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CHARACTERIZATION OF CLIFFORD-VALUED HARDY SPACES AND COMPENSATED COMPACTNESS

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Abstract. In this paper, the general Clifford $R_{n,s}$-valued Hardy spaces and conjugate Hardy spaces are characterized. In particular, each function in R_n-valued Hardy space can be determined by half of its function components through Riesz transform, and the explicit determining formulas are given. The products of two functions in the Hardy space give six kinds of compensated quantities, which correspond to six paracommutators, and their boundedness, compactness and Schatten-von Neumann properties are given.

1. Introduction

Clifford-valued (R_n-valued) Hardy spaces have been the subject of many works, in particular, by F. Sommen, F. Brackx, R. Delanghe, J. Ryan, T. Qian, M. Mitrea, K. Gürlebeck, W. Sprößig, S. Bernstein, etc.; see [10], [11], [3], [9], [5], [4], [1], etc. In this paper, we give the general Clifford-valued ($R_{n,s}$-valued) a new characterization.

To explain the Hardy space theory on the general Clifford algebras $R_{n,s}$, let us go back to the classical Hardy space theory on the complex numbers C. Denote the square-integrable complex-valued function space on the real line R by $L^2(R, C)$. The classical Hardy space $H^2(R, C)$ is defined to be

$$H^2(R, C) = \{f \in L^2(R, C) \mid F(x + iy) = P_y * f(x), \frac{\partial F}{\partial \overline{z}} = 0\},$$

where $P_y * f(x)$ is the Poisson integral of $f(x)$. By Fourier theory it turns out to be

$$H^2(R, C) = \{f + iHf \mid f \in L^2(R, R)\} = \{f \in L^2(R, C) \mid supp\widehat{f} \subset [0, +\infty)\}.$$

In this space, each complex-valued function $f + iHf$ is determined by its real component. There is a decomposition $L^2(R, C) = H^2(R, C) \bigoplus \overline{H}^2(R, C)$, where $\overline{H}^2(R, C)$ is the conjugate Hardy space.

In this paper, we generalize these to the general Clifford algebras $R_{n,s}$ and give the characterizations of the general Clifford-valued Hardy spaces and conjugate Hardy spaces. In particular, R_n-valued Hardy spaces are characterized, and each

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function in this kind of Hardy space is determined by half of its function components through Riesz transform. Moreover, for those representations, the explicit formulas are given. Comparing with the decomposition of square-integrable function space on the real line R into the direct sum of Hardy space and conjugate Hardy space, the square-integrable Clifford algebra-valued function spaces are decomposed into the orthogonal sum of Clifford-valued Hardy and conjugate Hardy spaces.

In the classical case, taking two functions $f = f_0 + iHf_0$ and $g = g_0 + iHg_0 \in H^2(R, C)$, one has studied the product $fg = f_0g_0 - Hf_0Hg_0 + i(f_0Hg_0 + Hf_0g_0)$; both its real part and imaginary part have compensated compactness. They are called compensated quantities. A one-to-one correspondence between paracommutator $T_b(A)f$ and compensated quantity $Q_A(f, g)$ has been established in [8]. The paracommutator was studied systematically by Janson, Peetre and Peng ([6], [7]). For the special case, the corresponding relation is as follows: $\langle fg, b\rangle = \langle H_b f, \overline{g}\rangle$. Therefore the BMO-boundedness, VMO-compactness and Schatten-von Neumann properties of $H_b f$ or fg can be read in theorems in [8]. In [13], [14], Zhijian Wu studied the product of a left monogenic function and a right monogenic function. In this paper, we shall consider the product of two left monogenic functions in Hardy spaces. This gives a nature method to obtain some examples of compensated quantities.

In §2 we give some preliminaries. In §3 we characterize the Hardy space $H^{(k)}(R^n, R_{n,s})$. By the characterization of Hardy space, we give some examples of the compensated quantities and paracommutators arising from the product of two functions in Hardy space.

2. Preliminaries

This section is an overview of some basic facts which are concerned with Clifford algebras. We set up the general formalism which will be used in the sequel (cf. [2]).

Let $V_{(n,s)}$ $(0 \le s \le n)$ be an n-dimensional $(n \ge 1)$ real linear space with basis $\{e_1, \cdots, e_n\}$, and provided with a bilinear form $(v, w), v, w \in V_{(n.s)}$, such that $(e_i, e_j) = 0, i \ne j$; $(e_i, e_i) = 1, i = 1, \cdots, s$; and $(e_i, e_i) = -1, i = s + 1, \cdots, n$.

If $v = \sum_{i=1}^n v_i e_i \in V_{(n,s)}$, then the associated quadratic form reads

$$(v, v) = \sum_{i=1}^{s} v_i^2 - \sum_{i=s+1}^{n} v_i^2.$$

Consider the 2^n-dimensional real linear space $C(V_{(n,s)})$ with basis

$$\{e_A = e_{h_1,\cdots,h_r} : A = (h_1, \cdots, h_r) \in PN, 1 \le h_1 < \cdots < h_r \le n\}$$

where N stands for the set $\{1, \cdots, n\}$ and $e_\emptyset = e_0$.

