PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 1, Pages 305–312
S 0002-9939(03)07018-7
Article electronically published on May 28, 2003

MONOID OF SELF-EQUIVALENCES AND FREE LOOP SPACES

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(Communicated by Paul Goerss)

ABSTRACT. Let M be a simply-connected closed oriented N-dimensional manifold. We prove that for any field of coefficients $\mathbb{k}$ there exists a natural homomorphism of commutative graded algebras $\Gamma : H_*(\Omega\,\mathrm{aut}_1 M) \to \mathbb{H}_*(M^{S^1})$ where $\mathbb{H}_*(M^{S^1}) = H_{*+N}(M^{S^1})$ is the loop algebra defined by Chas and Sullivan. As usual $\mathrm{aut}_1 X$ denotes the monoid of self-equivalences homotopic to the identity, and ΩX the space of based loops. When $\mathbb{k}$ is of characteristic zero, Γ yields isomorphisms $H^{n+N}_{(1)}(M^{S^1}) \xrightarrow{\cong} (\pi_n(\Omega\mathrm{aut}_1 M)\otimes \mathbb{k})^\vee$ where $\bigoplus_{l=1}^\infty H^n_{(l)}(M^{S^1})$ denotes the Hodge decomposition on $H^*(M^{S^1})$.

1. INTRODUCTION

Let M be a simply connected N-dimensional closed oriented manifold with base point m_0. We denote by M^{S^1} the space of free loops on M, by ΩM the space of based loops of M at m_0, by $\mathrm{aut}\,M$ the monoid of (unbased) self equivalences of M, by $\mathrm{aut}_1 M$ the connected component of Id_M in $\mathrm{aut}M$, and by $H_*(-)$ the singular homology functor with coefficients in the fixed field $\mathbb{k}$. The composition of loops induce a commutative graded algebra structure on $H_*(\Omega\mathrm{aut}_1 M)$.

It is convenient to write

$$\mathbb{H}_*(M) = H_{*+N}(M) \text{ (resp. } \mathbb{H}_*(M^{S^1}) = H_{*+N}(M^{S^1})).$$

Indeed $\mathbb{H}_*(M)$ becomes a commutative graded algebra with the intersection product, and $\mathbb{H}_*(M^{S^1})$ a commuative graded algebra with the loop product defined by Chas and Sullivan [1]. The definition of the loop product works as follows: Let $\alpha : \triangle^n \to M^{S^1}$ and $\beta : \triangle^m \to M^{S^1}$ be simplices of M^{S^1} and assume that $q\circ\alpha : \triangle^n \to M$ and $q\circ\beta : \triangle^m \to M$ are transverse in some sense. Then the intersection product $(q\circ\alpha)\cdot(q\circ\beta)$ makes sense, and at each point $(s,t)\in\triangle^n\times\triangle^m$ such that $q\sigma(s) = q\tau(t)$, the composition of the loops $\alpha(s)$ and $\beta(t)$ can be performed. This gives a chain $\alpha\cdot\beta\in\mathcal{C}_{m+n-N}(M^{S^1})$ and leads to a commutative and associative multiplication ([1], Theorem 3.3):

$$\mathbb{H}_k(M^{S^1})\otimes\mathbb{H}_l(M^{S^1})\to\mathbb{H}_{k+l}(M^{S^1})\,,\quad a\otimes b\mapsto a\cdot b\,.$$

Our first result reads:

Received by the editors May 5, 2002 and, in revised form, August 30, 2002.
2000 *Mathematics Subject Classification.* Primary 55P35, 55P62, 55P10.

