ASYMPTOTICALLY FLAT AND SCALAR FLAT METRICS
ON $\mathbb{R}^3$ ADMITTING A HORIZON

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Abstract. We give a new construction of asymptotically flat and scalar flat
metrics on $\mathbb{R}^3$ with a stable minimal sphere. The existence of such a metric
gives an affirmative answer to a question raised by R. Bartnik (1989).

1. Introduction and main results

The existence of an asymptotically flat and scalar flat metric on $\mathbb{R}^3$ with a
stable minimal sphere is closely related to R. Bartnik’s quasilocal mass definition
[2] restricted to scalar flat metrics in general relativity. It also offers an example
of a globally regular and asymptotically flat initial data for the Einstein vacuum
equations containing a trapped surface. R. Beig and N. Ó Murchadha first proved
the existence of such a metric in [4] by studying the behavior of a critical sequence
of metrics. A similar observation was also made independently by R. Schoen at a
later time.

In this paper we give a new approach to the existence problem, and we prove a
slightly stronger result.

Theorem. There exists an asymptotically flat and scalar flat metric on $\mathbb{R}^3$ which
is conformally flat outside a compact set and contains a horizon.

Combining this Theorem and the work of J. Corvino [6], we easily get an inter-
esting corollary.

Corollary. There exists a scalar flat metric on $\mathbb{R}^3$ which is Schwarzschild in a
neighborhood of infinity and contains a horizon.

Before giving the proof, we first introduce some relevant definitions. Interested
readers may refer to [1], [9] and [10] for more discussions on asymptotically flat
manifolds.

Definition 1 ([9]). A complete Riemannian manifold $(M^3, g)$ is said to be asymptotically flat if there is a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \{|x| \leq 1\}$, and a diffeomorphism $\Phi : M \setminus K \rightarrow \mathbb{R}^3 \setminus \{|x| \leq 1\}$ such that, in

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the coordinate chart defined by $\Phi$,

$$g = \sum_{i,j=1}^{3} g_{ij}(x)dx^i dx^j,$$

where

$$g_{ij}(x) = \delta_{ij} + O(|x|^{-p}),$$

$$|x||\partial_k g_{ij}(x)| + |x|^2 |\partial_k^2 g_{ij}(x)| = O(|x|^{-p}),$$

$$|R(g)(x)| = O(|x|^{-q}),$$

for some $p > \frac{1}{2}$ and some $q > 3$, where $R(g)$ is the scalar curvature of $(M^3, g)$.

**Definition 2.** A complete metric $g$ on $\mathbb{R}^3$ is said to be **asymptotically flat** if $(\mathbb{R}^3, g)$ is an asymptotically flat manifold.

**Definition 3.** A **horizon** of an asymptotically flat manifold $(M^3, g)$ is simply a stable minimal sphere in $(M^3, g)$.

## 2. Proof of the Theorem

Our construction of the metric is essentially based on the following scalar deformation lemma due to J. Lohkamp [8].

**Lemma 1.** Let $(M, g)$ be a smooth Riemannian manifold with dimension $\geq 3$. Let $U \subset M$ be an open subset and $f$ be any smooth function on $M$ with

$$f < R(g) \text{ on } U \text{ and } f = R(g) \text{ on } M \setminus U,$$

where $R(g)$ is the scalar curvature of $g$. Then $\forall \epsilon > 0$, $\exists$ a smooth metric $g_{\epsilon}$ on $M$ with

$$g_{\epsilon} = g \text{ on } M \setminus U_{\epsilon}, \quad f - \epsilon \leq R(g_{\epsilon}) \leq f \text{ on } U_{\epsilon}, \text{ and } \|g_{\epsilon} - g\|_{C^0(M)} < \epsilon,$$

where $U_{\epsilon}$ is the $\epsilon$-neighborhood of $U$ in $M$ with respect to the metric $g$.

To apply this lemma, we start with a metric on $\mathbb{R}^3$ with a horizon whose scalar curvature is nonnegative on $\mathbb{R}^3$ and zero outside a precompact open set. Then we apply Lemma 1 to get a new metric with well controlled scalar curvature and Sobolev constant. Finally, we use a small conformal perturbation to make the metric scalar flat while keeping the horizon nearly fixed.

To make the argument precise, we need a few more lemmas.

**Lemma 2.** For all $m > 0$, there exists a smooth spherically symmetric and conformally flat metric $\bar{g}$ on $\mathbb{R}^3$ with nonnegative scalar curvature such that

$$\bar{g} = (1 + \frac{m}{2r})^4 g_{\text{flat}} \text{ outside } B_{\frac{m}{r}}(0),$$

where $r = |x|$, $B_{\frac{m}{r}}(0)$ is the open ball centered at the origin with radius $\frac{m}{r}$ and $g_{\text{flat}}$ represents the usual Euclidean metric.

We note that the Schwarzschild metric $(1 + \frac{m}{2r})^4 g_{\text{flat}}$ contains a strictly minimizing sphere at $r = \frac{m}{2r}$.
Proof. It suffices to construct a smooth spherically symmetric super-harmonic function on $\mathbb{R}^3$. To do that, we adapt an argument in [5] by H. Bray.