Now a product on $C(V_{(n,s)})$ is defined by

$$e_A e_B = (-1)^{n((A\cap B)\setminus S)}(-1)^{p(A,B)} e_{A\triangle B}$$

where S stands for the set $\{1, \cdots, n\}$, $n(A) = |A|$,

$$p(A, B) = \sum_{j\in B} p(A, j),\ p(A, j) = \{i \in A : i > j\}$$

and the sets A, B and $A \triangle B$ are ordered in the prescribed way. It is easy to check that $C(V_{n,s})$ turned into a linear, associative, but non-commutative algebra over R; it is called the universal Clifford algebra over $V_{(n,s)}$.

In this paper, we use the real Clifford algebra $R_{n,s}$, which means that in R^n the basis $\{e_1, \cdots, e_n\}$ satisfies $e_i e_i = 1, i = 1, \cdots, s$, and $e_i e_i = -1, i = s+1, \cdots, n$. When $s = 0$, we simply set $R_{n,s} = R_n$.

If f is Lipschitz continuous, then at any point of differentiability $x \in U$ (U is an open subset of R^{n+1}) of $f(x) = \sum_I f_I(x) e_I$, we have

$$(Df)(x) = \sum_I \sum_{j=0}^{n} \frac{\partial f_I}{\partial x_j}(x) e_j e_I, \qquad (fD)(x) = \sum_I \sum_{j=0}^{n} \frac{\partial f_I}{\partial x_j}(x) e_I e_j,$$
$$D^k f = D(D^{k-1})f, \ fD^k = (fD^{k-1})D.$$

We shall call f left k-monogenic (right k-monogenic, or two-sided k-monogenic, respectively) if $D^k f = 0$ ($fD^k = 0$, or $D^k f = fD^k = 0$, respectively); D is called the Cauchy-Riemann operator.

3. Hardy space $H^{(k)}(R^n, R_{n,s})$

Definition 1. Let $R_{n,s}$ be a real Clifford algebra. A Clifford module $L^2(R^n, R_{n,s})$ is defined to be

$$L^2(R^n, R_{n,s}) = \{f : R^n \to R_{n,s}, f(x) = \sum_I f_I(x) e_I \mid, f_I \in L^2(R^n, R), \forall I\}.$$

The Hardy space $H^{(k)}(R^n, R_{n,s})$ is defined to be

$$\begin{aligned} H^{(k)}(R^n, R_{n,s}) = \{f \in L^2(R^n, R_{n,s}) \mid F(x_0, x) = P_{x_0} * f(x), x_0 > 0, \\ D^{k-1} F(x, x_0) \neq 0, \ D^k F(x_0, x) = 0\}, \end{aligned}$$

where $k \in N$, $0 \le s \le n$, and P_{x_0} is the Poisson kernel.

In these Hilbert spaces, the inner product is denoted by

$$(f, g) = \int_{R^n} \sum_I f_I(x) g_I(x) dx,$$

where, $f(x), g(x) \in L^2(R^n, R_{n,s})$.

In detail, for arbitrary $f(x) = \sum_I f_I(x) e_I \in H^{(1)}(R^n, R_{n,s})$, let

$$F(x_0, x) = P_{x_0} * f(x) = \sum_I (P_{x_0} * f_I(x)) e_I$$

and

$$F_I(x_0, x) = P_{x_0} * f_I(x), \ \partial_j F_I(x, x_0) = \frac{\partial F_I(x, x_0)}{\partial x_j}.$$

Then we have

$$F(x_0, x) = \sum_I F_I(x_0, x) e_I, \ DF(x_0, x) = \sum_I \sum_{j=0}^{n} \partial_j F_I e_j e_I.$$

The latter can be rewritten as

$$DF(x_0, x) = (e_0, e_1, \cdots, e_n, \cdots, e_{(1,2,\cdots,n)}) A (F_0, F_1, \cdots, F_n, \cdots, F_{(1,2,\cdots,n)})^T,$$

where $e_0, e_1, \cdots, e_n, \cdots, e_{(1,2,\cdots,n)}$ is an ordered basis of $R_{n,s}$. It is arranged according to I in lexicographical order. A is a $2^n \times 2^n$ matrix uniquely determined when the order of this basis is set.

