

ON THE SECOND EIGENVALUE OF THE LAPLACE OPERATOR ON A SPHERICAL BAND

CHUNG-TSUN SHIEH

(Communicated by Carmen C. Chicone)

ABSTRACT. In this paper we prove that the second eigenvalue of the Laplacian for a spherical band on the unit sphere S^2 has multiplicity 2. We also show that among all spherical bands of given fixed area less than 2π the second eigenvalue is maximized at the band which is symmetrical with respect to the equator.

1. INTRODUCTION

On the unit sphere S^2 , if we introduce the Euler coordinate

$$\begin{aligned}x &= \sin \phi \cos \theta, & y &= \sin \phi \sin \theta, & z &= \cos \phi, \\0 &\leq \phi \leq \pi, & 0 &\leq \theta < 2\pi,\end{aligned}$$

then the Laplace operator Δ_{S^2} on S^2 can be written as

$$(1.1) \quad \Delta_{S^2} u(\phi, \theta) = \frac{1}{\sin \phi} \left[\frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \phi} \frac{\partial u}{\partial \theta} \right) \right](\theta, \phi).$$

For $0 < \phi_1 < \phi_2 \leq \pi$, the set

$$\mathcal{B}(\phi_1, \phi_2) = \{(x, y, z) \in S^2 : \phi_1 \leq \phi \leq \phi_2\}$$

shall be called a *spherical band* in S^2 . For $\mathcal{B} = \mathcal{B}(\phi_1, \phi_2)$, consider the following problem:

$$(1.2) \quad \Delta_{S^2} u + \lambda u = 0, \quad u = 0 \text{ on } \partial \mathcal{B}.$$

Let $\{\lambda_n(\mathcal{B})\}_{n=1}^{\infty}$ denote the set of eigenvalues of (1.2), which are arranged in ascending order as

$$\lambda_1(\mathcal{B}) < \lambda_2(\mathcal{B}) \leq \dots.$$

The *multiplicity* of $\lambda_n(\mathcal{B})$ is the dimension of the space of eigenfunctions of (1.2) corresponding to $\lambda_n(\mathcal{B})$. In [7] we proved that, among all bands \mathcal{B} of given area, $\lambda_1(\mathcal{B})$ attains its maximum when \mathcal{B} is symmetric with respect to the equator. More precisely, we proved that $\lambda_1(\mathcal{B})$ is decreasing as \mathcal{B} moves from the equatorially symmetric band towards a pole. In this paper we prove that the multiplicity of $\lambda_2(\mathcal{B})$ is two for any band, and that among all bands of given fixed area smaller than 2π the function $\lambda_2(\mathcal{B})$ attains its maximum at the band which is symmetric with respect to the equator.

Received by the editors April 27, 2001 and, in revised form, July 8, 2002 and August 21, 2002.
2000 *Mathematics Subject Classification*. Primary 35P15.

2. THE MULTIPLICITY OF THE SECOND EIGENVALUE
OF A SPHERICAL BAND

Employing the technique of separation of variables in (1.2) by letting $u(\phi, \theta) = v(\phi)w(\theta)$ we separate (1.2) into two equations,

$$(2.1) \quad v''(\phi) + \cot \phi v'(\phi) + \left(\lambda - \frac{c^2}{\sin^2 \phi}\right)v(\phi) = 0, \quad v(\phi_1) = v(\phi_2) = 0,$$

$$(2.2) \quad w''(\theta) + c^2 w(\theta) = 0, \quad w^{(j)}(0) = w^{(j)}(2\pi) = 0, \quad j = 0, 1.$$

Note that the eigenvalues of the periodic eigenvalue problem (2.2) are of the form $c_n^2 = n^2$, $n = 0, 1, \dots$; 0 is a simple eigenvalue of (2.2), for $n > 0$ the multiplicity of n^2 is two. If we denote the k th eigenvalue and the corresponding eigenfunction of the Dirichlet eigenvalue of (2.1) by $\lambda_k(c)$ and $v_k(\phi; c)$, respectively, then the spectrum of (1.1) consists of $\lambda_k(n)$, where $k = 1, 2, \dots$ and $n = 0, 1, 2, \dots$. Note that $\lambda_1 = \lambda_1(0)$, which is a simple eigenvalue, and the eigenfunctions corresponding to $\lambda_k(n)$ are linear combinations of $v_k(\phi; n) \cos n\theta$ and $v_k(\phi; n) \sin n\theta$. To determine the multiplicity of an eigenvalue λ_* of (1.1) it is sufficient to find how many k 's and n 's are such that $\lambda_k(n) = \lambda_*$. This problem is in general difficult to deal with, but for the second eigenvalue of (1.1) the problem is much simpler. Indeed, Courant's nodal domain theorem tells us that the second eigenfunction of (1.1) has two nodal domains. On the other hand, among the eigenfunctions of (1.1) constructed above, only $v_2(\phi; 0)$, $v_1(\phi; 1) \cos \theta$ and $v_1(\phi; 1) \sin \theta$ have two nodal domains. Thus if we know the smallest between $\lambda_2(0)$ and $\lambda_1(1)$, then we can determine the multiplicity of λ_2 . The following theorem answers this question.