Let $v$ be a piecewise smooth function defined by

$$v(x) = \begin{cases} 3, & x \in B_{\mathbb{R}^3}(0), \\ (1 + \frac{m}{r}), & x \notin B_{\mathbb{R}^3}(0). \end{cases}$$

Choose a standard spherically symmetric mollifier $\phi$ with support in $B_{\mathbb{R}^3}(0)$ and, for $\sigma > 0$, we define

$$v_\sigma(x) = v * \phi^\sigma(x) = \int_{\mathbb{R}^3} v(y) \left( \frac{1}{\sigma^3} \phi\left( \frac{x - y}{\sigma} \right) \right) dy.$$

Since $v$ is a weakly super-harmonic function, $v_\sigma$ is a smooth super-harmonic function. Furthermore

$$v_\sigma(x) = \begin{cases} 3, & x \in B_{\mathbb{R}^3-\sigma}(0), \\ 1 + \frac{m}{r}, & x \notin B_{\mathbb{R}^3+\sigma}(0), \end{cases}$$

because of the mean value property of harmonic functions.

We conclude that $\bar{g} = v_\sigma^4 g_{\text{flat}}$ satisfies the lemma, when $\sigma < \frac{m}{12}$.

For the purpose of conformal deformation, we introduce the following existence lemma which is a special case of Lemmas 3.2 and 3.3 in [10]. The reader may refer to [10] for a detailed proof.

**Lemma 3 ([10])**. Let $g$ be a smooth asymptotically flat metric on $\mathbb{R}^3$ and $R(g)$ be the scalar curvature of $g$. There is a number $\epsilon_0 > 0$ depending only on the maximum and minimum norm of the eigenvalues of $g$ with respect to $g_{\text{flat}}$, and the rate of decay of $g$, $\partial g$ and $\partial^2 g$ at infinity so that if

$$\frac{1}{8} \left( \int_{\mathbb{R}^3} |R(g)| \frac{2}{\epsilon} dg \right)^\frac{2}{3} < \epsilon_0,$$

then

$$\left\{ \begin{array}{ll} \triangle_g u - \frac{1}{8} R(g) u &= 0, \\ \lim_{x \to \infty} u &= 1 \end{array} \right.$$  

has a unique smooth positive solution defined on $\mathbb{R}^3$ such that

$$u = 1 + \frac{A}{r} + \omega$$

for some constant $A$ and some function $\omega$, where

$$\omega = O(r^{-2}), \quad \partial \omega = O(r^{-3}), \quad \partial \partial \omega = O(r^{-4}).$$

Now we are in a position to prove our Theorem.

Proof. Fix an $m > 0$, and let $\bar{g}$ be the metric constructed in Lemma 2. For any $\epsilon > 0$, we apply Lemma 3 to $\bar{g}$ with $U = B_{\mathbb{R}^3}(0)$ and $f_\epsilon$ an arbitrary smooth function such that

$$f_\epsilon = 0 \text{ outside } B_{\mathbb{R}^3}(0), \quad -\epsilon < f_\epsilon < 0 \text{ everywhere else}.$$

We then get a smooth metric $g_\epsilon$ with

$$g_\epsilon = \bar{g} \text{ on } \mathbb{R}^3 \setminus U_\epsilon, \quad f_\epsilon - \epsilon \leq R(g_\epsilon) \leq f_\epsilon \leq 0 \text{ and } \| g_\epsilon - \bar{g} \|_{C^0(B_m(0))} < \epsilon.$$
Choosing \( \epsilon \) to be small, we might assume that \( U_\epsilon \subset B_{2m}(0) \). Now (11) and (12) imply

\[
\left( \int_{\mathbb{R}^3} |R(g_\epsilon)|^2 dg_\epsilon \right)^{\frac{1}{2}} = \left( \int_{B_{2m}} |R(g_\epsilon)|^2 dg_\epsilon \right)^{\frac{1}{2}} \\
\leq C \left( \int_{B_{2m}} |2\epsilon|^{\frac{2}{3}} \tilde{g} \right)^{\frac{1}{2}} \\
\leq C(m, \tilde{g}) \epsilon.
\]

(13)

It follows from Lemma 3 that we are able to solve

\[
\begin{cases}
\Delta g_\epsilon u_\epsilon - \frac{1}{8} R(g_\epsilon) u_\epsilon = 0, \\
\lim_{x \to \infty} u_\epsilon = 1
\end{cases}
\]

for each \( \epsilon \) provided \( \epsilon < \epsilon_0 \) for some \( \epsilon_0 \) depending only on \( \tilde{g} \) because of (12).

Now applying the Proposition below, we have

\[
1 \leq u_\epsilon \leq 1 + C(\epsilon), \text{ where } \lim_{\epsilon \to 0} C(\epsilon) = 0.
\]

(15)

On the other hand, since \( g_\epsilon = \tilde{g} \) outside \( B_{2m}(0) \), we have

\[
\Delta g_\epsilon u_\epsilon - \frac{1}{8} R(g_\epsilon) u_\epsilon = \Delta \tilde{g} u_\epsilon = 0 \quad \text{for } x \notin B_{2m}(0).
\]

(16)

The standard linear theory together with (15) and (19) then implies that, passing to a subsequence, \( u_\epsilon \) converges to 1 in \( C^2 \) norm on any compact set outside \( B_{2m}(0) \).

