

## SOME REMARKS ON TOTALLY IMPERFECT SETS

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ABSTRACT. We prove the following two theorems.

**Theorem 1.** *Let  $X$  be a strongly meager subset of  $2^{\omega \times \omega}$ . Then it is dual Ramsey null.*

We will say that a  $\sigma$ -ideal  $\mathcal{I}$  of subsets of  $2^\omega$  satisfies the condition  $(\ddagger)$  iff for every  $X \subseteq 2^\omega$ , if

$$\forall f \in \omega^{\uparrow \omega} \{g \in \omega^{\uparrow \omega} : \neg(f \prec g)\} \cap X \in \mathcal{I},$$

then  $X \in \mathcal{I}$ .

**Theorem 2.** *The  $\sigma$ -ideals of perfectly meager sets, universally meager sets and perfectly meager sets in the transitive sense satisfy the condition  $(\ddagger)$ .*

### 0. PRELIMINARIES

In this paper we continue our investigation of totally imperfect subsets of  $2^\omega$  (see [N], [NW1], and [NW2] for related results).

We divide it into two parts, the first one is devoted to some properties of strongly meager sets and the second one deals with combinatorial properties of perfectly meager sets.

We use standard set theoretic notation. We recall here the main notation and definitions from the paper [CS].

By a *partition* we mean a collection of pairwise disjoint, nonempty subsets of  $\omega$  whose union is  $\omega$ . By  $(\omega)^{\leq \omega}$ ,  $(\omega)^{< \omega}$  and  $(\omega)^\omega$  we denote the set of all partitions, the set of all finite partitions and the set of all infinite partitions, respectively. If  $X, Y \in (\omega)^{\leq \omega}$ , then we say that  $Y$  is *coarser* than  $X$  iff  $\forall A \in X \exists B \in Y A \subseteq B$  and we write  $Y \leq X$ .

For any collection of sets  $\mathcal{A}$ , let us define  $\|\mathcal{A}\| = \bigcup \mathcal{A}$ . For  $X \in (\omega)^\omega$  and  $n \in \omega$ , we put  $X[n] = \{x \cap n : x \in X\} \setminus \{\emptyset\}$ . Suppose that  $s$  is a (finite) partition of some  $n \in \omega$ . We write  $s \prec X$  iff  $s = X[n]$ , where  $n = \|s\|$ . We say that  $s \leq t$  iff  $\|s\| = \|t\|$  and  $s$  is coarser than  $t$ . Finally,  $s \leq X$  iff  $s \leq X[\|s\|]$ . For every  $s, X$  such that  $s \leq X$ , we define  $(s, X) = \{Y \in (\omega)^\omega : s \prec Y \leq X\}$ , and following [CS] we call it a *dual Ellentuck neighborhood*.

A set  $D \subseteq (\omega)^\omega$  is said to be a *dual Ramsey null* iff for each dual Ellentuck neighborhood  $(s, X)$ , there exists  $Y \in (s, X)$  such that  $(s, Y) \cap D = \emptyset$ .

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Notice that each  $X \in (\omega)^\omega$  can be identified with an element  $\tilde{X} \in 2^{\omega \times \omega}$  by the following correspondence:  $\tilde{X}((m, n)) = 1$  iff  $\exists x \in X m, n \in x$ . Thus, we can assume that  $(\omega)^\omega$  is a subset of  $2^{\omega \times \omega}$ .

We recall that  $E \subseteq 2^{\omega \times \omega}$  is said to be *strongly meager* iff for each measure zero set  $N$  included in  $2^{\omega \times \omega}$ ,  $E + N \neq 2^{\omega \times \omega}$  (see for example [BJ]).

### 1. STRONGLY MEAGER SETS

The proof of our main result of this part is based on the following well-known Lorentz's theorem (see [L]) which has many interesting applications in the area of special subsets of the reals (see, for example, [NSW] and [NW2]).