Theorem 1. *The natural map*

$$g : M \times \Omega\ \mathrm{aut}_1 M \to M^{S^1}\,, \quad g(x,\gamma)(t) = \gamma(t)(x)\,,$$

induces a morphism of commutative graded algebras

$$H_*(g) : \mathbb{H}_*(M) \otimes H_*(\Omega \mathrm{aut}_1 M) \to \mathbb{H}_*(M^{S^1})\,.$$

Denote by ω the fundamental class of M in homology. Then $\omega \in \mathbb{H}_0(M) = H_N(M) \cong \Bbbk\omega$ is the unit of the algebra $\mathbb{H}_*(M)$. The homomorphism $H_*(g)$ restricts to a morphism of commutative graded algebras

$$\Gamma : H_*(\Omega \mathrm{aut}_1 M) \to \mathbb{H}_*(M^{S^1})\,, \quad \Gamma(a) = H(g)(\omega \otimes a)\,.$$

The composition of Γ with the Hurewicz map $h : \pi_*(\Omega \mathrm{aut}_1 M) \otimes \Bbbk \to H_*(\Omega \mathrm{aut}_1 M)$ is a morphism of graded vector spaces

$$\Gamma_1 = \Gamma \circ h : \pi_*(\Omega \mathrm{aut}_1 M) \otimes \Bbbk \to H_{*+N}(M^{S^1})\,,$$

which in turn induces the dual morphism

$$\Gamma_1^\vee : H^{*+N}(M^{S^1}) \to (\pi_*(\Omega \mathrm{aut}_1 M \otimes \Bbbk))^\vee\,.$$

Now recall that $H^*(M^{S^1})$ is isomorphic as a graded vector space with the Hochschild homology of the cochain algebra $\mathcal{C}^*(M)$ ([10]):

$$H^*(M^{S^1}) \cong HH_*(\mathcal{C}^*(M); \mathcal{C}^*(M)).$$

Also recall that if $\Bbbk$ is a field of characteristic zero and A is a commutative graded $\Bbbk$-algebra, then the Hochschild homology of A, $HH_*(A;A)$, admits a Hodge decomposition ([8]):

$$\mathbb{H}_*(A;A) = \bigoplus_{l\geq 0}^{\infty} \mathbb{H}_*^{(l)}(A;A)\,.$$

Since $\mathcal{C}^*(M)$ is quasi-isomorphic to a commutative graded differential algebra A, we derive from the previous considerations a Hodge decomposition on the free loop space cohomology of M,

$$\mathbb{H}^*(M^{S^1}) = \bigoplus_{l\geq 0} \mathbb{H}^*_{(l)}(M^{S^1})\,.$$

We prove:

Theorem 2. *If $\Bbbk$ is a field of characteristic zero, then*

- $\Gamma_1 : \pi_*(\Omega\,\mathrm{aut}_1 M) \otimes \Bbbk \to \mathbb{H}_*(M^{S^1})$ *is injective,*
- $\Gamma_1^\vee : \mathbb{H}^n_{(1)}(M^{S^1}) \xrightarrow{\cong} (\pi_n(\Omega\,\mathrm{aut}_1 M) \otimes \Bbbk)^\vee$ *is an isomorphism for $n \geq 0$,*
- $\Gamma_1^\vee$ *vanishes on the components $\mathbb{H}^*_{(p)}(M^{S^1})$ for $p \geq 2$.*

Theorems 1 and 2 are proved respectively in sections 2 and 3. Section 4 contains examples and final remarks.

2. Proof of Theorem 1

We denote by $q : M^{S^1} \to M$ the free loop space fibration and by Sect (q) the space of sections of q. The composition of loops makes Sect (q) into a monoid with multiplication μ defined by

$$\mu(\sigma,\tau)(m)(t) = \begin{cases} \sigma(m)(2t)\,, & t \leq \frac{1}{2}, \\ \tau(m)(2t-1)\,, & t \geq \frac{1}{2}, \end{cases} \qquad \sigma,\tau \in \mathrm{Sect}\,(q)\,, \quad t\in[0,1]\,, m \in M\,.$$

Clearly the map $\psi : \Omega(\mathrm{aut}_1 M, id_M) \to \mathrm{Sect}\,(q)$ defined by

$$\psi(f)(m)(t) = f(t)(m)$$

is a homeomorphism of monoids making commutative the diagram

$$\begin{array}{ccc} M\times \mathrm{Sect}\,(q) & \xrightarrow{ev} & M^{S^1} \\ {\scriptstyle 1\times\psi}\downarrow & & \| \\ M \times \mathrm{aut}_1 M & \xrightarrow{g} & M^{S^1}\,, \end{array}$$

where ev denotes the evaluation map.

