A NEW CHARACTERIZATION OF THE UNIT BALL OF $H^2$

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Abstract. We derive a new expression for the norm of $H^2$ functions; we present some well-known results in a different setting.

Introduction

In 1915, Pick [3] proved the following result.

Theorem 1. Let $g$ be an analytic function on the unit disc $\Delta$ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in $\Delta$ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)g(z_l)}{1 - z_k z_l} \lambda_k \lambda_l \geq 0.$$  

Ahlfors [1], page 3, gives an elegant proof of this characterization of the unit ball of $H^\infty$.

In this note we shall present a characterization of the unit ball of $H^2$. Our main tool will be an explicit solution of the “minimal interpolation problem” for $H^2$ (see [2], page 141). As a byproduct we obtain a new proof of Pick’s theorem.

Description of the main result

Let $z_1, z_2, \ldots, z_n$ be a sequence in $\Delta$, and let $b$ be the Blaschke product generated by the sequence

$$b(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - z_j z}.$$  

We shall prove that the following conditions are equivalent for continuous functions $f$ on $\Delta$:

1) $f$ lies in the unit ball of $H^2$.
2) For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)\overline{f(z_l)}}{1 - z_k z_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.$$  

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Preliminaries

For mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ and for $w_1, w_2, \ldots, w_n$ in $\mathbb{C}$ we define

$$\Lambda = \{ f \in H^2 : f(z_j) = w_j, \ j = 1, 2, \ldots, n \}.$$ 

$\Lambda$ is not empty; it contains the Lagrange interpolation polynomial

$$\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k)b'(z_k)},$$

where $l(z) = \prod_{j=1}^{n} (z - z_j)$.

In the context of $H^p$ spaces it is more natural to work with the Blaschke interpolation function

$$\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \overline{w}_kz}{z - z_k} \frac{w_k}{b'(z_k)(1 - |z_k|^2)},$$

with $b(z)$ defined as in (2). Of course $\beta \in \Lambda$. However, for our purposes we are better off with

$$\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k)b'(z_k)}.$$ 

$\varphi \in \Lambda$, and $\varphi$ is analytic on some neighbourhood of $\overline{\Delta}$. $\Lambda$ is a hyperplane in $H^2$. With $\varphi$ and $b$ defined as in (4) and (2) we have

$$\Lambda = \{ \varphi + bg; \ g \in H^2 \}.$$ 

Theorem 2. $\varphi$ is the unique solution of the “minimal interpolation problem”, i.e., for every $f \in \Lambda \setminus \{\varphi\}$ we have $\|f\|_2 > \|\varphi\|_2$.

Proof. It suffices to show that $\varphi \perp (f - \varphi)$ for every $f \in \Lambda$ (since under those circumstances $\|f\|^2 = \|\varphi\|^2 + \|f - \varphi\|^2$).

From the decomposition $f = \varphi + bg$ we have

$$\langle f - \varphi, \varphi \rangle = \langle bg, \varphi \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it})g(e^{it})\overline{\varphi(e^{it})}dt = \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it})g(e^{it}) \sum_{k=1}^{n} \frac{w_k}{(e^{-it} - \overline{w}_k)(1 - \overline{w}_k)^2} \frac{1}{b'(z_k)}dt.$$ 

Note that $|b(e^{it})|^2 = 1$. Thus,

$$\langle f - \varphi, \varphi \rangle = \sum_{k=1}^{n} \frac{w_k}{2\pi b'(z_k)} \frac{1}{2\pi} \int_{0}^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it}\overline{w}_k} dt = \sum_{k=1}^{n} \frac{w_k}{b'(z_k)} \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{1 - \overline{w}_kz} dz = 0,$$

because the integrand is analytic on $\Delta$. 

It will be convenient to have an explicit expression for $\| \varphi \|_2$:

$$
\| \varphi \|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \frac{1}{2\pi} \sum_{k=1}^n \sum_{l=1}^n \frac{w_k w_l}{b'(z_k) b'(z_l)} \int_0^{2\pi} \frac{dt}{(e^{it} - z_k)(e^{-it} - \overline{z_l})}
$$

$$
= \frac{1}{2\pi} \sum_{k=1}^n \sum_{l=1}^n \frac{w_k w_l}{b'(z_k) b'(z_l)} \int_{\Gamma} \frac{dz}{(z - z_k)(1 - \overline{z_l} z)}
$$

$$
= \sum_{k=1}^n \sum_{l=1}^n \frac{w_k w_l}{1 - z_k \overline{z_l}} \frac{1}{b'(z_k) b'(z_l)}.
$$

There are, of course, many other expressions for $\| \varphi \|_2$.