**Theorem 1.1** (Lorentz). *If  $(G, +)$  is a finite group and if  $H$  is a nonempty subset of  $G$ , then there is a subset  $F$  of  $G$  such that*

- (1) *the cardinality of  $F$  is at most  $|G| \cdot \left(\frac{1 + \ln(|H|)}{|H|}\right)$ , and*
- (2)  *$G = F + H$ .*

**Lemma 1.2.** *For any dual Ellentuck neighborhood  $(s, A)$ , there exists a measure zero set  $H \in \mathcal{N}(2^{\omega \times \omega})$  such that for each  $t \in 2^{\omega \times \omega}$ , one can find  $B \in (s, A)$  such that  $(s, B) \subseteq H + t$ .*

*Proof.* For every  $n \in \omega$ , choose  $K_n \in \omega$  such that

$$\frac{\ln(K_n) + 1}{K_n} \leq \frac{1}{2^n}.$$

Since one can easily find a partition  $A' \in (s, A)$  such that  $\forall x \in A' |x| = \omega$ , we may further assume, without loss of generality, that  $\forall x \in A |x| = \omega$ . Let  $(A_n^{(k)})_{n \in \omega, 0 \leq k \leq K_n}$  be an enumeration of all elements of the set  $A \setminus \{x \in A : x \cap \|s\| \neq \emptyset\}$ . For each  $n, k \in \omega$  pick an arbitrary  $a_n^{(k)} \in A_n^{(k)}$ . Next, for every  $n \in \omega$ , define

$$I_n = \{a_n^{(k)} : 0 \leq k \leq K_n\}$$

and  $J_n = I_n \times I_n$ . For each  $n \in \omega$  and  $k \in [0, K_n]$ , let a function  $b_n^{(k)} : J_n \rightarrow 2$  be defined by the following conditions:

$$b_n^{(k)}(\langle m_1, m_2 \rangle) = \begin{cases} 1 & \text{if } k(m_1), k(m_2) \leq k \vee k(m_1), k(m_2) > k, \\ 0 & \text{otherwise,} \end{cases}$$

where  $k(m)$  denotes a (unique)  $k \in [0, K_n]$  such that  $m \in A_n^{(k)}$ . Put

$$B_n = \{b_n^{(k)} : 0 \leq k < K_n\}.$$

Obviously,  $B_n \subseteq 2^{J_n}$ . By Lorentz's theorem, there exist sets  $H_n \subseteq 2^{J_n}$ , so that

(1)

$$H_n + B_n = 2^{J_n},$$

(2)

$$|H_n| \leq \frac{\ln(|B_n|) + 1}{|B_n|} \cdot 2^{|J_n|} \leq \frac{1}{2^n} \cdot 2^{|J_n|}.$$

Define

$$H = \{x \in 2^{\omega \times \omega} : \exists_n^\infty x \upharpoonright J_n \in H_n\}.$$

It is easy to see that  $H \in \mathcal{N}(2^{\omega \times \omega})$ . Let  $t \in 2^{\omega \times \omega}$  be arbitrary. For each  $n \in \omega$ , choose  $k(n) \in [0, K_n]$  such that  $b_n^{(k(n))} \in H_n + (t \upharpoonright J_n)$ . Notice that the relation

$b_n^{(k)}$  can be identified with the partition  $\{D_n^{(k)}, \tilde{D}_n^{(k)}\}$  of the set  $I_n$ , where  $D_n^{(k)} = \{a_n^{(l)} : l \leq k\}$  and  $\tilde{D}_n^{(k)} = \{a_n^{(l)} : l > k\}$ . Let us fix another partition  $\{B_n^{(k)}, \tilde{B}_n^{(k)}\}$  of the set  $\bigcup_{0 \leq k \leq K_n} A_n^{(k)}$ , where  $B_n^{(k)} = \bigcup_{l \leq k} A_n^{(l)}$  and  $\tilde{B}_n^{(k)} = \bigcup_{l > k} A_n^{(l)}$ . Finally, for  $k = k(n)$ , we define a partition  $B$  by

$$B = \{x \in A : x \cap \|s\| \neq \emptyset\} \cup \left\{ \bigcup_{n \in \omega} B_n^{(k)} \right\} \cup \{ \tilde{B}_n^{(k)} : n \in \omega \}.$$

