the ϵ-thin part that corresponds to a puncture is called the *cusp neighborhood*.

The conditions mentioned above are given as follows.

Definition 6. We say that R satisfies the *lower bound condition* if there exists a constant $\epsilon > 0$ such that the ϵ-thin part of R consists only of cusp neighborhoods and neighborhoods of geodesics which are homotopic to boundary components. Further we say that R satisfies the *upper bound condition* if there exist a constant $M > 0$ and a connected component R_M^* of R_M such that a homomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ that is induced by the inclusion map of R_M^* into R is surjective.

Remark 3. The lower and upper bound conditions are invariant under quasiconformal deformations ([4, Lemma 8]). In other words, they are regarded as conditions for the Teichmüller space.

The following theorem gives a sufficient condition on Riemann surfaces R for $\mathrm{Mod}^{\#}(R)$ to be of the first kind.

Theorem 2. *If R does not satisfy the lower bound condition, then $\mathrm{Mod}^{\#}(R)$ is of the first kind.*

Proof. Since R does not satisfy the lower bound condition, R has a sequence $\{c_{n*}\}$ of distinct simple closed geodesics that are not homotopic to boundary components with $\ell(c_{n*}) \to 0$ $(n \to \infty)$.

Let $[h_n]$ be an element of $\mathrm{Mod}^{\#}(R)$ that is the Dehn twist along c_n for each n. By the assumption, we can take a representative h_n so that $\lim_{n\to\infty} K(h_n) = 1$. Indeed, the collar $C(c_n)$ of c_n is conformally equivalent to an annulus $A_n = \{z \mid 1 < |z| < r_n\}$, and we can take r_n so that $\lim_{n\to\infty} r_n = \infty$ by the collar lemma (cf. [8]). Further we can take h_n so that h_n is the identity on the complement of $C(c_n)$ and that the restriction of h_n to $C(c_n)$ is conjugate to a map $\tilde{h}_n : A_n \to A_n$ defined by

$$\tilde{h}_n(z) = z \exp\left(2\pi i \frac{\log|z|}{\log r_n}\right).$$

Then $\lim_{n\to\infty} K(\tilde{h}_n) = 1$. Hence $\lim_{n\to\infty} d_T([h_n](p_0), p_0) = 0$, where $p_0 = [R, id]$ is the base point of $T^{\#}(R)$. This means that $p_0 \in \Lambda(\mathrm{Mod}^{\#}(R))$. Further $[h_n](p_0) \neq p_0$ implies that $p_0 \in \Lambda_0(\mathrm{Mod}^{\#}(R))$.

Let f be an arbitrarily quasiconformal map of R onto S. If f is a K-quasiconformal map, the geodesic $f(c_n)_*$ that is homotopic to $f(c_n)$ satisfies

$$K^{-1}\ell(c_{n*}) \leq \ell(f(c_n)_*) \leq K\ \ell(c_{n*}).$$

Then S also has the sequence $\{f(c_n)_*\}$ of distinct simple closed geodesics with $\ell(f(c_n)_*) \to 0$ $(n \to \infty)$, and the quasiconformal map $f \circ h_n^{-1} \circ f^{-1}$ is the Dehn twist along each $f(c_n)$. Hence, for any point $p = [S, f] \in T^{\#}(R)$, we have $\lim_{n\to\infty} d_T([h_n](p), p) = 0$, which means that $p \in \Lambda(\mathrm{Mod}^{\#}(R))$. In fact, $T^{\#}(R) = \Lambda(\mathrm{Mod}^{\#}(R)) = \Lambda_0(\mathrm{Mod}^{\#}(R))$. □

In [4], we gave a sufficient condition on Riemann surfaces R for some subgroup G of $\mathrm{Mod}^{\#}(R)$ to satisfy $\Lambda(G) = \emptyset$.

Proposition 2 ([4]). *Suppose that R satisfies the lower and upper bound conditions. For a simple closed geodesic c on R, we set*

$$\mathrm{Mod}_c^{\#}(R) = \{[f] \in \mathrm{Mod}^{\#}(R) \mid f(c) \text{ is freely homotopic to } c\}.$$

Then $\Lambda(\mathrm{Mod}_c^{\#}(R)) = \emptyset$.

