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INFIMUM PRINCIPLE

WŁADYSŁAW KULPA AND ANDRZEJ SZYMANSKI

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ABSTRACT. We utilize the technique of dual sets to prove a theorem on the attainment of a simultaneous infimum by a compatible family of functions. Corollaries to the theorem include, among others, the von Neumann Minimax Principle and Nash's Equilibrium Theorem.

1. INTRODUCTION

J. von Neumann's Minimax Principle (cf. [12]) is of fundamental significance in game theory. Over the years, it has found a multitude of applications in (mathematical) economics, linear programing, sociology, and other sciences. It has also been an object of intense research.

In 1950, J. Nash ([6], [7]) proved one of the most celebrated theorems of game theory. It was the further applications of the equilibrium theorem in economics that showered several awards on John Nash, including the Nobel Prize in 1994 (cf. [1], [2]).

In the second half of 1950, D. Gale (cf. [3]) and H. Nikaido (cf. [9]) obtained a theorem that was instrumental in proving the existence of a competitive equilibrium for excess supply functions for the workability of decentralized economies (in the Walras sense) (cf. [8]).

It seems rather unlikely that these three great classical theorems may have anything in common. It is then an unexpected discovery that in fact they are all closely related. For, in our paper, we offer a topological theorem that has the three aforementioned theorems (along with yet another one - Kakutani's Fixed Point Theorem) as its particular instances (see Corollaries 3, 4, 1, and 2). In another similar development, H. Reitberger [10] showed that the Nash equilibrium theorem, von Neumann minimax theorem, and Kakutani fixed point theorem follow from S. Smale's generalization of the Vietoris mapping theorem.

Our main theorem, the Infimum Principle (see Theorem 3), is fairly general so that it enables us to derive not only some classical theorems but also stimulates new research. In the paper, we prove a theorem (see Corollary 2) that can be seen as an n-dimensional version of the Minimax Principle.

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2. Dual families

Definition 1.[1] For given set X, let $\mathcal{F} = \{F(x) : x \in X\}$ be a family of non-empty subsets of X indexed by the elements of the set X. Such a family gives rise to a dual family, $\mathcal{F}'$, of subsets of the set X defined as follows. For a given $y \in X$, let $F'(y) = \{x \in X : y \in F(x)\}$. We set

$$\mathcal{F}' = \{F'(y) : y \in X\}.$$

We have the following dichotomy:

$$y \in F(x) \text{ if and only if } x \in F'(y). \tag{2.1}$$

Consequently,

Lemma 1. (i) *If the family $\mathcal{F}$ consists of non-empty sets, then the dual family $\mathcal{F}'$ is a covering of X.* (ii) *If the family $\mathcal{F}$ is a covering of Y, then the dual family $\mathcal{F}'$ consists of non-empty subsets of X.* (iii) $(\mathcal{F}')' = \mathcal{F}$. (iv) *Let $\mathcal{F}_1 = \{F_1(x) : x \in X\}$, $\mathcal{F}_2 = \{F_2(x) : x \in X\}$ and $\mathcal{F} = \{F_1(x) \cap F_2(x) : x \in X\}$. Then $\mathcal{F}' = \{F_1^{'}(y) \cap F_2^{'}(y) : y \in Y\}$.*

In the next two theorems we introduce tools for proving our main results. The first one is a theorem on indexed families (see [5]).

Theorem 1 (Theorem on Indexed Families)**.** *Let $y_0, \ldots, y_m$ be points of a linear space and let the sets $U_0, \ldots, U_m$ form an open covering of the convex hull, $conv\{y_0, \ldots, y_m\}$, of the points $y_0, \ldots, y_m$. Then there exists a non-empty set of indices $\{i_o, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ such that $conv\{y_{i_0}, \ldots, y_{i_k}\} \cap U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$.*

The second one is a fixed point type theorem for families of sets.