To prove Theorem 1, it therefore suffices to establish that the evaluation map $H_*(ev) : \mathbb{H}_*(M)\otimes H_*(\mathrm{Sect}\,(q)) \to \mathbb{H}_*(M^{S^1})$ is a morphism of graded algebras.

We first remark that Chas and Sullivan prove that the morphism $H_*(\sigma_0)$: $\mathbb{H}(M) \to \mathbb{H}(M^{S^1})$, induced by the trivial section σ_0, is a morphism of graded algebras ([1], Proposition 3.4). Therefore the restriction of $H_*(ev)$ to $\mathbb{H}_*(M)$ is a morphism of graded algebras.

Recall now that the unit of $\mathbb{H}_*(M)$ is the fundamental class $\omega \in \mathbb{H}_0 M = H_N M$. Therefore for a cycle $\sum_i n_i\alpha_i$, with $\alpha_i : \Delta^r \to \mathrm{Sect}\,(q)$, $H_*(ev)(\omega\otimes\alpha)$ is the homology class of the sum $\sum_i n_i\alpha_i'$ where α_i' denotes the composition

$$\alpha_i' : M\times\Delta^r \stackrel{id\times f}{\longrightarrow} M\times \mathrm{Sect}\,(q) \stackrel{ev}{\longrightarrow} M^{S^1}\,.$$

Thus let $\alpha : \triangle^r \to \mathrm{Sect}\,(q)$ and $\beta : \triangle^s \to \mathrm{Sect}\,(q)$ be simplices. Since the simplices $q\circ\alpha'$ and $q\circ\beta'$ are transverse in M, the Chas-Sullivan product

$$\alpha'\cdot\beta' : M\times\Delta^r\times\Delta^s \stackrel{id\times\alpha\times\beta}{\longrightarrow} M\times\mathrm{Sect}\,(q)\times\mathrm{Sect}\,(q) \stackrel{(ev,ev)}{\longrightarrow} M^{S^1}\times_M M^{S^1} \stackrel{c}{\longrightarrow} M^{S^1}$$

is well defined, c denoting pointwise composition of loops.

As the multiplication μ makes commutative the diagram

$$\begin{array}{ccc} M\times\mathrm{Sect}\,(q)\times\mathrm{Sect}\,(q) & \stackrel{(ev,ev)}{\longrightarrow} & M^{S^1}\times_M M^{S^1} \\ \downarrow {\scriptstyle id\times\mu} & & \downarrow {\scriptstyle c} \\ M\times\mathrm{Sect}\,(q) & \stackrel{ev}{\longrightarrow} & M^{S^1}\,, \end{array}$$

the map $\alpha'\cdot\beta'$ is equal to $\mu(\alpha,\beta)'$. Therefore the restriction of $H_*(ev)$ to the component $\mathbb{k}\omega\otimes H_*(Sect\,(q))$ is also a morphism of algebras.

Finally let $\alpha : \triangle^r \to M$ and $\beta : \Delta^s \to \mathrm{Sect}\,(q)$. Then the simplices α and $q\beta'$ are transverse and the Chas-Sullivan product $\alpha\cdot\beta$ is equal to $ev(\alpha\times\beta)$. Therefore $H_*(ev)(\alpha)\cdot H_*(ev)(\beta) = H_*(ev)(\alpha\otimes\beta)$. □