Theorem 2.1. $\lambda_2(\mathcal{B}) = \lambda_1(1)$. *The multiplicity of the second eigenvalue of (1.1) is two.*

Proof. To prove our assertion it is sufficient to prove that $\lambda_1(1) < \lambda_2(0)$. Since $\lambda_2(0)$ is the second eigenvalue of the eigenvalue problem

$$(2.3) \quad v''(\phi) + \cot \phi v'(\phi) + \lambda v(\phi) = 0, \quad v(\phi_1) = v(\phi_2) = 0,$$

by the classical Sturm-Liouville theory, $v_2(\phi; 0)$ have only one nodal point in (ϕ_1, ϕ_2) . Let ϕ_3 be the nodal point of $v_2(\phi; 0)$. Then there exist ξ, η such that $\phi_1 < \xi < \phi_3 < \eta < \phi_2$, and $v_2'(\xi; 0) = v_2'(\eta; 0) = 0$. Denote $f(\phi) = v_2'(\phi; 0)$. If we differentiate the equation in (2.3) with respect to ϕ , we find that

$$f''(\phi) + \cot \phi f'(\phi) + \left(\lambda_2(0) - \frac{1}{\sin^2 \phi}\right)f(\phi) = 0, \quad f(\xi) = f(\eta) = 0,$$

i.e., $\lambda_2(0)$ is an eigenvalue of the problem

$$(2.4) \quad f''(\phi) + \cot \phi f'(\phi) + \left(\lambda - \frac{1}{\sin^2 \phi}\right)f(\phi) = 0, \quad f(\xi) = f(\eta) = 0.$$

Note that $\lambda_1(1)$ is the first eigenvalue of the problem

$$(2.5) \quad v''(x) + \cot \phi v'(\phi) + \left(\lambda - \frac{1}{\sin^2 \phi}\right)v(\phi) = 0, \quad v(\phi_1) = v(\phi_2) = 0.$$

Since (ξ, η) is a proper subinterval of (ϕ_1, ϕ_2) , the first eigenvalue of (2.5) is strictly less than that of (2.4). Hence $\lambda_1(1) < \lambda_2(0)$. This completes the proof. \square

Following Theorem 2.1 we find that the nodal set of a second eigenfunction of a spherical band consists of two longitudes which separate the spherical band into two equal parts.

3. THE EXTREMAL PROBLEM FOR $\lambda_2(\mathcal{B})$

In this section, we prove the following theorem

Theorem 3.1. *Let $0 < A < 1$, and let \mathcal{B}_0 be the spherical band symmetric with respect to the equator and with area $2\pi A$. Then, for all bands \mathcal{B} with given area $2\pi A$ we have*

$$\lambda_2(\mathcal{B}) \leq \lambda_2(\mathcal{B}_0).$$

Before proving the theorem, we need some preliminary observations and results. Note that $\lambda_2(\mathcal{B})$ coincides with the first eigenvalue of the problem

$$(3.1) \quad \frac{d}{ds} \{(4s - s^2)f'(s)\} + \left(\nu - \frac{4}{4s - s^2}\right)f(s) = 0, \quad f(s_1) = f(s_2) = 0$$

which is obtained from (2.5) via change of variables

$$s = 2(1 - \cos \phi), \quad s_1 = 2(1 - \cos \phi_1), \quad s_2 = 2(1 - \cos \phi_2).$$

If $A < 2$ is fixed and $2\pi(s_2 - s_1) = 2\pi A$ is the area of the band, then the last problem is equivalent to

$$(3.2) \quad \begin{cases} \frac{d}{ds} \{(4(s + s_1) - (s + s_1)^2)f'(s)\} + \left(\nu - \frac{4}{4(s + s_1) - (s + s_1)^2}\right)f(s) = 0, \\ f(0) = f(2A) = 0, \end{cases}$$

simply by replacing s with $s_1 + s$.