Example 1. $H^{(1)}(R^2,R_{2,1})=\{f\in L^2(R^2,R_{2,1})\mid F(x_0,x)=P_{x_0}*f(x),x_0>0,DF(x_0,x)=0\}$, $DF=(e_0,e_1,e_2,e_{12})A(F_0,F_1,F_2,F_{12})^T$, where

$$A=\begin{bmatrix}\partial_0 & \partial_1 & -\partial_2 & 0\\ \partial_1 & \partial_0 & 0 & \partial_2\\ \partial_2 & 0 & \partial_0 & \partial_1\\ 0 & -\partial_2 & \partial_1 & \partial_0\end{bmatrix}.$$

3.1. **Characterization of $H^{(k)}(R^n,R_{n,s})$.**

Theorem 1. *In the Hardy space $H^{(1)}(R^n,R_n)=\{f\in L^2(R^n,R_n)\mid F(x_0,x)=P_{x_0}*f(x),\ x_0>0,D^{k-1}F(x_0,x)\neq 0,\ DF(x_0,x)=0\}$, each function $f=\sum_I f_Ie_I$ is determined by its 2^{n-1} linearly independent function components. Explicitly for $\Omega=\{A=(h_1,\cdots,h_r)\in PN,1\leq h_1<\cdots<h_r\leq n\}$, denote $\Lambda=\{A=(h_1,h_2,\cdots,h_{2k+1}),1\leq h_1<\cdots<h_{2k+1}\leq n\}$ when $n=2m$, $k=0,1,\cdots,m-1$, when $n=2m+1$, $k=0,1,\cdots,m$. Then each f_J can be represented by the Riesz transforms of all f_K; f_K can also be represented by the Riesz transforms of all f_J, where $J\in\Lambda,K\in\Omega\setminus\Lambda$.*

Proof. For convenience, we consider DF in its Fourier transform

$$\widehat{DF}(x_0,\xi)=\sum_I\sum_{j=0}^n\widehat{\partial_jF_I}(x_0,\xi)e_je_I.$$

Since $\widehat{\partial_0F_I}(x_0,\xi)=-|\xi|\widehat{F_I}(x_0,\xi)$, $\widehat{\partial_jF_I}(x_0,\xi)=i\xi_j\widehat{F_I}(x_0,\xi)$, $j=1,2,\cdots,n$, for all I, then

$$\begin{aligned}\widehat{DF}(x_0,\xi)&=\sum_I(-|\xi|\widehat{F_I}e_0+\sum_{j=1}^n i\xi_j\widehat{F_I}e_j)e_I\\ &=(e_0,e_1,\cdots,e_I,\cdots,e_{(1,2,\cdots,n)})B(\widehat{F_0},\widehat{F_1},\cdots,\widehat{F_I},\cdots,\widehat{F}_{(1,2,\cdots,n)})^T.\end{aligned}\tag{1}$$

1) Let us prove that B is a Hermitian matrix. Denote

$$B=(u_0,u_1,\cdots,u_I,\cdots,u_{(1,2,\cdots,n)})^T=(v_0,v_1,\cdots,v_I,\cdots,v_{(1,2,\cdots,n)}),$$

where u_0, u_1, $\cdots$, u_I, $\cdots$, $u_{(1,2,\cdots,n)}$ are the rows of B, and v_0, v_1, $\cdots$, v_I, $\cdots$, $v_{(1,2,\cdots,n)}$ are the columns of B.

Let

$$\overrightarrow{u_0}=-|\xi|e_0-\sum_{j=1}^n i\xi_je_j,\ \overrightarrow{u_I}=\overrightarrow{u_0}e_I,\overrightarrow{w_0}=-|\xi|e_0+\sum_{j=1}^n i\xi_je_j,\ \overrightarrow{w_I}=\overrightarrow{w_0}e_I,$$

for all I.

We will prove that u_I are the coordinates of $\overrightarrow{u_I}$ with respect to the basis $\{e_0,e_1,\cdots,e_{(1,2,\cdots,n)}\}$, and v_I are the transpose of w_I. Here w_I are the coordinates of $\overrightarrow{w_I}$ with respect to the basis, $I=(0),(1),\cdots,(1,2,\cdots,n)$.

In fact, let $\lambda=(\widehat{F}_0,\widehat{F}_1,\cdots,\widehat{F}_I,\cdots,\widehat{F}_{(1,2,\cdots,n)})^T$. From (1), we have

$$\widehat{DF}=(e_0,e_1,\cdots,e_{(1,2,\cdots,n)})(u_0\lambda,u_1\lambda,\cdots,u_I\lambda,\cdots,u_{(1,2,\cdots,n)}\lambda)^T.$$

Obviously, $u_I\lambda=u_I(\widehat{F}_0,\widehat{F}_1,\cdots,\widehat{F}_I,\cdots,\widehat{F}_{(1,2,\cdots,n)})^T$ is the coefficient of e_I in $\widehat{DF}$, for each I.

On the other hand, for each $j = 1, 2, \cdots, n$, there is a unique I_j satisfying $e_j e_{I_j} = (-1)^{I_j} e_I$.

