PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 1, Pages 217–222
S 0002-9939(03)07029-1
Article electronically published on May 9, 2003

ASYMPTOTICALLY FLAT AND SCALAR FLAT METRICS ON $\mathbb{R}^3$ ADMITTING A HORIZON

PENGZI MIAO

(Communicated by Bennett Chow)

ABSTRACT. We give a new construction of asymptotically flat and scalar flat metrics on $\mathbb{R}^3$ with a stable minimal sphere. The existence of such a metric gives an affirmative answer to a question raised by R. Bartnik (1989).

1. Introduction and main results

The existence of an asymptotically flat and scalar flat metric on $\mathbb{R}^3$ with a stable minimal sphere is closely related to R. Bartnik's quasilocal mass definition [2] restricted to scalar flat metrics in general relativity. It also offers an example of a globally regular and asymptotically flat initial data for the Einstein vacuum equations containing a trapped surface. R. Beig and N. Ó Murchadha first proved the existence of such a metric in [4] by studying the behavior of a critical sequence of metrics. A similar observation was also made independently by R. Schoen at a later time.

In this paper we give a new approach to the existence problem, and we prove a slightly stronger result.

Theorem. *There exists an asymptotically flat and scalar flat metric on $\mathbb{R}^3$ which is conformally flat outside a compact set and contains a horizon.*

Combining this Theorem and the work of J. Corvino [6], we easily get an interesting corollary.

Corollary. *There exists a scalar flat metric on $\mathbb{R}^3$ which is Schwarzschild in a neighborhood of infinity and contains a horizon.*

Before giving the proof, we first introduce some relevant definitions. Interested readers may refer to [1], [9] and [10] for more discussions on asymptotically flat manifolds.

Definition 1 ([9])**.** A complete Riemannian manifold (M^3, g) is said to be **asymptotically flat** if there is a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \{|x| \leq 1\}$, and a diffeomorphism $\Phi : M \setminus K \longrightarrow \mathbb{R}^3 \setminus \{|x| \leq 1\}$ such that, in

Received by the editors May 2, 2002 and, in revised form, August 23, 2002.
2000 *Mathematics Subject Classification.* Primary 53C80; Secondary 83C99.
Key words and phrases. Scalar flat metrics, horizon.

the coordinate chart defined by Φ,

$$g = \sum_{i,j=1}^{3} g_{ij}(x)dx^i dx^j,$$

where

$$g_{ij}(x) = \delta_{ij} + O(|x|^{-p}),$$
$$|x||\partial_k g_{ij}(x)| + |x|^2|\partial^2_{kl} g_{ij}(x)| = O(|x|^{-p}),$$
$$|R(g)(x)| = O(|x|^{-q}),$$

for some $p > \frac{1}{2}$ and some $q > 3$, where $R(g)$ is the scalar curvature of (M^3, g).

Definition 2. A complete metric g on $\mathbb{R}^3$ is said to be **asymptotically flat** if $(\mathbb{R}^3, g)$ is an asymptotically flat manifold.

Definition 3. A **horizon** of an asymptotically flat manifold (M^3, g) is simply a stable minimal sphere in (M^3, g).

2. Proof of the Theorem

Our construction of the metric is essentially based on the following scalar deformation lemma due to J. Lohkamp [8].

Lemma 1. *Let (M, g) be a smooth Riemannian manifold with dimension ≥ 3. Let $U \subset M$ be an open subset and f be any smooth function on M with*

$$f < R(g) \text{ on } U \text{ and } f = R(g) \text{ on } M \setminus U, \tag{1}$$

where $R(g)$ is the scalar curvature of g. Then $\forall\ \epsilon > 0$, $\exists$ a smooth metric g_ϵ on M with

$$g_\epsilon = g \text{ on } M \setminus U_\epsilon, \quad f - \epsilon \leq R(g_\epsilon) \leq f \text{ on } U_\epsilon, \text{ and } \| g_\epsilon - g \|_{C^0(M)} < \epsilon, \tag{2}$$

where U_ϵ is the ϵ-neighborhood of U in M with respect to the metric g.

To apply this lemma, we start with a metric on $\mathbb{R}^3$ with a horizon whose scalar curvature is nonnegative on $\mathbb{R}^3$ and zero outside a precompact open set. Then we apply Lemma 1 to get a new metric with well controlled scalar curvature and Sobolev constant. Finally, we use a small conformal perturbation to make the metric scalar flat while keeping the horizon nearly fixed.