Theorem 2. *Let $\mathcal{F} = \{F(x) : x \in X\}$ be a family of non-empty convex subsets of a compact convex subspace X of a linear space such that $F'(y)$ is an open subset of X for each $y \in X$. Then there exists a point $a \in X$ such that*

$$a \in F(a).$$

Proof. Since X is compact, there are points $y_0, y_1, ..., y_m \in Y$ such that

$$X \subseteq F'(y_0) \cup F'(y_1) \cup ... \cup F'(y_m).$$

From Theorem 1 there is a point $a \in conv\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \cap F'(y_{i_0}) \cap F'(y_{i_1}) \cap ... \cap F'(y_{i_k})$. It follows from (2.1) that $\{y_{i_0}, y_{i_1}, ...y_{i_k}\} \subseteq F(a)$. Since $F(a)$ is convex, $conv\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq F(a)$. Since $a \in conv\{y_{i_0}, y_{i_1}, ..., y_{i_k}\}$, $a \in F(a)$. $\square$

Definition 2. Let X be a linear space. A real function $f : X \to \mathbf{R}$ is said to be quasi-convex if $f^{-1}((-\infty, z))$ is convex for each $z \in \mathbf{R}$.

Definition 3. Let $f : X \times Y \to Z$ be a map defined on the product of two sets. If $x \in X$ and $y \in Y$, then f_x and f^y denote the partial maps of the map f, i.e.,

$$f_x(y) = f(x, y) = f^y(x).$$

[1] The definition of families $\{F(x) : x \in X\}$ and their duals can also be given in terms of set-valued mappings $F : X \to 2^Y$ or in terms of subsets of the product $X \times Y$. For instance, if F was considered as a map $F : X \to 2^Y$, then $F' : Y \to 2^X$ and F' would be a kind of inverse map to F. For our exposition, we prefer that both F and F' be regarded as families of sets.

Definition 4. For each $i = 1, 2, ..., n$, let $g_i : X \times Y \to Z$.

We say that the family of functions $\{g_i : i = 1, 2, ..., n\}$ is n-compatible with respect to the first variable if for each $y \in Y$ and for each $x_1, x_2, ..., x_n \in X$ there exists $x_0 \in X$ such that $g_i(x_i, y) = g_i(x_0, y)$ for each $i = 1, 2, ..., n$.

We say that the family of functions $\{g_i : i = 1, 2, ..., n\}$ is n-compatible with respect to the second variable if for each $x \in X$ and for each $y_1, y_2, ..., y_n \in Y$ there exists $y_0 \in Y$ such that $g_i(x, y_i) = g_i(x, y_0)$ for each $i = 1, 2, ..., n$.

Theorem 3 (The Infimum Principle)**.** *Let X and Y be compact and convex subspaces of linear spaces. Let continuous functions $g_i : X \times Y \to \mathbf{R}$, $i = 1, 2, ..., n + 1, ..., n + m$, satisfy conditions:*

(gn1) *The partial function $g_i^y : X \to \mathbf{R}$ is quasi-convex for each $y \in Y$ and $i = 1, 2, ..., n$. The partial function $g_{ix} : Y \to \mathbf{R}$ is quasi-convex for each $x \in X$ and $i = n + 1, n + 2, ..., n + m$.*

(gn2) *The family of functions $\{g_i : i = 1, 2, ..., n\}$ is n-compatible with respect to the first variable. The family of functions $\{g_i : i = n + 1, n + 2, ..., n + m\}$ is m-compatible with respect to the second variable.*

Then there exists a point $(a, b) \in X \times Y$ such that

$$g_i(a, b) = \inf_{x \in X} g_i(x, b) \quad \text{for each } i = 1, 2, ..., n, \text{ and} \tag{2.2}$$

$$g_{n+j}(a, b) = \inf_{y \in Y} g_{n+j}(a, y) \quad \text{for each } j = 1, 2, ..., m. \tag{2.3}$$

Proof. For each $i,\ i = 1, 2, ..., n$, let $\overline{g_i}$ be a function given by

$$\overline{g_i}(y) = \inf_{x \in X} g_i(x, y).$$

For each $j = 1, 2, ..., m$, let $\overline{g_{n+j}}$ be a function given by

$$\overline{g_{n+j}}(x) = \inf_{y \in Y} g_{n+j}(x, y).$$

By continuity of the functions g_i, the functions $\overline{g_i},\ i = 1, \ldots, n+m$, are continuous. Let $\varepsilon > 0$ be given. For each $y \in Y$ and for each $i = 1, ..., n$, the set $A_i(y)$ is given by

$$A_i(y) = \{x \in X : \overline{g_i}(y) + \varepsilon > g_i(x, y)\}.$$

By (gn1), $A_i(y)$ is non-empty and convex for each $y \in Y$ and $i = 1, ..., n$.