Clearly,  $B \in (\omega)^\omega$  and  $s \prec B \leq A$ ; thus,  $B \in (s, A)$ . Let  $C \in (s, B)$  be arbitrary. Identify a partition  $C$  with a function  $\tilde{C} : \omega \times \omega \rightarrow 2$ . Then for each  $n \in \omega$ , the relation  $\tilde{C} \upharpoonright J_n$  is equal to  $b_n^{(k(n))}$  or to  $\mathbf{1} \upharpoonright J_n$ , where  $\mathbf{1}(\langle m_1, m_2 \rangle) = 1$ . If we had  $\forall_{n \in \omega} C \upharpoonright J_n = \mathbf{1} \upharpoonright J_n$ , then  $C$  would be finite which is impossible, since  $C \in (\omega)^\omega$ . Thus, we have  $\exists_{n \in \omega} C \upharpoonright J_n = b_n^{(k(n))}$ . Hence,  $C \in H + t$  which finishes the proof.  $\square$

As a consequence we obtain the main theorem of this section.

**Theorem 1.3.** *Suppose that  $X$  is a strongly meager set in  $2^{\omega \times \omega}$ . Then it is dual Ramsey null.*

*Proof.* Immediately follows from Lemma 1.2.  $\square$

We will say that  $X \subseteq 2^{\omega \times \omega}$  is a very meager (VFC) set iff for each measure zero set  $N$ , there is a sequence  $\{X_n\}_{n \in \omega}$  such that  $X \subseteq \bigcup_{n \in \omega} X_n$  and  $X_n + N \neq 2^{\omega \times \omega}$ , for every  $n \in \omega$ .

Let us notice that the above argument gives us the following lemma.

**Lemma 1.4.** *For any dual Ellentuck neighborhood  $(s, A)$ , there exists a measure zero set  $H \in \mathcal{N}(2^{\omega \times \omega})$  such that for each sequence  $\{t_n\}_{n \in \omega} \subseteq 2^{\omega \times \omega}$ , one can find  $B \in (s, A)$  with  $\forall_{n \in \omega} (s, B) \subseteq H + t_n$ .*

**Corollary 1.5.** *Assume that  $X$  is a very meager set in  $2^{\omega \times \omega}$ . Then it is dual Ramsey null.*

*Proof.* Straightforward application of Lemma 1.4.  $\square$

## 2. PERFECTLY MEAGER SETS

Throughout this part we write  $f \prec g$ , where  $f, g \in \omega^{\uparrow \omega}$ , iff  $\forall_n f(n) \leq g(n)$ . Depending on the context, a function  $f \in \omega^{\uparrow \omega}$  is identified with the characteristic function of its range, and  $t \in 2^\omega$  is often conflated with a function that increasingly enumerates the set  $\{n : t(n) = 1\}$ . We assume that the reader is familiar with the notions of perfectly meager and universally meager sets (see, for example, [G1], [G2], and [NW1]). We recall below a definition of perfectly meager sets in the transitive sense which originally appeared in [NSW].

**Definition 2.1.** Let  $X$  be a subset of the Cantor set  $2^\omega$ . We say that  $X$  is an AFC' set (perfectly meager in the transitive sense) iff for every perfect set  $P \subseteq 2^\omega$ , one can find  $F$ , an  $F_\sigma$  set containing  $X$ , such that for every  $t \in 2^\omega$ ,  $(F + t) \cap P$  is meager in the relative topology of  $P$ .

We will use the following lemma which characterizes perfectly meager in the transitive sense subsets of  $2^\omega$ .

**Lemma 2.2.** *Let  $X \subseteq 2^\omega$ . The following conditions are equivalent:*

- (1)  $X \notin AFC'$ .
- (2) *There exists a sequence  $\{Q_n\}_{n < \omega}$  of perfect subsets of  $2^\omega$  such that if  $X \subseteq \bigcup_{n < \omega} F_n$ ,  $F_n = \overline{F_n}$ , then there exist  $n, m < \omega$ ,  $t \in 2^\omega$  such that  $Q_m + t \subseteq F_n$ .*

*Proof.* See [NW1], Lemma 6.

**Theorem 2.3.** *Let  $X \subseteq 2^\omega$  be a set which satisfies the condition*

$$\forall f \in \omega^{\uparrow\omega} \{g \in \omega^{\uparrow\omega} : f \not\prec g\} \cap X \in AFC'.$$

*Then  $X$  is an  $AFC'$  set.*

*Proof.* The following lemma will be needed to prove our theorem.