3. Proof of Theorem 2

Since $\mathbb{Q} \subset \Bbbk$ we may as well suppose that $\Bbbk = \mathbb{Q}$. Hereafter we will make extensive use of the theory of minimal models in the sense of Sullivan ([12]), for which we refer systematically to [5], §12. We denote by $(\wedge V, d)$ the minimal model of M. By [13] a relative minimal model for the fibration $q : M^{S^1} \to M$ is given by the extension

$$(\wedge V, d) \hookrightarrow (\wedge V \otimes \wedge sV, D)\,, |sv| = |v| - 1\,, D(v) = d(v)\,, D(sv) = -s(dv)\,,$$

where $s : \wedge V \to \wedge V \otimes \wedge sV$ is the unique derivation defined by $s(v) = sv$. The cochain complex $(\wedge V \otimes \wedge sV, D)$ decomposes into a direct sum of complexes

$$(\wedge V \otimes \wedge sV, D) = \bigoplus_{k\geq 0} (\wedge V \otimes \wedge^k sV, D)\,.$$

This induces a new graduation on $H^*(M^{S^1})$, $H^*(M^{S^1}) = \bigoplus_k H^*_{(k)}(M^{S^1})$ with

$$H^*_{(k)}(M^{S^1}) = H^*(\wedge V \otimes \wedge^k sV, D)\,.$$

In [14], Vigué proves that this decomposition coincides with the Hodge decomposition of the Hochschild homology $\mathbb{H}_*((\wedge V, d); (\wedge V, d))$:

$$H^*(\wedge V \otimes \wedge^k sV, D) \cong \mathbb{H}^{(k)}_*((\wedge V, d); (\wedge V, d))\,.$$

By the Milnor-Moore Theorem ([11]), $H_*(\mathrm{Sect}\,(q); \mathbb{Q})$ is isomorphic as a Hopf algebra to the universal enveloping algebra on the graded homotopy Lie algebra $\pi_*(\Omega\ \mathrm{aut}_1 M) \otimes \mathbb{Q}$. Thus Theorem 2 in the Introduction is a direct consequence of Theorem 3 below.

Theorem 3. *Let*

$$\Phi_1 : \pi_*(Sect\,(q)) \otimes \mathbb{Q} \to \mathbb{H}_*(M^{S^1}; \mathbb{Q})$$

denote the restriction of $H_*(ev)$ to $\omega \otimes \pi_*(Sect\,(q)) \otimes \mathbb{Q}$. Then,

- *Φ_1 is an injective morphism,*
- *the dual map ${\Phi_1}^\vee$ vanishes on each $H^*_{(p)}(M^{S^1}; \mathbb{Q})$, $p \geq 2$, and induces an isomorphism $\bigoplus_{q>N} H^q_{(1)}(M^{S^1}; \mathbb{Q}) \cong \pi_*(Sect\,(q))^\vee$.*

Proof. We first construct a quasi-isomorphism $\rho : (\wedge V, d) \to (A, d)$ with (A, d), a commutative differential graded algebra satisfying $A^0 = \mathbb{Q}\,, A^1 = 0$, $A^{>N} = 0$, $A^N = \mathbb{Q}\omega$, and $\dim\, A^i < \infty$ for all i.

For this we denote

$$Z^k = Ker(d : (\wedge V)^k \to (\wedge V)^{k+1})\,,$$

and we choose a supplement S^k of Z^k in $(\wedge V)^k$:

$$(\wedge V)^k = Z^k \oplus S^k\,.$$

The quotient $(\wedge V)^N/(S^N \oplus dS^{N-1}) \cong H^N(M)$ has dimension one. Since $V^1 = 0$, the subcomplex $I = S^{N-1} \oplus dS^{N-1} \oplus S^N \oplus (\wedge V)^{>N}$ is an acyclic ideal in $(\wedge V, d)$. Therefore the natural projection $\rho : (\wedge V, d) \to (\wedge V/I, d)$ is a quasi-isomorphism of differential graded algebras. We define $(A, d) = (\wedge V/I, d)$.