**Theorem 3.**

$$
\| \varphi \|_2 = \max \left\{ \sum_{k=1}^n \frac{w_k f(z_k)}{b'(z_k)} \mid f \in H^2, \| f \|_2 \leq 1 \right\}.
$$

**Proof.**

$$
\sum_{k=1}^n \frac{w_k f(z_k)}{b'(z_k)} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z) \varphi(z)}{b(z)} dz;
$$

hence, by Schwarz’s inequality we have

$$
\left| \sum_{k=1}^n \frac{w_k f(z_k)}{b'(z_k)} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})| \cdot |\varphi(e^{it})| dt \leq \| f \|_2 \cdot \| \varphi \|_2 \leq \| \varphi \|_2.
$$

Equality holds for the function $f : z \to \frac{1}{\| \varphi \|_2} \sum_{k=1}^n \frac{w_k}{(1 - z_k \overline{z_l}) b'(z_k)}$.

An immediate result from Theorem 2 is

**Corollary.** For every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points of $\Delta$ we have

$$
\sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k) b'(z_l)} \leq 1.
$$

**Proof.** Take $w_1 = w_2 = \ldots = w_n = 1$. Then $1 \in \Lambda$ and since $\| 1 \|_2 = 1$,

we have

$$
1 \geq \| \varphi \|_2^2 = \frac{\sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}}}{b'(z_k) b'(z_l)} \cdot \frac{1}{b'(z_k) b'(z_l)}.
$$

The equality sign certainly occurs if $0 \in \{z_1, z_2, \ldots, z_n\}$:

$$
1 = \varphi(0)^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \| \varphi \|_2^2 = \sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k) b'(z_l)}.
$$

If $0 \not\in \{z_1, z_2, \ldots, z_n\}$, there is strict inequality.

Because of the uniqueness of $\varphi$ there can be equality only if

$$
b(z) \sum_{k=1}^n \frac{1}{(z - z_k) b'(z_k)} = 1.
$$
In this identity for rational functions we let \( z \to \infty \). Since \( z_j \neq 0, \lim_{z \to \infty} b(z) \) has a finite value. Therefore, the left-hand side has limit zero.

**Remark.** The corollary shows that a function satisfying (1) also satisfies (3).

The fact that \( \varphi \in \Lambda \) has an interesting reformulation. We start with a lemma.

**Lemma 1.** The partial fraction decomposition of \( \varphi \) is

\[
\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \bar{z}_l z)(1 - \bar{z}_l z_k)b'(z_k)b'(z_l)}.
\]

**Proof.** An elegant way to prove this is to compute both sides of the following identity.

For \( z \in \Delta \) we have

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta) - \bar{\varphi}(\zeta)}{\zeta - z} \, dz = \varphi(z),
\]

while the right-hand side is equal to the complex conjugate of

\[
\frac{1}{2\pi i} \int_{\Gamma} b(\zeta) \sum_{k=1}^{n} \frac{\bar{w}_k}{(\zeta - z_k)b'(z_k)} \cdot \frac{1}{1 - \zeta \zeta_k} \cdot \frac{d\zeta}{\zeta},
\]

i.e., to the complex conjugate of

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{b(\zeta)} \sum_{k=1}^{n} \frac{\bar{\varphi}(\zeta)}{(1 - \bar{z}_l \zeta)(1 - \bar{z}_l z_k)b'(z_k)b'(z_l)} \cdot \frac{d\zeta}{1 - \zeta \zeta_l}.
\]

Calculation of the residues at the points \( z_1, z_2, \ldots, z_n \) lead to (5).

The condition \( \varphi \in \Lambda \) implies that \( \varphi(z_j) = w_j, j = 1, \ldots, n \), i.e.,

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \bar{z}_l z_j)(1 - \bar{z}_l z_k)b'(z_k)b'(z_l)} = w_j.
\]

This is equivalent to the assertion that the matrices

\[
B = (\beta_{lk})
\]

and its conjugate \( \overline{B} = (\overline{\beta}_{lk}) \) where

\[
\beta_{lk} = \frac{1}{(1 - \bar{z}_l z_k)b'(z_k)}
\]

are inverses of each other, i.e., \( B \) and \( \overline{B} \) are unitary.
Proof of the main result

**Theorem 4.** Let $f$ be a continuous function on the unit disc in the complex plane. Then the following conditions are equivalent:

1. $f$ is analytic and $f$ lies in the unit ball of $H^2$.
2. For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

$$
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
$$

**Proof.** We split up the proof into two lemmas.

**Lemma 2.** Let $f$ belong to the unit ball of $H^2$, and let a sequence of mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ be given. Then (3) holds.

**Proof.** Define $w_j = f(z_j)$. $f$ lies in the hyperplane $\Lambda$ and the element $\varphi$ of $\Lambda$ with minimal norm satisfies

$$
\|\varphi\|_2 \leq \|f\|_2 \leq 1.
$$

Use of the explicit expression for $\|\varphi\|_2$ leads to (3).