**Lemma 2.4.** *For every perfect set  $Q \subseteq 2^\omega$ , there exists  $f_Q \in \omega^{\uparrow\omega}$  such that*

$$\forall t \in 2^\omega (Q + t) \cap \{g \in \omega^{\uparrow\omega} : g \prec f_Q\} \neq \emptyset.$$

*Proof.* Let  $T \subseteq 2^{<\omega}$  be a perfect tree of the perfect set  $Q$ , i.e.,  $[T] = \{x \in 2^\omega : \forall k \in \omega x \upharpoonright k \in T\} = Q$ . Choose a sequence  $\{t_s : s \in 2^{<\omega}\} \subseteq T$  such that

$$\forall s \in 2^{<\omega} \forall i \in 2 t_{s \smallfrown \langle i \rangle} \supseteq t_s \smallfrown \langle i \rangle.$$

Define

$$f_Q(l) = \left( \max_{s \in 2^l} |t_s| \right).$$

It is easy to see that  $f_Q \in \omega^{\uparrow\omega}$ .

Suppose that  $t \in 2^\omega$ . We construct a sequence  $(s_k)_{k \in \omega}$  by induction as follows. Let  $s_0 = \emptyset$ . Define  $n_k = |t_{s_k}|$  and put  $s_{k+1} = s_k \smallfrown \langle 1 - t(n_k) \rangle$ . Finally, let  $z = \bigcup_{k \in \omega} t_{s_k}$ . Obviously,  $z \in [T] = Q$ . Since  $t_{s_{k+1}} = t_{s_k \smallfrown \langle 1 - t(n_k) \rangle} \supseteq t_{s_k} \smallfrown \langle 1 - t(n_k) \rangle$  and  $|t_{s_k}| = n_k$ , we obtain that  $t_{s_{k+1}}(n_k) = 1 - t(n_k)$ ; thus,  $z(n_k) + t(n_k) = 1$ . Moreover,  $|s_k| = k$  and therefore  $n_k = |t_{s_k}| \leq \max_{s \in 2^k} |t_s| = f_Q(k)$ . This implies that  $z + t \prec f_Q$ .  $\square$

Fix  $\{Q_n\}_{n \in \omega}$  to be a sequence of perfect sets, and choose  $\overline{f} \in \omega^{\uparrow\omega}$  such that

$$\forall n < \omega \overline{f} \succ f_{Q_n}.$$

Define

$$f^*(n) = \overline{f}(n) + 1.$$

Notice that the set  $\{f \in [\omega]^\omega : f \succ f^*\}$  is an  $F_\sigma$  set in the space  $\omega^{\uparrow\omega}$ . Let  $\hat{F}$  be defined as

$$\hat{F} = \{f \in [\omega]^\omega : f \succ f^*\} \cup [\omega]^{<\omega}.$$

We easily see that  $\hat{F}$  is an  $F_\sigma$  set. Now let  $t \in 2^\omega$  and  $n \in \omega$  be given. There exists  $q \in Q_n$  such that

$$q + t \in \{g \in \omega^{\uparrow\omega} : g \prec f_{Q_n}\} \subseteq \{g \in \omega^{\uparrow\omega} : g \prec \overline{f}\}.$$

We have that  $q + t \notin \hat{F}$ , since otherwise  $q + t \in [\omega]^\omega$  and  $q + t \succ f^*$ . In this case  $f^* \prec \overline{f}$  which contradicts the choice of  $f^*$ . This means that

$$\forall n < \omega \forall t \in 2^\omega Q_n + t \not\subseteq \hat{F}.$$

By our assumption,  $\{f \in \omega^{\uparrow\omega} : \neg(f^* \prec f)\} \cap X$  is an  $AFC'$  set, so by the characterization of  $AFC'$  sets, we obtain that there exists a sequence of closed sets  $(F_n)_{n \in \omega} \subseteq 2^\omega$  such that

(1)

$$\{f \in \omega^{\uparrow\omega} : \neg(f^* \prec f)\} \cap X \subseteq \bigcup_{n \in \omega} F_n,$$

(2)

$$\forall n, m < \omega \forall t \in 2^\omega t + Q_n \not\subseteq F_m.$$

We have that

$$X \subseteq \hat{F} \cup \left[ \{f \in \omega^{\uparrow\omega} : \neg(f \succ f^*)\} \cap X \right] \subseteq \left( \hat{F} \cup \bigcup_{n \in \omega} F_n \right).$$