Using this result, we have a sufficient condition on Riemann surfaces R for $\mathrm{Mod}^{\#}(R)$ to be of the second kind.

Theorem 3. *If R satisfies the lower and upper bound conditions, then* $\mathrm{Mod}^{\#}(R)$ *is of the second kind.*

To prove this theorem, we use the following lemma.

Lemma 5. *Let R be a Riemann surface satisfying the lower bound condition for a constant $\epsilon > 0$, and c_0 a simple closed geodesic on R. Then there exist a positive constant $\alpha < \epsilon$ and a quasiconformal map f of R such that $\ell(f(c_0)_*) < \alpha$ and $\ell(f(c)_*) > \alpha$ for any other simple closed geodesics $c \neq c_0$ on R.*

Proof. We take two adjacent pairs of pants P^j $(j = 1, 2)$ with three geodesic boundaries, c_0, c_1^j and c_2^j, so that the five geodesics are mutually disjoint. We fix j, and denote c_1 and c_2 instead of c_1^j and c_2^j respectively. Set $a_i = (1/2)\ell(c_i)$ and

$b_i = d_R(c_{i-1}, c_{i+1})$ for $i = 0, 1, 2$ (if $i = 0$, then we put $i - 1 = 2$, and if $i = 2$, then we put $i + 1 = 0$). Here $d_R(\cdot, \cdot)$ is the hyperbolic distance on R. Further, set

$$A_0 = \operatorname{arccosh}\left(\frac{1 + \cosh a_1 \cosh a_2}{\sinh a_1 \sinh a_2}\right)$$

and $\alpha = \min\{\epsilon/2, A_0, b_1, b_2\}$. We can take a quasiconformal map f on R so that $\ell(f(c_0)_*) < \alpha$ and the restriction of f to the complement of $P^1 \cup P^2$ is the identity. In particular, $f(c_i) = c_i$ for $i = 1, 2$, and $f(c) = c$ for any non-trivial simple closed curve c on R that is not through P^j $(j = 1, 2)$. Therefore, from the lower bound condition, $\ell(f(c)_*) > \epsilon$ for such c. Further, for any non-trivial simple closed curve c through P^j, we have $\ell(f(c)_*) > \alpha$.

Indeed, set $a'_i = (1/2)\ell(f(c_i)_*)$ and $b'_i = d_{f(R)}(f(c_{i-1})_*, f(c_{i+1})_*)$ for $i = 0, 1, 2$ (if $i = 0$, then we put $i-1 = 2$, and if $i = 2$, then we put $i+1 = 0$). Here $d_{f(R)}(\cdot, \cdot)$ is the hyperbolic distance on $f(R)$. Then $a'_i = a_i$ for $i = 1, 2$. For any non-trivial simple closed curve c through P^j, we have $\ell(f(c)_*) > b'_i$ for some $i = 0, 1, 2$. By the formula for a right-angled hexagon ([1, Theorem 7.19.2]), we have

$$\cosh b_i^{(\prime)} = \frac{\cosh a_i^{(\prime)} + \cosh a_{i-1}^{(\prime)} \cosh a_{i+1}^{(\prime)}}{\sinh a_{i-1}^{(\prime)} \sinh a_{i+1}^{(\prime)}}.$$

Hence $b'_0 > A_0$. Since $a'_0 < a_0$, we see that $b'_i > b_i$ for $i = 1, 2$. Then, for any non-trivial simple closed curve c through P^j, we have $\ell(f(c)_*) > \min\{A_0, b_1, b_2\}$. Hence $\ell(f(c)_*) > \alpha$. □