For each $x \in X$ and for each $j = 1, ..., m$, the set $B_j(x)$ is given by

$$B_j(x) = \{y \in Y : \overline{g_{n+j}}(x) + \varepsilon > g_{n+j}(x, y)\}.$$

By (gn1), $B_j(x)$ is non-empty and convex for each $x \in X$ and $j = 1, ..., m$.

By continuity of g_i's, the dual sets

$$A'_i(x) = \{y \in Y : \overline{g_i}(y) + \varepsilon > g_i(x, y)\} \tag{2.4}$$

and

$$B'_j(y) = \{x \in X : \overline{g_{n+j}}(x) + \varepsilon > g_{n+j}(x, y)\} \tag{2.5}$$

are open for each $x \in X$, $y \in Y$, $i = 1, \ldots, n$, and $j = 1, 2, ..., m$.

For each $y \in Y$, $A(y)$ is given by $A(y) = \bigcap\{A_i(y) : i = 1, 2, ..., n\}$. For each $x \in X$, $B(x)$ is given by $B(x) = \bigcap\{B_j(x) : j = 1, 2, ..., m\}$. We shall show that each of the sets $A(y)$ as well as $B(x)$ is non-empty.

Appealing to the definition of $\overline{g_i}(y)$, for each $i = 1, 2, ..., n$, there exists $x_i \in X$ such that $\overline{g_i}(y) + \varepsilon > g_i(x_i, y)$. Since the family $\{g_i : i = 1, 2, ..., n\}$ is n-compatible with respect to the first variable, there exists $x_0 \in X$ such that $g_i(x_i, y) = g_i(x_0, y)$ for each $i = 1, 2, ..., n$. Hence $\overline{g_i}(y) + \varepsilon > g_i(x_0, y)$ for each $i = 1, 2, ..., n$, which means that $x_0 \in \bigcap\{A_i(y) : i = 1, 2, ..., n\} = A(y)$.

For similar reasons the set $B(x)$ is non-empty for each $x \in X$.

Let $F(x, y) = A(y) \times B(x)$. Then $F(x, y)$ is a non-empty convex subset of the product of linear spaces for each $(x, y) \in X \times Y$. We have

$$\begin{aligned} F'(u, v) &= \{(x, y) \in X \times Y : (u, v) \in F(x, y)\} \\ &= \{(x, y) \in X \times Y : (u, v) \in A(y) \times B(x)\} \\ &= \{(x, y) \in X \times Y : (x, y) \in B'(v) \times A'(u)\} = B'(v) \times A'(u). \end{aligned}$$

By Lemma 1(iv) and (2.4), (2.5), the dual sets $F'(u, v)$ are open for each $(u, v) \in X \times Y$.

Theorem 2 applied to the product of compact and convex spaces yields a point $(a_\varepsilon, b_\varepsilon) \in X \times Y$ such that $(a_\varepsilon, b_\varepsilon) \in F(a_\varepsilon, b_\varepsilon) = \bigcap\{A_i(b_\varepsilon) : i = 1, 2, ..., n\} \times \bigcap\{B_j(a_\varepsilon) : j = 1, 2, ..., m\}$. Hence, for each $i = 1, ..., n$, $\overline{g_i}(b_\varepsilon) + \varepsilon > g_i(a_\varepsilon, b_\varepsilon)$, and, for each $j = 1, 2, ..., m$, $\overline{g_{n+j}}(a_\varepsilon) + \varepsilon > g_{n+j}(a_\varepsilon, b_\varepsilon)$.

For a given $\varepsilon > 0$, we set

$$K(\varepsilon) = \left\{ \begin{array}{c} (x, y) \in X \times Y : \forall_{i=1,\dots,n} \forall_{j=1,\dots,m} \;\; g_i(x, y) - \overline{g_i}(y) \le \varepsilon \\ \text{and } g_{n+j}(x, y) - \overline{g_{n+j}}(x) \le \varepsilon \end{array} \right\}.$$

We just showed that the sets $K(\varepsilon)$ are non-empty for each $\varepsilon > 0$, and since each of the functions g_i and $\overline{g_i}$ is continuous, the sets $K(\varepsilon)$ are also closed. By compactness of the product space $X \times Y$, there exists a point $(a, b) \in X \times Y$ such that $(a, b) \in K(\varepsilon)$ for each $\varepsilon > 0$. This is only possible if the point (a, b) is as required. $\square$

Remark 1. Condition (gn1) in versions of Theorem 3 when $n = 0, 1$ and/or $m = 0, 1$ is superfluous.