To make the argument precise, we need a few more lemmas.

Lemma 2. *For all $m > 0$, there exists a smooth spherically symmetric and conformally flat metric $\bar{g}$ on $\mathbb{R}^3$ with nonnegative scalar curvature such that*

$$\bar{g} = (1 + \frac{m}{2r})^4 g_{flat} \quad \text{outside } B_{\frac{m}{3}}(0), \tag{3}$$

where $r = |x|$, $B_{\frac{m}{3}}(0)$ is the open ball centered at the origin with radius $\frac{m}{3}$ and g_{flat} represents the usual Euclidean metric.

We note that the Schwarzschild metric $(1 + \frac{m}{2r})^4 g_{flat}$ contains a strictly minimizing sphere at $r = \frac{m}{2}$.

Proof. It suffices to construct a smooth spherically symmetric super-harmonic function on $\mathbb{R}^3$. To do that, we adapt an argument in [5] by H. Bray.

Let v be a piecewise smooth function defined by

$$v(x) = \begin{cases} 3, & x \in B_{\frac{m}{4}}(0), \\ (1+\frac{m}{2r}), & x \notin B_{\frac{m}{4}}(0). \end{cases} \tag{4}$$

Choose a standard spherically symmetric mollifier ϕ with support in $B_1(0)$ and, for $\sigma > 0$, we define

$$v_\sigma(x) = v * \phi^\sigma(x) = \int_{\mathbb{R}^3} v(y)\left(\frac{1}{\sigma^3}\phi(\frac{x-y}{\sigma})\right) dy. \tag{5}$$

Since v is a weakly super-harmonic function, v_σ is a smooth super-harmonic function. Furthermore

$$v_\sigma(x) = \begin{cases} 3, & x \in B_{\frac{m}{4}-\sigma}(0), \\ 1+\frac{m}{2r}, & x \notin B_{\frac{m}{4}+\sigma}(0), \end{cases} \tag{6}$$

because of the mean value property of harmonic functions.

We conclude that $\bar{g} = {v_\sigma}^4 g_{flat}$ satisfies the lemma, when $\sigma < \frac{m}{12}$. □

For the purpose of conformal deformation, we introduce the following existence lemma which is a special case of Lemmas 3.2 and 3.3 in [10]. The reader may refer to [10] for a detailed proof.

Lemma 3 ([10]). *Let g be a smooth asymptotically flat metric on $\mathbb{R}^3$ and $R(g)$ be the scalar curvature of g. There is a number $\epsilon_0 > 0$ depending only on the maximum and minimum norm of the eigenvalues of g with respect to g_{flat}, and the rate of decay of g, ∂g and $\partial\partial g$ at infinity so that if*

$$\frac{1}{8}\left(\int_{\mathbb{R}^3} |R(g)|^{\frac{3}{2}} dg\right)^{\frac{2}{3}} < \epsilon_0, \tag{7}$$

then

$$\begin{cases} \triangle_g u - \frac{1}{8}R(g)u &= 0, \\ \lim_{x\to\infty} u &= 1 \end{cases} \tag{8}$$

has a unique smooth positive solution defined on $\mathbb{R}^3$ such that

$$u = 1 + \frac{A}{r} + \omega \tag{9}$$

for some constant A and some function ω, where

$$\omega = O(r^{-2}),\ \partial\omega = O(r^{-3}),\ \partial\partial\omega = O(r^{-4}). \tag{10}$$

Now we are in a position to prove our Theorem.

Proof. Fix an $m > 0$, and let $\bar{g}$ be the metric constructed in Lemma 2. For any $\epsilon > 0$, we apply Lemma 1 to $\bar{g}$ with $U = B_{\frac{m}{3}}(0)$ and f_ϵ an arbitrary smooth function such that

$$f_\epsilon = 0 \text{ outside } B_{\frac{m}{3}}(0), \text{ and } -\epsilon < f_\epsilon < 0 \text{ everywhere else.} \tag{11}$$

We then get a smooth metric g_ϵ with

$$g_\epsilon = \bar{g} \text{ on } \mathbb{R}^3 \setminus U_\epsilon,\ f_\epsilon - \epsilon \le R(g_\epsilon) \le f_\epsilon \le 0 \text{ and } \| g_\epsilon - \bar{g} \|_{C^0(B_m(0))} < \epsilon. \tag{12}$$