From (1), the e_I component in $\widehat{DF}$ is

$$-|\xi|\widehat{F}_I e_I + \sum_{j=1}^{n} i\xi_j \widehat{F}_{I_j} e_j e_{I_j} = (-|\xi|\widehat{F}_I + \sum_{j=1}^{n} (-1)^{I_j} i\xi_j \widehat{F}_{I_j}) e_I,$$

so

$$u_I \lambda = -|\xi|\widehat{F}_I + \sum_{j=1}^{n} (-1)^{I_j} i\xi_j \widehat{F}_{I_j} \tag{2}$$

and

$$\begin{aligned} \overrightarrow{u_I} &= -|\xi| e_0 e_I - \sum_{j=1}^{n} i\xi_j e_j e_I \\ &= -|\xi| e_0 e_I - \sum_{j=1}^{n} (-1)^{I_j+1} i\xi_j e_{I_j} = -|\xi| e_I + \sum_{j=1}^{n} (-1)^{I_j} i\xi_j e_{I_j}. \end{aligned} \tag{3}$$

From (2) and (3), we obtain that u_I is the component of $\overrightarrow{u_I}$ with respect to the basis $\forall I$.

Now to prove that v_I is the transpose of w_I, i.e. to prove

$$(e_0, e_1, \cdots, e_{(1,2,\cdots,n)}) v_I = \overrightarrow{w_J}.$$

Let $\lambda = (0, 0, \cdots, \hat{f}, \cdots, 0)^T$ be the vector in R^{2^n} with all zeros except for an $\hat{f}$ ($\hat{f} \in L^2(R^n, R)$) in the Ith entry. From (1), we have

$$(-|\xi| e_0 e_I + \sum_{j=1}^{n} i\xi_j e_j e_I)\hat{f} = (e_0, e_1, \cdots, e_I, \cdots, e_{(1,2,\cdots,n)}) v_I \hat{f}.$$

This is $\overrightarrow{w_I} = (e_0, e_1, \cdots, e_I, \cdots, e_{(1,2,\cdots,n)}) v_I$, so the transpose of w_I is precisely v_I.

These cases and

$$\begin{aligned} \overrightarrow{u_I} &= -|\xi| e_0 e_I + \sum_{j=1}^{n} (-1)^{I_j} i\xi_j e_{I_j}, \\ \overrightarrow{w_I} &= -|\xi| e_1 e_I - \sum_{j=1}^{n} (-1)^{I_j} i\xi_j e_{I_j} \end{aligned} \tag{4}$$

show that u_I is the conjugate transpose of v_I for each I. So $\overline{B}^T = B$, i.e. B is a Hermitian matrix.

2) Let us prove $BB = -2|\xi|B$. By the properties of Fourier transform, we know that the Cauchy-Riemann operator D corresponds to the operator $\hat{D} = -|\xi| + \sum_{j=1}^{n} i\xi_j e_j$, and the latter corresponds to B, so we get

$$BB = -2|\xi|B. \tag{5}$$

3) We are going to prove that $r_B = 2^{n-1}$. From (5), we get

$$(B + 2|\xi| E_{2^n})B = -B^T B = 0,$$

i.e. $BB^T = \overline{B^T B} = 0$. Therefore, $-2|\xi|$ and 0 are eigenvalues of B. Denote by $V_0, V_{-2|\xi|}$ the eigenspace of B associated with $0, -2|\xi|$, respectively; V_0' is the eigenspace of B^T associated with 0. Then

$$dimV_{-2|\xi|} = dimV_0' \geq r_B. \tag{6}$$

Applying the Plus Nullity Theorem, we get

$$r_B + dimV_0 = 2^n \tag{7}$$

and

$$r_{B^T} + dimV_0' = 2^n.$$

Threfore

$$dimV_0 = dimV_0' = dimV_{-2|\xi|}.$$

Since B is a hermitian matrix, B can be diagonalized. If B has other eigenvalues different from $-2|\xi|$ and 0, let d be the sum of the dimensions of their eigenspaces. Then

$$dimV_{-2|\xi|} + dimV_0 + d = 2^n.$$

Together with (6) and (7), we have

$$2^n = dimV_{-2|\xi|} + dimV_0 + d \geq 2^n + d > 2^n,$$

which is a contradiction. So the eigenvalues of B are $-2|\xi|$ and 0, and we get

$$dimV_{-2|\xi|} = r_B.$$

Thus from $dimV_{-2|\xi|} = dimV_0'$ and (7), we know $r_B = 2^{n-1}$. Therefore each f can be determined by 2^{n-1} of its function components.

In fact, the rows u_J are the maximum linearly independent in all rows of B, where

$$J \in \Lambda = \{A = (h_1, h_2, \cdots, h_{2k+1}), 1 \leq h_1 < \cdots < h_{2k+1} \leq n\}.$$

This can be deduced from $\overrightarrow{u_J} = -|\xi| e_J - \sum_{j=1}^n i\xi_j e_j e_J$.

If

$$\sum_{J\in\Lambda} a_J u_J = 0,$$

there must be an $a_J = 0$. Hence the u_J are linearly independent and, together with $r_B = 2^{n-1}$, we know all the rows of B can be determined by these u_J.