The homomorphism ρ extends to a quasi-isomorphism $\rho \otimes 1 : (\wedge V \otimes \wedge sV, D) \to (A \otimes \wedge sV, D)$ with $D(a \otimes sv) = d(a) \otimes sv - (-1)^{|a|} a \cdot (\rho \otimes 1)(Dsv)$. The complex $(A \otimes \wedge sV, D)$ also decomposes into the direct sum of the complexes $(A \otimes \wedge^k sV, D)$.

Denote by $(a_i)\,, i=1,\dots,n$, a homogeneous linear basis of A with $a_n=\omega$, and by $(a_i^\vee)$ the dual basis, i.e. the linear basis of $A^\vee = \mathrm{Hom}\,(A,\mathbb{Q})$ such that

$$\langle a_i^\vee, a_j\rangle = \delta_{ij}\,.$$

In [9], Haefliger proved that a model for the evaluation map $ev : M\times \mathrm{Sect}\,(q)\to M^{S^1}$ is given by the morphism

$$\theta : (A\otimes \wedge sV, D)\to (A,d)\otimes (\wedge(A^\vee\otimes sV),\delta)\,,\quad \theta(a\otimes sv)=\sum_i aa_i\otimes (a_i^\vee\otimes sv)\,.$$

Since $D(sV)\subset A\otimes sV$ and θ is a morphism of differential graded algebras, then $\delta(A^\vee\otimes sV)\subset A^\vee\otimes sV$. We now fix some notations:

- $\rho_1 : (\wedge(A^\vee\otimes sV),\delta)\to (A^\vee\otimes sV,\delta)$ denotes the projection on the complex of indecomposable elements,
- $P:(A,d)\to(\mathbb{Q}\omega,0)$ is the homogeneous projection onto the component of degree N,
- $\pi_1 : (A\otimes\wedge sV,D)\to (A\otimes sV,D)$ is the canonical projection on the subcomplex $(A\otimes sV,D)$.

The dual of Φ_1,

$$\Phi_1^\vee : H^{*+d}(M^{S^1};\mathbb{Q})\to (\pi_*(\mathrm{Sect}\,(q))\otimes\mathbb{Q})^\vee,$$

therefore coincides with $H^*(P\otimes\rho_1)\circ H^*(\theta)$:

$$(A\otimes\wedge sV,D)\xrightarrow{\theta}(A,d)\otimes(\wedge(A^\vee\otimes sV),\delta)\xrightarrow{P\otimes\rho_1}\mathbb{Q}\omega\otimes(A^\vee\otimes sV,\delta)\,,$$

and vanishes on $(A\otimes\wedge^{\geq 2}sV,D)$.

Lemma. *The duality map $\Delta : A\to A^\vee$ defined by*

$$\langle\Delta(a),b\rangle = P(ab)\in\mathbb{Q}\omega\cong\mathbb{Q}$$

extends into a quasi-isomorphism of complexes

$$\Delta\otimes 1 : (A\otimes sV,D)\to (A^\vee\otimes sV,\delta)\,.$$

Proof. Denote by α_{ij}^k and β_i^j rational numbers defined by the relations

$$\begin{cases} a_i\cdot a_j = \sum_k \alpha_{ij}^k a_k,\\ d(a_i)=\sum_j \beta_i^j a_j.\end{cases}$$

Recall that $\{a_i^\vee\}_i$ denotes the dual basis of $\{a_i\}_i$. Then straightforward computations show that

- $d(a_i^\vee) = -(-1)^{|a_i|}\sum_j \beta_j^i\, a_j^\vee$.
- $\sum_r \alpha_{ij}^r\,\alpha_{rk}^t = \sum_s \alpha_{jk}^s\,\alpha_{is}^t$, for $i,j,k,t=1,\dots,n$ (associativity of the multiplication law).
- $\sum_r \alpha_{ij}^r\,\beta_r^s = \sum_t \beta_i^t\,\alpha_{tj}^s + (-1)^{|a_i|}\sum_l \beta_j^l\,\alpha_{il}^s$, for $i,j,l=1,\dots,n$ (compatibility of the differential d with the multiplication).
- $\delta(a_j^\vee\otimes sv) = (-1)^{|a_j|}\Big[\sum_{i,l}\alpha_{il}^j\,(a_l^\vee\otimes sv_i) - \sum_r \beta_r^j\,(a_r^\vee\otimes sv)\Big]$.
- $\Delta(a_i) = \sum_j \alpha_{ij}^n\, a_j^\vee$.