Choosing ϵ to be small, we might assume that $U_\epsilon \subset B_{\frac{2m}{5}}(0)$. Now (11) and (12) imply

$$\begin{aligned} \left(\int_{\mathbb{R}^3} |R(g_\epsilon)|^{\frac{3}{2}} dg_\epsilon\right)^{\frac{2}{3}} &= \left(\int_{B_{\frac{2m}{5}}} |R(g_\epsilon)|^{\frac{3}{2}} dg_\epsilon\right)^{\frac{2}{3}} \\ &\leq C\left(\int_{B_{\frac{2m}{5}}} |2\epsilon|^{\frac{3}{2}} d\bar{g}\right)^{\frac{2}{3}} \\ &\leq C(m, \bar{g})\epsilon. \end{aligned} \tag{13}$$

It follows from Lemma 3 that we are able to solve

$$\begin{cases} \triangle_{g_\epsilon} u_\epsilon - \frac{1}{8} R(g_\epsilon) u_\epsilon &= 0, \\ \lim_{x\to\infty} u_\epsilon &= 1 \end{cases} \tag{14}$$

for each ϵ provided $\epsilon < \epsilon_0$ for some ϵ_0 depending only on $\bar{g}$ because of (12).

Now applying the Proposition below, we have

$$1 \leq u_\epsilon \leq 1 + C(\epsilon), \text{ where } \lim_{\epsilon\to 0} C(\epsilon) = 0. \tag{15}$$

On the other hand, since $g_\epsilon = \bar{g}$ outside $B_{\frac{2m}{5}}(0)$, we have

$$\triangle_{g_\epsilon} u_\epsilon - \frac{1}{8} R(g_\epsilon) u_\epsilon = \triangle_{\bar{g}} u_\epsilon = 0 \;\text{ for } x \notin B_{\frac{2m}{5}}(0). \tag{16}$$

The standard linear theory together with (15) and (16) then implies that, passing to a subsequence, u_ϵ converges to 1 in C^2 norm on any compact set outside $B_{\frac{2m}{5}}(0)$.

Define

$$\bar{g}_\epsilon = {u_\epsilon}^4 g_\epsilon. \tag{17}$$

It follows from (14), (12) and (15) that $\bar{g}_\epsilon$ is scalar flat, conformally flat at infinity and C^2 close to $\bar{g}$ on any compact set outside $B_{\frac{2m}{5}}(0)$. Since $\bar{g}$ coincides with the Schwarzschild metric $(1 + \frac{m}{2r})^4 g_{flat}$ outside $B_{\frac{2m}{5}}(0)$, which admits a strictly minimizing sphere at $\{r = \frac{m}{2}\}$, we conclude that $\bar{g}_\epsilon$ is forced to have a stable minimal sphere near $\{r = \frac{m}{2}\}$ for ϵ sufficiently small. □

Therefore, our proof will be complete provided we prove (15), which is given by the Proposition below.

Proposition. *For the solution $\{u_\epsilon\}$ above, we have*

$$1 \leq u_\epsilon \leq 1 + C(\epsilon), \text{ where } \lim_{\epsilon\to 0} C(\epsilon) = 0.$$

Proof. The first inequality follows directly from the maximum principle since u_ϵ is super-harmonic and goes to 1 near infinity.

To see the second inequality, we write $v_\epsilon = u_\epsilon - 1$. From (14) we have

$$\triangle_{g_\epsilon} v_\epsilon - \frac{1}{8} R(g_\epsilon) v_\epsilon = \frac{1}{8} R(g_\epsilon) \;\text{ and } v_\epsilon = \frac{A_\epsilon}{r} + \omega_\epsilon \tag{18}$$

for some constants A_ϵ and some function ω_ϵ with the decay property described in Lemma 3. Multiplying (18) by v_ϵ and integrating over $\mathbb{R}^3$,

$$\int_{\mathbb{R}^3} (v_\epsilon \triangle_{g_\epsilon} v_\epsilon - \frac{1}{8} R(g_\epsilon) {v_\epsilon}^2) dg_\epsilon = \int_{\mathbb{R}^3} \frac{1}{8} R(g_\epsilon) v_\epsilon dg_\epsilon. \tag{19}$$

Since $R(g_\epsilon)$ has compact support, both integrals above are finite. Integrating by parts and using the Hölder Inequality we have that