On the other hand, for each $j = 1, \cdots, n$, there is a $J_j \in \Omega \setminus \Lambda$, which uniquely satisfies $e_j e_J = (-1)^{J_j} J_j$, so

$$\overrightarrow{u_J} = -|\xi| e_J - \sum_{j=1}^n (-1)^{J_j} i\xi_j e_{J_j}.$$

Solving the equations $u_J(\widehat{F_0}, \widehat{F_1}, \cdots, \widehat{F_{(1,\cdots,n)}})^T = 0$, we get

$$f_J(x) = -\sum_{j=1}^n (-1)^{J_j} R_j f_{J_j},$$

where $J \in \Lambda$, i.e. each f_J can be represented by the Riesz transforms of all f_K, where $J \in \Lambda$, $K \in \Omega \setminus \Lambda$. Similarly, f_K can be represented by the Riesz transforms of all f_J. The proof is finished. $\square$

Remark. From the proof of the above theorem, we can write each function $f \in H^{(1)}(R^n, R_n)$ as follows:

1) When n is even,

$$
\begin{aligned}
f &= \sum_I f_I e_I \\
&= (f_0 + R_1 f_0 e_1 + R_2 f_0 e_2 + \cdots + R_n f_n e_n) \\
&+ (f_{12} + R_1 f_{12} e_1 + R_2 f_{12} e_2 + \cdots + R_n f_{12} e_n) e_{12} + \cdots \\
&+ (f_{n-1,n} + R_1 f_{n-1,n} e_1 + R_2 f_{n-1,n} e_2 + \cdots + R_n f_{n-1,n} e_n) e_{n-1,n} + \cdots \\
&+ (f_{12\cdots n} + R_1 f_{12\cdots n} e_1 + R_2 f_{12\cdots n} e_2 + \cdots + R_n f_{12\cdots n} e_n) e_{12\cdots n}.
\end{aligned} \tag{8}
$$

2) When n is odd,

$$
\begin{aligned}
f &= \sum_I f_I e_I \\
&= (f_0 + R_1 f_0 e_1 + R_2 f_0 e_2 + \cdots + R_n f_n e_n) \\
&+ (f_{12} + R_1 f_{12} e_1 + R_2 f_{12} e_2 + \cdots + R_n f_{12} e_n) e_{12} + \cdots \\
&+ (f_{n-1,n} + R_1 f_{n-1,n} e_1 + R_2 f_{n-1,n} e_2 + \cdots + R_n f_{n-1,n} e_n) e_{n-1,n} + \cdots \\
&+ (f_{2\cdots n} + R_1 f_{2\cdots n} e_1 + R_2 f_{2\cdots n} e_2 + \cdots + R_n f_{2\cdots n} e_n) e_{2\cdots n}.
\end{aligned} \tag{9}
$$

In fact, we can represent the function by using other function components.

Example 2. The Hardy space $H^{(1)}(R^1, R_1) = \{f \in L^2(R^1, R_1) \mid F(x_0, x) = P_{x_0} * f(x), x_0 > 0,\ DF(x_0, x) = 0\}$ is exactly the classical Hardy space. For each $f \in H^{(1)}(R^1, R_1)$, it can be represented by the Hilbert transform (when $n = 1$, there is only one Riesz transform, namely the Hilbert transform) of one of its function components.

Example 3. For each $f \in H^{(1)}(R^3, R_3)$,

$$
\begin{aligned}
&\widehat{DF}(x_0, \xi) \\
&= (e_0, e_1, e_2, e_3, e_{(1,2)}, e_{(1,3)}, e_{(2,3)}, e_{(1,2,3)}) \\
&B(\widehat{F}_0, \widehat{F}_1, \widehat{F}_2, \widehat{F}_3, \widehat{F}_{(1,2)}, \widehat{F}_{(1,3)}, \widehat{F}_{(2,3)}, \widehat{F}_{(1,2,3)})^T,
\end{aligned}
$$

where

$$
B = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_{(1,2)} \\ u_{(1,3)} \\ u_{(2,3)} \\ u_{(1,2,3)} \end{bmatrix} = \begin{bmatrix}
-|\xi| & -i\xi_1 & -i\xi_2 & -i\xi_3 & 0 & 0 & 0 & 0 \\
i\xi_1 & -|\xi| & 0 & 0 & i\xi_2 & i\xi_3 & 0 & 0 \\
i\xi_2 & 0 & -|\xi| & 0 & -i\xi_1 & 0 & i\xi_3 & 0 \\
i\xi_3 & 0 & 0 & -|\xi| & 0 & -i\xi_1 & -i\xi_2 & 0 \\
0 & -i\xi_2 & i\xi_1 & 0 & -|\xi| & 0 & 0 & -i\xi_3 \\
0 & -i\xi_3 & 0 & i\xi_1 & 0 & -|\xi| & 0 & i\xi_2 \\
0 & 0 & -i\xi_3 & i\xi_2 & 0 & 0 & -|\xi| & -i\xi_1 \\
0 & 0 & 0 & 0 & i\xi_3 & -i\xi_2 & i\xi_1 & -|\xi|
\end{bmatrix}.
$$