Let $\nu_1(s_1) = \lambda_2(\mathcal{B})$, which is the first eigenvalue of the above problem. Let $\mathbb{H}_0 = W_0^{1,2}(0, 2A)$, and define for $f \in \mathbb{H}_0$

$$\mathfrak{R}(s_1; f) = \int_0^{2A} (4(s + s_1) - (s + s_1)^2)(f'(s))^2 ds + \int_0^{2A} \frac{4f^2(s)}{4(s + s_1) - (s + s_1)^2} ds.$$

By the classical Rayleigh-Ritz variational principle (see [1])

$$(3.3) \quad \nu_1(s_1) = \inf_{f \in \mathbb{H}_0} \frac{\mathfrak{R}(s_1; f)}{\int_0^{2A} f^2(s) ds}.$$

Lemma 3.1. *Given $0 < A < 1$, then for all $s_1 \geq 0$ with $2A + s_1 < 2$, if f is a C^2 -function defined in $[0, 2A]$, which is symmetric with respect to $s = A$ and such that $f(0) = f(2A) = 0$ and $f(x)$ is not identically zero for all $x \in (0, 2A)$, we have*

$$(3.4) \quad \frac{\int_0^{2A} (4 - 2(s + s_1))(f'(s))^2 dx}{\int_0^{2A} \frac{(1 - \frac{2(s + s_1)}{4})f^2(s)}{(s + s_1)^2(1 - \frac{s + s_1}{4})^2} ds} > 4 \frac{\int_0^A (f'(s))^2 ds}{\int_0^A \frac{f^2(s)}{(s + s_1)^2(1 - \frac{s + s_1}{4})^2} ds}.$$

Proof. Since $(f'(s))^2$ is symmetric with respect to $s = A$ on $[0, 2A]$, we have

$$(3.5) \quad \begin{aligned} & \int_0^{2A} (4 - 2(s + s_1))(f'(s))^2 ds \\ &= \int_0^A (4 - 2(s + s_1))(f'(s))^2 ds + \int_0^A (4 - 2(2A - s + s_1))(f'(s))^2 ds \\ &= 4[2 - (A + s_1)] \int_0^A (f'(s))^2 ds. \end{aligned}$$

On the other hand, for $0 < s < A$, $2A + s_1 < 2$, we have

$$(3.6) \quad (s + s_1)\left(1 - \frac{s + s_1}{4}\right) = 1 - \frac{(s + s_1 - 2)^2}{4} < 1 - \frac{(2A - s + s_1 - 2)^2}{4}.$$

Thus if we denote

$$\mathcal{G}_{s_1}(s) = (s + s_1)\left(1 - \frac{s + s_1}{4}\right),$$

then we have, for $0 < s < A$,

$$(3.7) \quad \mathcal{G}_{s_1}(s) < \mathcal{G}_{s_1}(2A - s),$$

where s_1 satisfies the condition $2A < 2A + s_1 < 2$. Now if $f(s)$ is an even function in $[0, 2A]$, using (3.7) we have

$$(3.8) \quad \begin{aligned} & \int_0^{2A} \frac{\left(1 - \frac{2(s+s_1)}{4}\right)f^2(s)}{(s+s_1)^2\left(1 - \frac{s+s_1}{4}\right)^2} ds \\ &= \int_0^A \frac{\left(1 - \frac{2(s+s_1)}{4}\right)f^2(s)}{(s+s_1)^2\left(1 - \frac{s+s_1}{4}\right)^2} ds + \int_0^A \frac{\left(1 - \frac{2(2A-s+s_1)}{4}\right)f^2(s)}{(2A-s+s_1)^2\left(1 - \frac{2A-s+s_1}{4}\right)^2} ds \\ &< \int_0^A \frac{\left[\left(1 - \frac{2(s+s_1)}{4}\right) + \left(1 - \frac{2(2A-s+s_1)}{4}\right)\right]f^2(s)}{(s+s_1)^2\left(1 - \frac{s+s_1}{4}\right)^2} ds \\ &= \int_0^A \frac{[2 - (A + s_1)]f^2(s)}{(s+s_1)^2\left(1 - \frac{s+s_1}{4}\right)^2} ds. \end{aligned}$$

(3.4) follows immediately from (3.5) and (3.8). \square

Lemma 3.2. *Given $0 < A < 1$, and $s_1 \geq 0$ with $0 < 2A + s_1 < 2$, let $f(x) \in C^2[0, A]$ with $f(0) = f'(A) = 0$ and $f(x)$ is not identically zero. Then*

$$(3.9) \quad \int_0^A (f'(s))^2 ds \geq \frac{1}{4} \int_0^A \frac{f^2(s)}{(s+s_1)^2\left(1 - \frac{s+s_1}{4}\right)^2} ds.$$

"=" could possibly hold only when $s_1 = 0$.