**Lemma 3.** Let $f$ be continuous and assume that $f$ satisfies (3). Then $f$ is analytic and $f$ lies in the unit ball of $H^2$.

**Proof.** We apply (3) for the case $n = 1$; an easy computation shows that

$$
|f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}}
$$

for every choice of $z \in \Delta$.

Let $0 < r < \rho < 1$, and let $z_1, z_2, z_3, \ldots$ be an enumeration of the rational points of $\Delta_\rho$. For every $n$ there is a function $\varphi_n$ with

$$
\varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n,
$$

and

$$
\|\varphi_n\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
$$

Thus, $\varphi_n$ lies in the unit ball of $H^2$, and so by Lemma 2 we have for every sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$ in $\Delta$

$$
\sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\varphi_n(\zeta_k)\overline{\varphi_n(\zeta_l)}}{1 - \zeta_k \overline{\zeta_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
$$

It follows from (6) that

$$
|\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1 - |\zeta|^2}}
$$

hence the sequence $\varphi_1, \varphi_2, \ldots$ is uniformly bounded on $\Delta_\rho$. Therefore, it contains a locally uniformly convergent subsequence $\varphi_{n_j}$. At the points $z_1, z_2, \ldots$ the subsequence converges to $f$. By the continuity of $f$ and the fact that $\{z_1, z_2, \ldots\}$ is dense in $\Delta_\rho$ we see that

$$
\lim_{n_j \to \infty} \varphi_{n_j} = f.
$$
This shows that $f$ is analytic on $\Delta_\rho$ for all $\rho < 1$. Because of uniform convergence on $\Gamma_r$, we have
\[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{n_j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \leq 1. \]
Thus, $f \in H^2$ and $\|f\|_2^2 \leq 1$.
Lemma 2 and Lemma 3 together constitute a proof of the theorem.

**Corollary.** For $f \in H^2$ we define
\[ \nu(f) = \sup \left\{ \sum_{k=1}^n \sum_{l=1}^n \frac{f(z_k)\overline{f(z_l)}}{1 - z_k\overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} : z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta \right\}. \]
Then $\nu(f) = \|f\|_2^2$.

**Proof.** Assume that $\nu(f) = 1$. Then by Lemma 3 $\|f\|_2^2 \leq 1$. If $\|f\|_2^2 < \lambda^2 < 1$ for some $\lambda$, then we have $\left\| \frac{1}{\lambda^2} f \right\| < 1$ but $\nu \left( \frac{1}{\lambda^2} f \right) > 1$ which is impossible by Lemma 2.
In a similar way we can show that $\|f\|_2^2 = 1$ implies that $\nu(f) = 1$. By the homogeneity of $\nu$ and $\| \cdot \|_2$ it follows that for all $f \in H^2$: $\nu(f) = \|f\|_2^2$.

**Pick’s theorem**

As an application of our results we shall give a proof of Pick’s theorem.
Let $g$ belong to the unit ball of $H^\infty$, and let $z_1, z_2, \ldots, z_n$ be a sequence of mutually distinct points in $\Delta$. Let $w_1, w_2, \ldots, w_n$ be an arbitrary sequence of complex numbers. We consider the hyperplanes $\Lambda$ and $\Lambda_g$ where
\[ \Lambda_g = \{ f \in H^2 : f(z_j) = w_j g(z_j), j = 1, 2, \ldots, n \}. \]
Of course, if $f \in \Delta$, then $g \cdot f \in \Delta_g$, and by Theorem 2 applied to $\Lambda_g$ we have
\[ \|gf\|_2^2 \geq \sum_{k=1}^n \sum_{l=1}^n \frac{w_k g(z_k) w_l g(z_l)}{1 - z_k\overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}. \]
Let $\varphi$ be, as before, the element of $\Lambda$ with smallest norm. From $\|g\|_\infty \leq 1$ we obtain
\[ \|g\varphi\|_2 \leq \|\varphi\|_2. \]
Combining these steps leads to
\[ \sum_{k=1}^n \sum_{l=1}^n \frac{1 - g(z_k)g(z_l)}{1 - z_k\overline{z_l}} \cdot \frac{w_k \overline{w_l}}{b'(z_k)b'(z_l)} \geq 0, \]
i.e.,
\[ \sum_{k=1}^n \sum_{l=1}^n \frac{1 - g(z_k)g(z_l)}{1 - z_k\overline{z_l}} \cdot \frac{w_k \overline{w_l}}{b'(z_k)b'(z_l)} \geq 0, \]
and since the sequence \( w_1, w_2, \ldots, w_n \) is arbitrary, we have for all choices of \( \lambda_1, \lambda_2, \ldots, \lambda_n \),

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k \overline{z_l}} \cdot \lambda_k \overline{\lambda_l} \geq 0.
\]

By the choice \( n = 1, \lambda_1 = 1 \) we see that the converse is trivial.

References


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