Define

\[
\tilde{g}_\epsilon = u_\epsilon^4 g_\epsilon.
\]

(17)

It follows from (14), (12) and (15) that \( \tilde{g}_\epsilon \) is scalar flat, conformally flat at infinity and \( C^2 \) close to \( \tilde{g} \) on any compact set outside \( B_{2m}(0) \). Since \( \tilde{g} \) coincides with the Schwarzschild metric \( (1 + \frac{m}{r})^4 \tilde{g}_{\text{flat}} \) outside \( B_{2m}(0) \), which admits a strictly minimizing sphere at \( \{ r = \frac{m}{2} \} \), we conclude that \( \tilde{g}_\epsilon \) is forced to have a stable minimal sphere near \( \{ r = \frac{m}{2} \} \) for \( \epsilon \) sufficiently small.

Therefore, our proof will be complete provided we prove (15), which is given by the Proposition below.

**Proposition.** For the solution \( \{ u_\epsilon \} \) above, we have

\[
1 \leq u_\epsilon \leq 1 + C(\epsilon), \text{ where } \lim_{\epsilon \to 0} C(\epsilon) = 0.
\]

**Proof.** The first inequality follows directly from the maximum principle since \( u_\epsilon \) is super-harmonic and goes to 1 near infinity.

To see the second inequality, we write \( v_\epsilon = u_\epsilon - 1 \). From (14) we have

\[
\Delta g_\epsilon v_\epsilon - \frac{1}{8} R(g_\epsilon) v_\epsilon = \frac{1}{8} R(g_\epsilon) \quad \text{and } v_\epsilon = \frac{A_\epsilon}{r} + \omega_\epsilon
\]

for some constants \( A_\epsilon \) and some function \( \omega_\epsilon \) with the decay property described in Lemma 3. Multiplying (18) by \( v_\epsilon \) and integrating over \( \mathbb{R}^3 \),

\[
\int_{\mathbb{R}^3} (v_\epsilon \Delta g_\epsilon v_\epsilon - \frac{1}{8} R(g_\epsilon) v_\epsilon^2) dg_\epsilon = \int_{\mathbb{R}^3} \frac{1}{8} R(g_\epsilon) v_\epsilon^2 dg_\epsilon.
\]

(19)
Since $R(g_e)$ has compact support, both integrals above are finite. Integrating by parts and using the Hölder Inequality we have that

$$
\int_{\mathbb{R}^3} |\nabla v_e|^2 \, dg_e \leq \int_{\mathbb{R}^3} \frac{1}{8} |R(g_e)| v_e^2 \, dg_e + \int_{\mathbb{R}^3} \frac{1}{8} |R(g_e)| \cdot |v_e| \, dg_e
$$

$$
\leq \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{2}}
$$

$$
+ \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{2}}.
$$

(20)

On the other hand, by the Sobolev Inequality, we have

$$
\left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{3}} \leq C_s(\epsilon) \int_{\mathbb{R}^3} |\nabla v_e|^2 \, dg_e
$$

(21)

where $C_s(\epsilon)$ denotes the Sobolev constant of the metric $g_e$. Hence, it follows from (20), (21) and the elementary inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ that

$$
\left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{3}} \leq C_s(\epsilon) \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{2}}
$$

$$
+ C_s(\epsilon) \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{2}}
$$

$$
\leq C_s(\epsilon) \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{2}}
$$

$$
+ \frac{C_s(\epsilon)^2}{2} \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} + \frac{1}{2} \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{3}}.
$$

(22)

We note that (12) implies that $C_s(\epsilon)$ is uniformly close to $C_s(\bar{g})$, which is the Sobolev constant of $\bar{g}$. Hence, we have

$$
\left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{3}} \leq C(\bar{g}) \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{2}}
$$

$$
+ C(\bar{g}) \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} + \frac{1}{2} \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{3}},
$$

(23)

which together with (13) and (12) implies that

$$
\left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{3}} \leq C(\bar{g}) \left( \int_{\mathbb{R}^3} |R(g_e)| \frac{2}{3} \, dg_e \right)^{\frac{3}{2}} = o(1), \quad \text{as} \quad \epsilon \to 0.
$$

(24)

This $L^6$ estimate and (13) then imply the desired supremum estimate for $v_e$ by the standard linear theory (say Theorem 8.17 in [7]),

$$
\sup_{\mathbb{R}^3} |v_e| \leq C \left( \int_{\mathbb{R}^3} v_e^6 \, dg_e \right)^{\frac{1}{3}} + C \left( \int_{\mathbb{R}^3} |R(g_e)|^3 \, dg_e \right)^{\frac{1}{3}} = o(1) \quad \text{as} \quad \epsilon \to 0,
$$

(25)

which finishes the proof. □
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References


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