Proof. At first we treat only the case for $s_1 > 0$. For the case $s_1 = 0$, inequality follows simply from the $s_1 > 0$ inequality by letting $s_1 \rightarrow 0$. Let η_1 denote the first eigenvalue of the problem

$$(3.10) \quad w''(s) + \eta \frac{1}{s^2\left(1 - \frac{s}{4}\right)^2} w(s) = 0, \quad w(s_1) = w'(s_1 + A) = 0.$$

To prove our assertion for $s_1 > 0$ it is sufficient to prove that $\eta_1 > 1/4$. If the following transformations are made

$$\begin{aligned} z &= \frac{1}{K} \int_{s_1}^s \left(\frac{1}{s} + \frac{1}{4-s}\right) ds, \quad u = \left(\frac{1}{s} + \frac{1}{4-s}\right)^{1/2} w, \\ \rho &= K^2 \eta, \quad K = \frac{1}{\pi} \int_{s_1}^{s_1+A} \left(\frac{1}{s} + \frac{1}{4-s}\right) ds, \end{aligned}$$

(3.10) can be transformed into

$$(3.11) \quad u''(z) + [\rho - q(z)]u(z) = 0, \quad u(0) = u'(\pi) = 0,$$

where $q(z) = \theta''(z)/\theta(z)$, and $\theta(z) = \left(\frac{1}{s} + \frac{1}{4-s}\right)^{1/2}$. By a direct calculation we obtain $q(z) = \frac{K^2}{4}$, so that the first eigenvalue of (3.11) is $\rho_1 = 1/4 + K^2/4$. Hence we have $\eta_1 = \frac{1}{4} + \frac{1}{4K^2} > 1/4$, since $\eta_1 = \rho_1/K^2$. \square

Remark. Inequality (3.9) belongs to a class of general Wirtinger-type inequalities studied in [2]. In fact, if we take $p(x) = \eta_1/[s^2(1 - \frac{x}{4})^2]$, then the equation

$$y''(x) + p(x)y(x) = 0, \quad s_1 \leq x \leq s_1 + A$$

possesses a positive solution $y_1(x)$ with $y_1(s_1) = 0$, $y_1'(s_1) > 0$, since η_1 is the first eigenvalue of (3.10) and the corresponding eigenfunction never vanishes in $(s_1, s_1 + A)$. Inequality (3.9) follows from Theorem 1.1 in [2].

Now, let us go into the proof of our main theorem.

Proof of Theorem 3.1. It is enough to show that the first eigenvalue $\nu_1(s_1)$ of (3.2) attains its maximum at $s_1 = 2 - A$. Let $(\nu(s_1), f_1(s_1; s))$ denote the first normalized eigenpair of (3.2) with $f_1(s_1; s) > 0$ in $(0, 2A)$. Taking $f(s) \in \mathbb{H}_0$, we then have

$$(3.12) \quad \begin{aligned} \frac{\partial \mathfrak{R}(s_1; f)}{\partial s_1} &= \int_0^{2A} [4 - 2(s + s_1)](f'(s))^2 ds - \int_0^{2A} \frac{(1 - \frac{s+s_1}{2})f^2(s)}{(s + s_1)^2(1 - \frac{s+s_1}{4})^2} ds \\ &= \int_0^{2A} [4 - 2(s + s_1)](f'(s))^2 ds - \int_0^{2A} \frac{f^2(s)}{(s + s_1)^2} ds \\ &\quad + \int_0^{2A} \frac{f^2(s)}{[4 - (s + s_1)]^2} ds, \end{aligned}$$

$$(3.13) \quad \frac{\partial^2 \mathfrak{R}(s_1; f)}{\partial s_1^2} = -2 \int_0^{2A} (f'(s))^2 ds + \int_0^{2A} \frac{2f^2(s)}{(s + s_1)^3} ds + \int_0^{2A} \frac{2f^2(s)}{[4 - (s + s_1)]^3} ds,$$

and

$$(3.14) \quad \frac{\partial^3 \mathfrak{R}(s_1; f)}{\partial s_1^3} = -6 \int_0^{2A} \frac{f^2(s)}{(s + s_1)^4} ds + 6 \int_0^{2A} \frac{f^2(s)}{[(4 - (s + s_1))]^4} ds.$$