Remark 2. If the version of our main Theorem 3 with conclusions (2.2) and (2.3) was called the inf/inf theorem, then one could have three more versions of the theorem specified as: (1) sup/inf, (2) inf/sup, and (3) sup/sup. To state and to prove one of the three new versions, one would have to replace quasi-convexity by quasi-concavity (cf. Definition 5) in part (gn1) that pertains to supremum and make the appropriate adjustment in the original proof. This is because of the following trivial fact: $\inf(-g) = -\sup g$. One can also observe that compatibility (with respect to any of the two variables) of a family $\{g_k : k = 1, ..., l\}$ remains in place for the family $\{-g_k : k = 1, ..., l\}$. Since we do not need any of the other three versions of our main Theorem 3 in the forthcoming study, we decided not to pursue this venue.

3. Consequences

We are going to derive four important results as easy corollaries from the Infimum Principle. They will be proved for normed linear spaces despite the feasibility of obtaining substantially more general statements.

Definition 5. A real function $f : X \to \mathbf{R}$ is said to be quasi-concave if $f^{-1}((z, \infty))$ is convex for each $z \in \mathbf{R}$.

Corollary 1 (Nash Equilibrium Theorem)**.** *Let $X_1, \ldots, X_n$ be non-empty compact convex subsets of normed spaces and let $X = X_1 \times \ldots \times X_n$ be their product. For points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of X, the point $x \overset{i}{\cdot} y$ is given by*

$$x \overset{i}{\cdot} y = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, ..., x_n).$$

Suppose that $f_1, \ldots, f_n$ are continuous real functions defined on the space X and let $h_i : X \times X \to \mathbf{R}$, $i = 1, 2, ..., n$, be given by

$$h_i(x, y) = f_i\left(y \overset{i}{\cdot} x\right).$$

If the partial function h_{ix} is quasi-concave for each $x \in X$ and $i = 1, 2, ..., n$, then there is a point $a \in X$ such that

$$f_i(a) = \sup\left\{f_i\left(a \overset{i}{\cdot} x\right) : x \in X\right\} \text{ for each } i = 1, 2, ..., n.$$

Proof. Let us begin by verifying that the family of functions $\{h_i(x, y) \colon i = 1, 2, ..., n\}$ is *n-compatible* with respect to the first variable.

Take n points $x_i = (x_1^i, \ldots, x_n^i)$, $i = 1, 2, ..., n$, from the space X. Then consider the diagonal point $x_0 = (x_1^1, x_2^2, \ldots, x_n^n)$. It is obvious that

$$h_i\left(x_i, y\right) = f_i\left(y \overset{i}{\cdot} x_i\right) = f_i\left(y \overset{i}{\cdot} x_0\right) = h_i\left(x_0, y\right) \text{ for each } y \in X \text{ and } i = 1, 2, ..., n.$$

Let the function g_i, $i = 1, 2, ..., n$, be given by

$$g_i(x, y) = -h_i(x, y),$$

and let g_{n+1} be given by

$$g_{n+1}(x, y) = ||x - y||.$$

Theorem 3 applied to the functions g_i, $i = 1, 2, ..., n$, and g_{n+1} yields a point $(a, b) \in X \times X$ such that

$$g_i(a, b) = \inf_{x \in X} g_i(x, b) \text{ for each } i = 1, 2, ..., n, \text{ and } g_{n+1}(a, b) = \inf_{y \in Y} g_{n+1}(a, y).$$

Hence

$$h_i(a, b) = \sup_{x \in X} h_i(x, b) \text{ for each } i = 1, 2, ..., n, \text{ and } \|a - b\| = \inf_{y \in Y} \|a - y\| = 0.$$