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla v_\epsilon|^2 dg_\epsilon &\leq \int_{\mathbb{R}^3}\frac{1}{8}|R(g_\epsilon)|{v_\epsilon}^2 dg_\epsilon + \int_{\mathbb{R}^3}\frac{1}{8}|R(g_\epsilon)|\cdot|v_\epsilon| dg_\epsilon \\ &\leq \left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{3}{2}}dg_\epsilon\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}} \\ &\quad +\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{6}{5}}dg_\epsilon\right)^{\frac{5}{6}}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{6}}. \end{aligned} \tag{20}$$

On the other hand, by the Sobolev Inequality, we have

$$\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}} \leq C_s(\epsilon)\int_{\mathbb{R}^3}|\nabla v_\epsilon|^2 dg_\epsilon \tag{21}$$

where $C_s(\epsilon)$ denotes the Sobolev constant of the metric g_ϵ. Hence, it follows from (20), (21) and the elementary inequality $ab \leq \frac{a^2}{2}+\frac{b^2}{2}$ that

$$\begin{aligned}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}} &\leq C_s(\epsilon)\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{3}{2}}dg_\epsilon\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}} \\ &\quad +C_s(\epsilon)\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{6}{5}}dg_\epsilon\right)^{\frac{5}{6}}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{6}} \\ &\leq C_s(\epsilon)\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{3}{2}}dg_\epsilon\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}} \\ &\quad +\frac{C_s(\epsilon)^2}{2}\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{6}{5}}dg_\epsilon\right)^{\frac{5}{3}} + \frac{1}{2}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}}. \end{aligned} \tag{22}$$

We note that (12) implies that $C_s(\epsilon)$ is uniformly close to $C_s(\bar{g})$, which is the Sobolev constant of $\bar{g}$. Hence, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}} &\leq C(\bar{g})\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{3}{2}}dg_\epsilon\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}} \\ &\quad +C(\bar{g})\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{6}{5}}dg_\epsilon\right)^{\frac{5}{3}} + \frac{1}{2}\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}}, \end{aligned} \tag{23}$$

which together with (13) and (12) implies that

$$\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{3}} \leq C(\bar{g})\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^{\frac{6}{5}}dg_\epsilon\right)^{\frac{5}{3}} = o(1), \quad \text{as } \epsilon \to 0. \tag{24}$$

This L^6 estimate and (18) then imply the desired supremum estimate for v_ϵ by the standard linear theory (say Theorem 8.17 in [7]),

$$\sup_{\mathbb{R}^3}|v_\epsilon| \leq C\left(\int_{\mathbb{R}^3}{v_\epsilon}^6 dg_\epsilon\right)^{\frac{1}{6}} + C\left(\int_{\mathbb{R}^3}|R(g_\epsilon)|^3 dg_\epsilon\right)^{\frac{1}{3}} = o(1) \ \text{ as } \epsilon \to 0, \tag{25}$$

which finishes the proof. □

Acknowledgments

I thank my advisor, Professor Richard Schoen, for suggesting this problem. I also thank Professor Hubert Bray for many stimulating discussions. Finally I thank Professor Robert Bartnik for showing me the work of [4].

References

1. Robert Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693. MR **88b:**58144
2. ———, *New definition of quasilocal mass*, Phys. Rev. Lett. **62** (1989), no. 20, 2346–2348. MR **90e:**83041
3. ———, *Some open problems in mathematical relativity*, Conference on Mathematical Relativity (Canberra, 1988), Austral. Nat. Univ., Canberra, 1989, pp. 244–268. MR **90g:**83001
4. R. Beig and N. Ó Murchadha, *Trapped surfaces due to concentration of gravitational radiation*, Phys. Rev. Lett. **66** (1991), no. 19, 2421–2424. MR **92a:**83005
5. Hubert Bray, *The penrose inequality in general relativity and volume comparison theorems involving scalar curvature*, Thesis, Stanford University (1997).
6. Justin Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Comm. Math. Phys. **214** (2000), no. 1, 137–189. MR **2002b:**53050
7. David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Berlin: Springer-Verlag, 1983. MR **86c:**35035
8. Joachim Lohkamp, *Scalar curvature and hammocks*, Math. Ann. **313** (1999), no. 3, 385–407. MR **2000a:**53059
9. Richard Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in the Calculus of Variations, Lecture Notes in Math. 1365, Berlin: Springer-Verlag, 1987, pp. 120–154. MR **90g:**58023
10. Richard Schoen and Shing Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76. MR **80j:**83024

Department of Mathematics, Stanford University, Palo Alto, California 94305
E-mail address: `mpengzi@math.stanford.edu`