Notice that for the case $s_1 = 2 - A$ in (3.2), the corresponding spherical band is symmetric with respect to the equator, hence the corresponding first eigenfunction $f_1(2 - A; s)$ is symmetric with respect to $s = A$. Hence

$$\frac{\partial^3 \mathfrak{R}(s_1; f_1(2 - A; s))}{\partial s_1^3} = -6 \int_0^{2A} \frac{f_1^2(2 - A; s)}{(s + s_1)^4} ds + 6 \int_0^{2A} \frac{f_1^2(2 - A; s)}{[(s + (4 - 2A - s_1))]^4} ds.$$

Since $0 < s_1 < 2 - A$, we have $s_1 < 4 - 2A - s_1$, which implies that

$$\frac{1}{(s + s_1)^4} > \frac{1}{[s + (4 - 2A - s_1)]^4}$$

for all s in $[0, 2A]$. Hence for $0 < s_1 < 2 - A$,

$$(3.15) \quad \frac{\partial^3 \mathfrak{R}}{\partial s_1^3}(s_1; f_1(2 - A; s)) < 0.$$

Also notice that, as

$$\frac{\partial \mathfrak{R}}{\partial s_1}(2 - A, f_1(2 - A; s)) = \int_0^{2A} (2A - 2s)(f_1'(2 - A; s))^2 ds,$$

$(f_1'(2 - A; s))^2$ is symmetric with respect to $s = A$ on $[0, 2A]$ and $2A - 2s$ is anti-symmetric with respect $s = A$ on $[0, 2A]$, we have

$$(3.16) \quad \frac{\partial \mathfrak{R}}{\partial s_1}(2 - A, f_1(2 - A; s)) = 0.$$

Moreover, by Lemmas 3.1 and 3.2 and (3.15), we have that

$$(3.17) \quad \frac{\partial \mathfrak{R}}{\partial s_1}(s_1, f_1(2-A; s)) > 0, \text{ for } 0 < s_1 < 2-2A.$$

Hence, we have that $\frac{\partial \mathfrak{R}(s_1; f_1(2-A; s))}{\partial s_1}$ is concave, $\frac{\partial \mathfrak{R}(0^+; f_1(2-A; s))}{\partial s_1} \geq 0$ and $\frac{\partial \mathfrak{R}(2-A; f_1(2-A; s))}{\partial s_1} = 0$, so

$$(3.18) \quad \frac{\partial \mathfrak{R}}{\partial s_1}(s_1, f_1(2-A; s)) > 0, \text{ for } 0 < s_1 < 2-A.$$

Therefore, for $0 \leq s_1 < 2-A$,

$$(3.19) \quad \nu_1(2-A) = \mathfrak{R}(2-A, f_1(2-A; s)) \geq \mathfrak{R}(s_1, f_1(2-A; s)) > \nu_1(s_1).$$

This completes the proof. \square

We believe that Lemmas 3.1 and 3.2, and hence Theorem 3.1, can be extended to the case $0 < A < 2$. We also conjecture that λ_2 is monotonic in s_1 . We will prove the monotonicity of λ_2 for the flat 2-dimensional case in the following section.

4. MONOTONICITY OF THE SECOND DIRICHLET EIGENVALUE ON A FLAT 2-DIMENSIONAL ANNULUS OF GIVEN AREA

Let A_1 and A_2 be two fixed positive constants, $A_1 < A_2$ and $\mathcal{A}(A_1, A_2)$ denote the annulus $2A_1 \leq x^2 + y^2 \leq 2A_2$ in \mathbb{R}^2 . We denote $2\pi A$ as the area of $\mathcal{A}(A_1, A_2)$ and $\{\lambda_n(A_1)\}$ as the eigenvalues of the problem

$$(4.1) \quad \Delta u + \lambda u = 0 \text{ on } \mathcal{A}(A_1, A_2), \quad u|_{\partial \mathcal{A}(A_1, A_2)} = 0,$$

which is arranged in ascending order. Then by the same arguments as that in Theorem 2.1, we can show that the second eigenvalue of (4.1) is equal to the first eigenvalue ν_1 of the problem