Since

$$\forall t \in 2^\omega \forall m, n < \omega Q_n + t \not\subseteq F_m$$

and

$$\forall t \in 2^\omega \forall n < \omega Q_n + t \not\subseteq \hat{F},$$

it follows that  $X$  is an  $AFC'$  set. □

**Conclusion 2.5.** Assume  $b = d$ . Let  $\{f_\alpha : \alpha < b\} \subseteq \omega^{\uparrow\omega}$  be a dominating family of elements from  $\omega^{\uparrow\omega}$  such that

$$\forall \alpha < \beta f_\alpha \prec f_\beta.$$

Then  $\{f_\alpha : \alpha < b\}$  is an  $AFC'$  set.

*Proof.* This follows immediately from the fact that  $non(AFC') \geq \mathfrak{d}$  (see [N]). □

For a finite set  $s \in [\omega]^{<\omega}$  and an infinite  $A \subseteq \omega$  with  $\max(s) < \min(A)$ , let  $[s, A] = \{B \in [\omega]^\omega : s \subseteq B \subseteq s \cup A\}$ . A tree  $T \subseteq \omega^{<\uparrow\omega}$  is said to be a Laver tree iff for each  $s$ , with  $\text{stem}(T) \subseteq s$ , the set of all immediate successors of  $s$  is infinite (see [NW2] for details).

**Definition 2.6.** We shall say that  $X \subseteq 2^\omega$  is a completely Ramsey null ( $CR_0$ ) set iff for every  $[s, A]$ , there is an infinite  $B \subseteq A$  such that  $[s, B] \cap X = \emptyset$ . An  $X \subseteq 2^\omega$  will be called an  $l_0$ -set iff for every Laver tree  $T$ , there exists a Laver tree  $S \subseteq T$  with  $\{\text{ran}(x) : x \in [S]\} \cap X = \emptyset$ .

We have the following consequence of Theorem 2.3.

**Theorem 2.7.** Assume  $CH$ . Then there is an  $AFC'$  set  $X \subseteq 2^\omega$  which is not a  $CR_0$  set.

*Proof.* Assume that  $\{g_\alpha\}_{\alpha < \omega_1}$  is an enumeration of  $[\omega]^\omega$ . Given  $\{x_\alpha\}_{\alpha < \beta < \omega_1}$ , so that

(1)

$$\forall \alpha < \beta x_\alpha \subseteq g_\alpha,$$

(2)

$$\forall \alpha < \alpha' < \beta x_\alpha \prec x_{\alpha'},$$

find  $x_\beta \subseteq g_\beta$  which satisfies

$$\forall \alpha < \beta x_\alpha \prec x_\beta.$$

Clearly,  $\{x_\alpha\}_{\alpha < \omega_1}$  is an  $AFC'$  set which is not  $CR_0$ . □

**Corollary 2.8.** *Assume CH. Then there is an AFC' set  $X \subseteq 2^\omega$  which is not an  $l_0$ -set.*

*Proof.* First notice that for a given Laver tree  $T$ , one can find an  $x \in [T]$  that dominates a countable family of elements from  $\omega^{\uparrow\omega}$  and then repeat the above argument.  $\square$

Motivated by Theorem 2.3 we introduce the following notion.

**Definition 2.9.** We say that a  $\sigma$ -ideal  $\mathcal{I}$  of subsets of  $2^\omega$  satisfies condition  $(\ddagger)$  if and only if the following holds.

For every  $X \subseteq 2^\omega$ , if

$$(1) \quad \forall_{f \in \omega^{\uparrow\omega}} \{g \in \omega^{\uparrow\omega} : \neg(f \prec g)\} \cap X \in \mathcal{I},$$

then  $X \in \mathcal{I}$ .

Let us denote for brevity the set  $\{g \in \omega^{\uparrow\omega} : \neg(f \prec g)\}$  as  $G_f$ .