By using Theorem 1,

$$\begin{aligned} f_1(x) &= R_1 f_0(x) + R_2 f_{(1,2)}(x) + R_3 f_{(1,3)}(x), \\ f_2(x) &= R_2 f_0(x) - R_1 f_{(1,2)}(x) + R_3 f_{(2,3)}(x), \\ f_3(x) &= R_3 f_0(x) - R_1 f_{(1,3)}(x) - R_2 f_{(2,3)}(x), \\ f_{(1,2,3)}(x) &= R_3 f_{(1,2)}(x) - R_2 f_{(1,3)}(x) + R_1 f_{(2,3)}(x). \end{aligned}$$

In fact, $f_0,\ f_{(1,2)},\ f_{(1,3)},\ f_{(2,3)}$ can be represented by the Riesz transform of other function components.

Corollary 1. *In the Hardy space $H^{(k)}(R^n, R_n) = \{f(x) \in L^2(R^n, R_n) | F(x_0, x) = P_{x_0} * f(x), x_0 > 0, D^{k-1}F(x_0, x) \neq 0,\ D^k F(x_0, x) = 0\}$, each f is determined by 2^{n-1} linearly independent function components.*

Proof. Let $F(x_0, x) = \sum_I F_I(x_0, x) e_I$ and $DF(x_0, x) = \sum_I \sum_{j=0}^n \partial_j F_I e_j e_I$. For convenience, we consider DF in its Fourier transform. Using the formula

$$\widehat{DF}(x_0, \xi) = (e_0, e_1, \cdots, e_I, \cdots, e_{(1,2,\cdots,n)}) B(\widehat{F_0}, \widehat{F_1}, \cdots, \widehat{F_I}, \cdots, \widehat{F}_{(1,2,\cdots,n)})^T$$

k times, we obtain

$$\widehat{D^k F}(x_o, \xi) = (e_0, e_1, \cdots, e_I, \cdots, e_{(1,2,\cdots,n)}) B^k(\widehat{F_0}, \widehat{F_1}, \cdots, \widehat{F_I}, \cdots, \widehat{F}_{(1,2,\cdots,n)})^T.$$

Now we prove $r_{B^k} = r_B = 2^{n-1}$. From Theorem 1, we know $BB = -2|\xi|B$. Therefore

$$B^k = -2|\xi| B^{k-1} = (-2|\xi|)^2 B^{k-2} = \cdots = (-2|\xi|)^{k-1} B.$$

Then $r_{B^k} = 2^{n-1}$. This ends the proof. □

Theorem 2. $H^{(1)}(R^n, R_{n,n}) = \{0\}$.

Proof. For convenience, we still consider DF in its Fourier transform. Denote $B = (u_0, u_1, \cdots u_I, \cdots, u_{(12\cdots n)})^T = (v_0, v_1, \cdots, v_I, \cdots, v_{(1,2,\cdots,n)})$. Let $\overrightarrow{u_0} = -|\xi| e_0 + \sum_{j=1}^n i\xi_j e_j,\ \overrightarrow{u_I} = \overrightarrow{u_0} e_I$, and $\overrightarrow{w_0} = -|\xi| e_0 + \sum_{j=1}^n i\xi_j e_j,\ \overrightarrow{w_I} = \overrightarrow{w_0} e_I$, for all I.

As in Theorem 1, we can prove that u_I are the coordinates of $\overrightarrow{u_I}$ with respect to the basis $\{e_0, e_1, \cdots, e_{(1,2,\cdots,n)}\}$, and v_I are the transpose of w_I. Here w_I are the coordinates of $\overrightarrow{w_I}$ with respect to the basis, $I = (0), (1), \cdots, (1, 2, \cdots, n)$. Then we get

1) B is a symmetric matrix.
2) $B\overline{B} = 2|\xi|^2 E_{2^n}$.

The method to prove this is similar to Theorem 1.

Therefore $r_B = 2^n$, i.e. $F(x_0, x) = 0$. Thus $H^{(1)}(R^n, R_{n,n}) = \{0\}$. □

Corollary 2.

$$H^{(k)}(R^n, R_{n,n}) = \{0\}, k \in N.$$

More generally, we have the following theorem.