The duality morphism has degree N. A standard computation then shows that

$$\delta \circ (\Delta \otimes 1) = (-1)^N (\Delta \otimes 1) \circ d\,.$$

Since $H^*(M)$ is a Poincaré duality algebra and since $H^*(\Delta) : H^*(M) \to H_*(M)$ is the Poincaré duality, $\Delta \otimes 1$ is a quasi-isomorphism. □

End of the proof of Theorem 3. It is easy to check the commutativity of the following diagram of complexes:

$$\begin{array}{ccccc} (A \otimes \wedge sV, D) & \xrightarrow{\theta} & (A, d) \otimes (\wedge(A^\vee \otimes sV), \delta) & \xrightarrow{P\otimes\rho_1} & \mathbb{Q}\omega \otimes (A^\vee \otimes sV, \delta) \\ {\scriptstyle \pi_1} \downarrow & & & & \uparrow {\scriptstyle 1\otimes(\Delta\otimes 1)} \\ (A \otimes sV, D) & & \xrightarrow{\sigma} & & \mathbb{Q}\omega \otimes (A \otimes sV, D)\,, \end{array}$$

with $\sigma(a \otimes sv) = \omega \otimes a \otimes sv$. By the above lemma, $H_*(1 \otimes \Delta \otimes 1)$ is an isomorphism. Therefore $H^*((1 \otimes (\Delta \otimes 1)) \circ \sigma \circ \pi_1)$ is surjective and this implies the surjectivity of $\Phi_1^\vee = H_*(P \otimes \rho_1) \circ H^*(\theta)$. □

4. Examples and further comments

Remark 1. The morphism $\Gamma : H_*(\Omega\,\mathrm{aut}_1 M) \to H_*(M^{S^1})$ is not injective in general, as we shall now explain.

Denote by $ev_0 : \mathrm{aut}_1 M \to M$ the evaluation at the base point. The image of the morphism $\pi_n(ev_0) : \pi_n(\mathrm{aut}_1 M) \to \pi_n M$ is known as the n-th Gottlieb group of M, $G_n(M)$ ([5]). Since $\Omega ev_0 : \Omega\,\mathrm{aut}_1 M \to \Omega M$ is an H-map, $H_*(\Omega ev_0; \mathbb{Q}) = U(\pi_*(\Omega ev_0) \otimes \mathbb{Q})$ is the enveloping algebra on $\pi_*(\Omega ev_0) \otimes \mathbb{Q}$, whose image is the enveloping algebra on the abelian graded Lie algebra $\overline{G}_*(X)$ that corresponds by duality to $G_*(X) \otimes \mathbb{Q}$.

Denote by $I : \mathbb{H}_*(M^{S^1}) \to H_*(\Omega M)$ the intersection morphism defined in ([1], Proposition 3.4), and let ψ be defined as in the beginning of section 2. The commutativity of the following diagram

$$\begin{array}{ccc} H_*(\Omega\,\mathrm{aut}_1 M) & \xrightarrow{\pi_*(\psi)} & H_*(\mathrm{Sect}\,(q)) \\ {\scriptstyle H_*(\Omega ev_0)} \downarrow & & \downarrow {\scriptstyle H_*(ev)(\omega\otimes -)} \\ H_*(\Omega M) & \xleftarrow{I} & \mathbb{H}_*(M^{S^1}) \end{array}$$

shows that the image of $I \circ \Phi_1$ is the universal enveloping algebra on $\overline{G}_*(X)$.