**Observation 2.10.** *The  $\sigma$ -ideals AFC, of perfectly meager sets, and  $\overline{AFC} = UM$ , of universally meager sets, satisfy condition  $(\ddagger)$ .*

*Proof.* Suppose that a set  $X \subseteq 2^\omega$  satisfies condition (1) from above for  $\mathcal{I} = AFC$ . Let  $Q$  be a perfect set and a sequence  $(Q_n)_{n \in \omega}$  be an enumeration of all nonempty elements of the class  $\{Q \cap [s] : s \in 2^{<\omega}\}$ . For each  $n \in \omega$ , choose a function  $f_n \in \omega^{\uparrow\omega} \cap Q_n$ . Let  $f^* \in \omega^{\uparrow\omega}$  be such that  $\forall_{n \in \omega} f_n \prec f^*$  and define  $\bar{f}(n) = f^*(n) + 1$ . Assume that there is an  $n \in \omega$  such that  $Q_n \subseteq Q \setminus G_{\bar{f}}$ . Then  $Q_n \subseteq Q \setminus G_{\bar{f}} \subseteq Q \setminus G_{f_n+1}$ , since  $f_n + 1 \prec \bar{f}$ . But  $f_n \in Q_n$ , so  $f_n \notin G_{f_n+1}$  and hence  $f_n + 1 \prec f_n$  which is a contradiction. This finishes the proof that  $Q \setminus G_{\bar{f}} \in MGR(Q)$ , the set of meager subsets of  $Q$  in the relative topology of  $Q$ . Therefore,  $X \in AFC$ .  $\square$

We now proceed to prove that  $\overline{AFC}$  satisfies  $(\ddagger)$ . To begin this proof we quote a characterization of  $\overline{AFC}$  sets which may be found in [B].

**Characterization 2.11** (T. Bartoszyński [B]). *A set  $X$  is an  $\overline{AFC}$  set iff for every sequence of perfect sets  $(P_n)_{n \in \omega}$  there exists an  $F_\sigma$  set  $E$  such that  $X \subseteq E$  and  $E$  is meager in  $P_n$ , for every  $n \in \omega$ .*

Let  $(P_n)_{n \in \omega}$  be a sequence of perfect sets. For  $n \in \omega$ , let  $(Q_{n,m})_{m \in \omega}$  be an enumeration of all nonempty elements of the family  $\{P_n \cap [s] : s \in 2^{<\omega}\}$ . For every  $n, m \in \omega$  choose  $f_{n,m} \in \omega^{\uparrow\omega} \cap Q_{n,m}$ . Let  $f^* \in \omega^{\uparrow\omega}$  be such that  $\forall_{n,m \in \omega} f_{n,m} \prec f^*$  and define  $\bar{f} = f^* + 1$ . Suppose that  $Q_{n,m} \subseteq 2^\omega \setminus G_{\bar{f}}$ . Since  $f_{n,m} \notin G_{\bar{f}}$ ,  $f^* + 1 \prec f_{n,m}$  and hence  $f^* + 1 \prec f_{n,m} \prec f^*$ , which is impossible. Therefore,  $\forall_{n \in \omega} (2^\omega \setminus G_{\bar{f}}) \cap P_n \in MGR(P_n)$ . Let us denote  $E = 2^\omega \setminus G_{\bar{f}}$ . By assumption, we have  $G_{\bar{f}} \cap X \in \overline{AFC}$ , so there is an  $F_\sigma$  set  $\tilde{E}$  such that  $G_{\bar{f}} \cap X \subseteq \tilde{E}$  and  $\forall_{n \in \omega} P_n \cap \tilde{E} \in MGR(P_n)$ . Define  $E^* = E \cup \tilde{E}$ . Obviously,  $E^*$  is an  $F_\sigma$  set. Moreover,  $X \subseteq (X \cap G_{\bar{f}}) \cup (2^\omega \setminus G_{\bar{f}}) \subseteq \tilde{E} \cup E = E^*$  and for each  $n \in \omega$ ,  $E^* \cap P_n = (\tilde{E} \cap P_n) \cup (E \cap P_n) \in MGR(P_n)$ . This proves that  $X \in \overline{AFC}$ .  $\square$

We finish this paper with the following question we are unable to answer.

**Problem 2.12.** *Is it true that the  $\sigma$ -ideal of  $\lambda'$  subsets of  $2^\omega$  satisfies condition  $(\ddagger)$ ?*

## ADDED IN PROOF

A. Miller proved recently (see [M]) that it is independent from the ZFC axioms whether or not every  $\lambda'$ -set satisfies condition ( $\ddagger$ ).

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