Theorem 3. *For each $f \in H^{(1)}(R^n, R_{n,s}) = \{f \in L^2(R^n, R_{n,s}) \mid F = P_{x_0} * f, x_0 > 0, DF(x_0, x) = 0\}$, if $\sum_{j=1}^s x_j^2 \neq 0$, then $f = 0$; otherwise, f can be determined by 2^{n-1} linearly independent function components.*

Proof. We consider DF in its Fourier transform. Denote

$$B = (u_0, u_1, \cdots, u_I, \cdots, u_{(12\cdots n)})^T = (v_0, v_1, \cdots, v_I, \cdots, v_{(1,2,\cdots,n)}).$$

Using a method similar to that in Theorem 1, we get $B\overline{B} = 2\sum_{j=1}^{s} \xi_j^2 E_{2^n}$. If $\sum_{j=1}^{s} \xi_j^2 \neq 0$, then $r_B = 2^n$. Therefore $f = 0$. If $\sum_{j=1}^{s} \xi_j^2 = 0$, then B is a hermitian matrix. Using Theorem 1, we have $r_B = 2^{n-1}$. Therefore f is determined by 2^{n-1} linearly independent function coefficients. □

Corollary 3. *For each* $f \in H^{(k)}(R^n, R_{n,s}) = \{f \in L^2(R^n, R_{n,s}) \mid F = P_{x_0} * f, x_0 > 0, D^{k-1}F(x_0, x) \neq 0,\ D^k F(x_0, x) = 0\}$, *if* $\sum_{j=1}^{s} x_j^2 \neq 0$, *then* $f = 0$; *otherwise,* f *can be determined by* 2^{n-1} *linearly independent function components.*

Similarly, we denote the conjugate Hardy space $\overline{H}^{(k)}(R^n, R_{n,s})$ by

$$\begin{aligned}\overline{H}^{(k)}(R^n, R_{n,s}) = \{f \in L^2(R^n, R_{n,s}) \mid F(x_0, x) = P_{x_0} * f(x),\\ x_0 > 0, \overline{D}^{k-1}F(x_0, x) \neq 0,\ \overline{D}^k F = 0\},\end{aligned}$$

where $k \in N$, $0 \leq s \leq n$.

From the above theorems, we have

Corollary 4. *In the conjugate Hardy space* $\overline{H}^{(k)}(R^n, R_n) = \{f(x) \in L^2(R^n, R_n) \mid F(x_0, x) = P_{x_0} * f(x), x_0 > 0, \overline{D}^{k-1}F(x_0, x) \neq 0\ \overline{D}^k F(x_0, x) = 0\}$, *each* f *is determined by* 2^{n-1} *linearly independent function components.*

Corollary 5. $\overline{H}^{(k)}(R^n, R_{n,n}) = \{0\}$.

F. Sommen proved a decomposition of $L^2(R^n, R_n)$ first in [12]. The inner product he used was Clifford algebra-valued. Here we can get the orthogonal decomposition of $L^2(R^n, R_n)$ by using our characterization of Hardy space and conjugate Hardy space:

$$L^2(R^n, R_n) = H^{(1)}(R^n, R_n) \oplus \overline{H}^{(1)}(R^n, R_n).$$

Now we give a natural method to obtain some compensated quantities. For each $f,\ g \in H^{(1)}(R^n, R_n)$, for convenience, we assume n is an even number. By using (8), in the product fg, we have

$$\begin{aligned}&Rf_0 Rg_I e_I\\ &= (f_0 + R_1 f_0 e_1 + R_2 f_0 e_2 + \cdots + R_n f_0 e_n)\\ &\quad \cdot (g_I + R_1 g_I e_1 + R_2 g_I e_2 + \cdots + R_n g_I e_n) e_I\\ &= [(f_0 g_I - \sum_{j=1}^{n} R_j f_0 R_j g_I) + \cdots + \sum_{j,k=1, j<k}^{n} (R_j f_0 R_k g_I - R_k f_0 R_j g_I) e_{jk}] e_I\end{aligned}$$

and

$$\begin{aligned}
&Rf_Ie_IRg_Je_J\\
&= (f_I + R_1f_Ie_1 + R_2f_Ie_2 + \cdots + R_nf_Ie_n)e_I\\
&\quad \cdot (g_J + R_1g_Je_1 + R_2g_Je_2 + \cdots + R_ng_Je_n)e_J\\
&= (f_I + R_1f_Ie_1 + R_2f_Ie_2 + \cdots + R_nf_Ie_n)\\
&\quad \cdot (g_J - \sum_{j\in I} R_jg_Je_j + \sum_{j\notin I} R_jg_Je_j)e_Ie_J\\
&= [(f_Ig_J + \sum_{j\in I} R_jf_IR_jg_J - \sum_{j\notin I} R_jf_IR_jg_J)\\
&\quad + \sum_{j\in I}(g_JR_jf_I - f_IR_jg_J)e_j\\
&\quad + \sum_{k\notin I}(g_JR_kf_I + f_IR_kg_J)e_k\\
&\quad + \sum_{j\in I, k\notin I, j<k} (R_jf_IR_kg_J + R_kf_IR_jg_J)e_{jk}\\
&\quad + \sum_{k,j\notin I, j<k} (R_jf_IR_kg_J - R_kf_IR_jg_J)e_{jk}\\
&\quad - \sum_{j,k\in I, j<k} (R_jf_IR_kg_J - R_kf_IR_jg_J)e_{jk}]e_Ie_J,
\end{aligned}$$

where $I \in \Omega \setminus \Lambda,\ I \neq \{0\}$.