Thus $a = b$ and since $a \overset{i}{\cdot} a = a$,

$$f_i(a) = \sup\left\{f_i\left(a \overset{i}{\cdot} x\right) : x \in X\right\} \text{ for each } i = 1, 2, ..., n.$$

□

Corollary 2 (n-Minimax Principle)**.** *Let X and Y be compact and convex subsets of normed spaces and let $h_i : X \times Y \to \mathbf{R}$, $i = 1, 2, ..., n$, be a continuous function. Suppose further that the partial function h_i^y is quasi-concave for each $y \in Y$ and for each $i = 1, 2, ..., n$, and that the partial function h_{ix} is quasi-convex for each $x \in X$ and $i = 1, 2, ..., n$. If the family $\{h_i : i = 1, 2, ..., n\}$ is* n-compatible *with respect to the first and with respect to the second variable, then there exists a point $(a, b) \in X \times Y$ such that*

$$\max_{x \in X} \min_{y \in Y} h_i(x, y) = \min_{y \in Y} \max_{x \in X} h_i(x, y) = h_i(a, b) \text{ for each } i = 1, 2, ..., n.$$

Proof. The following inequality holds true for arbitrary function. In particular for every function h_i,

$$\sup_{x\in X}\inf_{y\in Y} h_i(x,y) \;\le\; \inf_{y\in Y}\sup_{x\in X} h_i(x,y). \tag{3.1}$$

Let the function g_i be given by

$$g_i(x,y)=\begin{cases} -h_i(x,y) & \text{if } i=1,2,...,n,\\ h_i(x,y) \text{ if } i=n+1,n+2,...,2n. \end{cases}$$

We can apply our Theorem 3 to the functions $g_i,\ i=1,2,...,2n$. There exists a point $(a,b)\in X\times Y$ such that

$$g_i(a,b)=\inf_{x\in X} g_i(x,b) \quad \text{for each } i=1,2,...,n,$$

and

$$g_{n+j}(a,b)=\inf_{y\in Y} g_{n+j}(a,y) \quad \text{for each } j=1,2,...,n.$$

Hence

$$-h_i(a,b)=-\sup_{x\in X} h_i(x,b) \quad \text{and} \quad h_i(a,b)=\inf_{y\in Y} h_i(a,y). \tag{3.2}$$

Incorporating (3.2) into (3.1) we get

$$\begin{aligned} h_i(a,b)=\inf_{y\in Y} h_i(a,y) &\le \sup_{x\in X}\inf_{y\in Y} h_i(x,y) \;\le\; \inf_{y\in Y}\sup_{x\in X} h_i(x,y)\\ &\le \sup_{x\in X} h_i(x,b)=h_i(a,b). \end{aligned}$$

Because of compactness of X and Y and continuity of the functions h_i, both $\max\limits_{x\in X}\min\limits_{y\in Y} h_i(x,y)$ and $\min\limits_{y\in Y}\max\limits_{x\in X} h_i(x,y)$ exist. Hence

$$\max_{x\in X}\min_{y\in Y} h_i(x,y)=\min_{y\in Y}\max_{x\in X} h_i(x,y)=h_i(a,b) \quad \text{for each } i=1,2,...,n.$$

□

Definition 6. Let X and Y be topological spaces and let $T: X\to 2^Y$ be a set valued map. The map T is said to be upper semicontinuous if $T^{-1}(V)=\{x\in X : T(x)\subseteq V\}$ is an open set in X provided that V is an open set in Y.

Theorem 4. *Let X and Y be compact convex subspaces of a normed space. Suppose that $g: X\times Y\to \mathbf{R}$ and $T: X\to 2^Y$ satisfy the following conditions:*

(kk1) *g is continuous and the partial function g^y is quasi-convex for each $y\in Y$;*
(kk2) *T is upper semicontinuous and $T(x)$ is a non-empty closed convex subset of Y for each $x\in X$.*