Proof of Theorem 3. Let c_0 be a simple closed geodesic on R, and let f be a quasiconformal map on R obtained by Lemma 5. Setting $S = f(R)$, we will show that the point $p = [S, f] \in T^{\#}(R)$ belongs to $\Omega(\mathrm{Mod}^{\#}(R))$. Suppose that $p \in \Lambda(\mathrm{Mod}^{\#}(R))$. Then there exist distinct elements $[h_n]$ in $\mathrm{Mod}^{\#}(R)$ such that $[h_n](p) \to p$ $(n \to \infty)$. Let g_n be an extremal quasiconformal automorphism of S in the homotopy class of $f \circ h_n^{-1} \circ f^{-1}$. Then $K(g_n) \to 1$. Letting α be a constant in Lemma 5, we may assume that $K(g_n) < \alpha/\ell(f(c_0)_*)$ for all n. Then the geodesic $g_n(f(c_0))_*$ satisfies $\ell(g_n(f(c_0))_*) \leq K(g_n)\ell(f(c_0)_*) < \alpha$. By Lemma 5, $f(c_0)_*$ is the only geodesic on S whose length is less than α. Thus $g_n(f(c_0))$ is homotopic to $f(c_0)$, which implies that $[h_n] \in \mathrm{Mod}^{\#}_{c_0}(R)$. However this contradicts Proposition 2. □

The following proposition gives examples of Riemann surfaces that satisfy the lower and upper bound conditions.

Proposition 3. *Let $\hat{R}$ be an analytically finite Riemann surface, and R a normal covering surface of $\hat{R}$ which is not a universal cover. Then R satisfies the lower and upper bound conditions.*

Proof. The lower bound condition is clearly satisfied. We set $\hat{R}_{\geq\epsilon} = \hat{R} - \hat{R}_\epsilon$, where $\hat{R}_\epsilon$ is the ϵ-thin part of $\hat{R}$. The lift R_0 of $\hat{R}_{\geq\epsilon}$ to R is connected and a homomorphism of $\pi_1(R_0)$ to $\pi_1(R)$ which is induced by the inclusion map of R_0 into R is surjective. We will show that $R_0 \subset R_M$ for some $M > 0$. Then the upper bound condition is satisfied for the constant M. Since R is a normal covering surface of $\hat{R}$ which is not a universal cover, we can take a simple closed geodesic $\hat{c}_*$ on $\hat{R}$ so that the lifts of $\hat{c}_*$ to R are closed geodesics. For an arbitrary point $p \in R_0$, let $\hat{p}$ be the projection of p. We connect $\hat{p}$ and $\hat{c}_*$ by the shortest geodesic $\hat{\ell}$. Since $\hat{R}_{\geq\epsilon}$ is compact, there exists a constant M_1 such that the hyperbolic length of $\hat{\ell}$ is less than M_1 for all

$\hat{p} \in \hat{R}_{\geq \epsilon}$. Hence there exists a non-trivial simple closed curve $\hat{c}_p$ through $\hat{p}$ whose hyperbolic length is less than $M = 2M_1 + M_2$, where M_2 is the hyperbolic length of $\hat{c}_*$. Considering the lift c_p of $\hat{c}_p$ which is through p, we conclude that $p \in R_M$. $\square$

By Theorem 3 and Proposition 3, we have the following.

Corollary 3. *Let $\hat{R}$ be an analytically finite Riemann surface, and R a normal covering surface of $\hat{R}$ which is not a universal cover. Then* $\mathrm{Mod}^{\#}(R)$ *is of the second kind.*

Example 4. Let R be a Riemann surface as in Example 3. Then the base point $[R, id]$ belongs to $\Lambda^1_\infty(\mathrm{Mod}^{\#}(R))$. On the other hand, $\mathrm{Mod}^{\#}(R)$ is of the second kind by Corollary 3. Thus both $\Lambda(\mathrm{Mod}^{\#}(R)) \neq \emptyset$ and $\Omega(\mathrm{Mod}^{\#}(R)) \neq \emptyset$ are satisfied.

We conjecture that the sufficient condition for $\mathrm{Mod}^{\#}(R)$ to be of the second kind can be weakened as follows. A partial solution will be given in the author's forthcoming paper.

Conjecture. *If R satisfies the lower bound condition, then* $\mathrm{Mod}^{\#}(R)$ *is of the second kind. That is, considering Theorem 2, we conjecture that* $\mathrm{Mod}^{\#}(R)$ *is of the second kind if and only if R satisfies the lower bound condition.*

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