On the other hand, the kernel of I is a nilpotent ideal with nilpotency index less than or equal to N ([6]).

Now consider the manifold $M = S^3 \times S^3 \times S^{11}$. A simple computation using minimal models shows that $\pi_5(\mathrm{aut}_1 M) \otimes \mathbb{Q} \neq 0$ and $G_5(M) \otimes \mathbb{Q} = 0$. Then denote by x a nonzero element in $\pi_4(\Omega\,\mathrm{aut}_1 M) \otimes \mathbb{Q}$. Since $H_*(\Omega\,\mathrm{aut}_1 M; \mathbb{Q})$ is a free commutative graded algebra, some power of x belongs in the kernel of Γ.

Remark 2. In [2] Cohen and Jones prove that $\mathbb{H}_*(M^{S^1})$ is isomorphic as an algebra to the Hochschild cohomology $HH^*(\mathcal{C}^*(M), \mathcal{C}^*(M))$. On the other hand, in [7], Gatsinzi establishes for any space M an algebraic isomorphism between $\pi_*(\mathrm{aut}_1 M) \otimes \mathbb{Q}$ and a sub-vector space of $HH^*(\mathcal{C}^*(M), \mathcal{C}^*(M))$. Our Theorem 2 relates these two results.

Problem. We would like to know if the homomorphism

$$\Gamma : \mathbb{H}_*(M) \otimes H_*(\Omega \mathrm{aut}_1 M) \to \mathbb{H}_*(M^{S^1})$$

is surjective. It is true for example when $M = \mathbb{C}P^{2N}$. When Γ is surjective there is a strong connection between the behaviour of the sequences of Betti numbers $\dim H_i(M^{S^1})$ and $\dim \pi_i(\mathrm{aut}\, M) \otimes \mathbb{Q}$.

Example 1. Let G be a Lie group. The minimal model of G is $(\wedge V, 0)$ with V finite dimensional and concentrated in odd degrees ([5], §12(a)). Therefore a model of the free loop space G^{S^1} is $(\wedge V \otimes \wedge sV, 0)$ and the Haefliger model for the space $\mathrm{Sect}\,(q)$ is $(\wedge((\wedge V)^\vee \otimes sV), 0)$. Since the model θ of the evaluation map ev is injective, $H_*(ev) : H_*(M) \otimes H_*(\mathrm{Sect}\,(q)) \to H_*(M^{S^1})$ is surjective. This implies the existence of an isomorphism of graded algebras,

$$\mathbb{H}_*(M^{S^1}) \cong \mathbb{H}_*(M) \otimes H_*(\Omega M)\,.$$

Here the multiplication on the right is the product of the intersection product on $\mathbb{H}_*(M)$ with the usual Pontryagin product on $H_*(\Omega M)$.

Example 2. Let us assume that M is a $\mathbb{Q}$-hyperbolic space satisfying either $(H^+(M))^3 = 0$ or $(H^+(M))^4 = 0$, and M is a coformal space.

Recall that a space M is $\mathbb{Q}$-hyperbolic if $\dim \pi_*(M) \otimes \mathbb{Q} = \infty$ and is coformal if the differential graded algebras $\mathcal{C}_*(\Omega M)$ and $(H_*(\Omega M), 0)$ are quasi-isomorphic. Under the above hypothesis, in [15] Vigué proves that there exist an integer n_0 and some constants $C_1 \geq C_2 > 1$ such that

$$C_2^n \leq \sum_{i=1}^n \dim H^i_{(1)}(X^{S^1}) \leq C_1^n\,, \text{ for all } n \geq n_0\,.$$

We deduce from Theorem 3 that the same relations hold for the sequence of dimensions of $\pi_i(\mathrm{aut}\, M) \otimes \mathbb{Q}$, i.e., in both cases the sequences of Betti numbers have exponential growth.

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