Then there exists a point $(a,b)\in X\times Y$ such that

$$g(a,b)=\inf_{x\in X} g(x,b) \quad \textit{and} \quad b\in T(a).$$

Proof. (I). First assume that $T: X\to Y$ is a continuous map. Let g_1 and g_2 be given by

$$g_1(x,y)=g(x,y) \quad \text{and} \quad g_2(x,y)=||y-T(x)||.$$

The functions $g_1, g_2 : X\times Y\to \mathbf{R}$ are continuous. Since any open ball in a normed space is convex, the partial functions g_{2x} and g_1^y (by assumption) are quasi-convex for each $x \in X$ and $y \in Y$. As 1-*compatibility* always holds, we can apply our Theorem 3 (in the case $n = 1 = m$). There is a point $(a, b) \in X \times Y$ such that

$$g(a, b) = g_1(a, b) = \inf_{x \in X} g(x, b) \text{ and } ||b - T(a)|| = \inf_{y \in Y} ||y - T(a)|| = 0.$$

Hence $b = T(a)$ and $g(a, b) = \inf_{x \in X} g(x, b)$.

(II). Now, let $T : X \to 2^Y$ be a set-valued map satisfying the assumptions (kk2). From part (I) and Lemma 11.2 of [4], there are sequences $\{a_n\}$, $\{a'_n\}$, $\{b_n\}$, $\{b'_n\}$ of points of X and there are continuous maps $T_n : X \to Y$, $n = 1, 2, \ldots$, such that $||a_n - a'_n|| + ||b_n - b'_n|| < \frac{1}{n}$ and

$$g(a_n, b_n) = \overline{g}(b_n) = \inf_{x \in X} g(x, b_n), \; b_n = T_n(a_n), \text{ and } b'_n \in T(a'_n). \tag{3.3}$$

Since X and Y are compact we may assume that $a_n \to a$ and $b_n \to b$, and, in consequence, $a'_n \to a$ and $b'_n \to b$. From continuity of g and $\overline{g}$ we infer that $g(a, b) = \overline{g}(b) = \inf_{x \in X} g(x, b)$, and by upper semicontinuity of T, $b \in T(a)$. □

Corollary 3 (Kakutani's Fixed Point Theorem)**.** *Let $T : X \to 2^X$ be an upper semicontinuous map from a compact convex subspace X of a normed space such that $T(x)$ is a non-empty closed and convex subset of X for each $x \in X$. Then the map T has a fixed point, i.e., there exists a point $a \in X$ such that $a \in T(a)$.*

Proof. Apply Theorem 4 to $X = Y$, the given set-valued map T, and g_1 given by

$$g_1(x, y) = ||x - y||.$$

Clearly, g_1^y satisfies the condition (kk1). There are points $a, b \in X$ such that

$$\|a - b\| = g_1(a, b) = \inf\{||x - b|| : x \in X\} = 0 \text{ and } b \in T(a).$$

Hence $a = b$, so $a \in T(a)$. □

Corollary 4 (Gale-Nikaido Theorem)**.** *Let $T : \Delta_n \to 2^C$ be an upper semicontinuous map from the n-dimensional standard simplex Δ_n such that $T(x)$ is a non-empty closed and convex subset of a compact convex set $C \subseteq \mathbf{R}^n$. Suppose further, the Walras law in the general sense holds:*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i \geq 0 \text{ for each } x \in \Delta_n \text{ and } y \in T(x). \tag{3.4}$$

Then there exists $a \in \Delta_n$ and $b \in T(a)$ such that $b_i \geq 0$ for each $i = 1, \ldots, n$.

Proof. Apply Theorem 4 to $X = \Delta_n$, $Y = C$, the given set-valued map T, and g_1 given by $g_1(x, y) = \langle x, y\rangle$. Since g_1^y is a linear map restricted to the simplex Δ_n, the condition (kk1) holds. There is a point $(a, b) \in \Delta_n \times C$ such that

$$\langle a, b\rangle = g_1(a, b) = \inf\{\langle x, b\rangle : x \in \Delta_n\} \text{ and } b \in T(a). \tag{3.5}$$

By (3.4), $\langle a, b\rangle \geq 0$. By (3.5), $0 \leq \langle a, b\rangle \leq \langle x, b\rangle$ for each $x \in \Delta_n$. Since $e_i \in \Delta_n$, $0 \leq \langle e_i, b\rangle = b_i$ for each $i = 1, 2, ..., n$. □

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Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland

E-mail address: `kulpa@ux2.math.us.edu.pl`

Department of Mathematics, Slippery Rock University, Slippery Rock, Pennsylvania 16057

E-mail address: `andrzej.szymanski@sru.edu`