If we denote

$$\begin{aligned}
&Q_0(u,v) = uv - \sum_{j=1}^{n} R_juR_jv, Q_j(u,v) = uR_jv - R_juv,\\
&Q_{j,k}(u,v) = R_juR_kv - R_kuR_jv,\\
&P_0(u,v) = uv + \sum_{j\in I} R_juR_jv - \sum_{k\notin I} R_kuR_kv, P_j(u,v) = R_juv - uR_1v,\\
&P_{j,k}(u,v) = R_juR_kv + R_kuR_jv,
\end{aligned}$$

where $j,\ k = 1, \cdots, n,\ j < k$, for real-valued functions $u, v \in L^2(R^n)$, then in the product fg, there are six kinds of components as above.

For a real-valued symbol function b, write

$$\langle Q_j(u,v), b\rangle = \langle H_b^j u, v\rangle,\ \langle P_j(u,v), b\rangle = \langle T_b^j u, g\rangle, \text{where } j = 0, 1, \cdots, n,$$
$$\langle Q_{j,k}(u,v), b\rangle = \langle H_b^{j,k} u, v\rangle,\ \langle P_{j,k}(u,v), b\rangle = \langle T_b^{j,k} u, v\rangle,$$

where, $j, k =, 1, 2, \cdots, n,\ j < k$.

Denote the corresponding Fourier kernels of H_b^j by $A_j(\xi,\eta)$, and the Fourier kernels of $H_b^{j,k}$ by $A_{j,k}(\xi,\eta)$. Then we obtain

$$A_0(\xi,\eta) = 1 - \frac{\xi\cdot\eta}{|\xi||\eta|} = \frac{1}{2}(\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|})^2,\ A_j(\xi,\eta) = -i(\frac{\xi_j}{|\xi|} - \frac{\eta_j}{|\eta|}),$$

where $j = 1, \cdots, n$,

$$A_{j,k}(\xi,\eta) = \frac{\eta_j\xi_k - \eta_k\xi_j}{|\xi||\eta|},\ j,k = 1,\cdots,n,\ j < k.$$

Similarly, denote the corresponding Fourier kernels of $T_b^j,\ T_b^{j,k}$ by $B_j(\xi,\eta),\ j = 0, 1, \cdots, n$, and $B_{j,k}(\xi,\eta)$, respectively. Then

$$B_0(\xi,\eta) = 1 + \sum_{j\in I}\frac{\xi_j\eta_j}{|\xi||\eta|} - \sum_{k\notin I}\frac{\xi_k\eta_k}{|\xi||\eta|},\ B_j(\xi,\eta) = i(\frac{\xi_j}{|\xi|} + \frac{\eta_j}{|\eta|}),\ j = 1,\cdots,n,$$

$$B_{j,k}(\xi,\eta) = \frac{\eta_j\xi_k + \eta_k\xi_j}{|\xi||\eta|},\ j,k = 1,\cdots,n, j < k.$$

Therefore we have six kinds of kernels: A_0; $A_j, j = 1,\cdots,n$; $A_{j,k}, j,k = 1,\cdots,n$, $j < k$; B_0; $B_j, j = 1,\cdots,n$; and $B_{j,k}, j,k = 1,\cdots,n, j < k$.

Using theorems in [8], we get $\langle Q_0(u,v), b\rangle$ is a compensated quantity which belongs to the real Hardy space $H^1(R^2)$, and its corresponding paracommutator is bounded if and only if $b \in BMO(R^2)$. It is compact if and only if $b \in VMO(R^2)$; for $\frac{2}{2} = 1 < p < \infty$, it belongs to S_p if and only if $b \in B_p(R^2)$; for $0 < p \le \frac{2}{2} = 1$, it belongs to S_p if and only if b is a polynomial.

$\langle Q_j(u,v), b\rangle,\ j = 1,\cdots,n$, are compensated quantities which belong to the real Hardy space $H^1(R^2)$, and their corresponding paracommutators are bounded if and only if $b \in BMO(R^2)$. They are compact if and only if $b \in VMO(R^2)$; for $2 < p < \infty$, they belong to S_p if and only if $b \in B_p(R^2)$; for $0 < p \le 2$, they belong to S_p if and only if b is a polynomial.

Similarly $Q_{j,k},\ j,k = 1,\cdots,n,\ j < k$, have the same properties. The quantities $P_j(u,v), j = 0,1,\cdots,n,\ P_{j,k}(u,v),\ j,k = 1,\cdots,nj < k$, are not compensated, they belong to $L^1(R^2)$ only. Their corresponding paracommutators are bounded iff $b \in L^\infty(R^2)$, and can never be compact unless $b \equiv 0$.

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