$$(4.2) \quad z''(r) + \frac{1}{r}z'(r) + \left(\nu - \frac{1}{r^2}\right)z(r) = 0, \quad z(\gamma_1) = z(\gamma_2) = 0,$$

where $\gamma_j = (2A_j)^{1/2}$. Following the theory of Bessel functions (see [3, ch. 9] or [6]), we see that $\lambda_2(A_1) = \nu_1$ is the first zero of

$$\mathcal{F}(\lambda) = Y_1(\sqrt{\lambda}\gamma_1)J_1(\sqrt{\lambda}\gamma_2) - Y_1(\sqrt{\lambda}\gamma_2)J_1(\sqrt{\lambda}\gamma_1),$$

where $J_1(x)$ and $Y_1(x)$ denote the first and second kinds of Bessel functions of order 1, respectively. So if we know how the zeros of $\mathcal{F}(\lambda)$ varies as A_1 changes, we can know how $\lambda_2(A_1)$ changes as A_1 tends to infinity. However, the investigation for the zeros of $\mathcal{F}(\lambda)$ is a little bit complicated, so we avoid this approach. Note that (4.2) can be transformed into

$$(4.3) \quad \frac{d}{ds}(4sw'(s)) + \left(\nu - \frac{1}{s}\right)w(s) = 0, \quad w(s_1) = w(s_2) = 0$$

via the change of variables

$$s = r^2, \quad s_j = \gamma_j^2 = 2A_j, \quad j = 1, 2.$$

For convenience, we simply replace s with $s_1 + s$ to get

$$(4.4) \quad \frac{d}{ds}(4(s_1 + s)w'(s)) + \left(\nu - \frac{1}{s_1 + s}\right)w(s) = 0, \quad w(0) = w(2A) = 0.$$

Then we have

Theorem 4.1. *Given a positive constant $A > 0$, let $0 < A_1 < A_2$ be two positive numbers with $\mathcal{A}(A_1, A_2) = 2\pi A$, where $\mathbb{A}(A_1, A_2)$ is as that defined in section 2. Denote $\lambda_2(A_1)$ to be the second eigenvalue of*

$$(4.5) \quad \begin{cases} \Delta u + \lambda u = 0 & \text{on } \mathbb{A}(A_1, A_2), \\ u|_{\partial\mathbb{A}(A_1, A_2)} = 0. \end{cases}$$

Then, as a function of A_1 , $\lambda_2(A_1)$ is monotonically increasing. Moreover

$$\lim_{A_1 \rightarrow \infty} \lambda_2(A_1) = \infty.$$

Proof. Let $2A_j = s_j$, $j = 1, 2$, and let $w(s)$ be the normalized eigenfunction corresponding to the first eigenvalue $\nu_1(s_1) = \lambda_2(A_1)$ of (4.4). Then

$$\nu_1(s_1) = \int_0^{2A} 4(s + s_1)[w'(s)]^2 ds + \int_0^{2A} \frac{[w(s)]^2}{s + s_1} ds.$$

Define the functional

$$\mathcal{K}(s_1; w) = \int_0^{2A} 4(s + s_1)[w'(s)]^2 ds + \int_0^{2A} \frac{[w(s)]^2}{s + s_1} ds.$$

Then by the following inequality due to Hardy and Littlewood ([4], Thm. 253)

$$\int_0^\infty 4(f'(x))^2 dx > \int_0^\infty \frac{f^2(x)}{x^2} dx$$

for all f with $f' \in L^2$ and $f(0) = 0$, we have

$$\frac{\partial \mathcal{K}(s_1; w)}{\partial s_1} = \int_0^{2A} 4[w'(s)]^2 ds - \int_0^{2A} \frac{w(s)^2}{(s + s_1)^2} ds > 0.$$

Hence, for $\tilde{s} < s_1$,

$$\nu_1(\tilde{s}) < \mathcal{K}(\tilde{s}; w) < \mathcal{K}(s_1; w) = \nu_1(s_1),$$

i.e., $\lambda_2(A_1) = \nu_1(s_1)$ is monotonically increasing.

Next, let us compute the limit of $\lambda_2(A_1)$. For s_1 very large and $f \in W_0^{1,2}(0, 2A)$ with $\int_0^{2A} f^2 ds = 1$, we have

$$(4.6) \quad 4s_1 \int_0^{2A} [w'(s)]^2 ds \leq \int_0^{2A} 4(s + s_1)[w'(s)]^2 ds + \int_0^{2A} \frac{[w(s)]^2}{s + s_1} ds.$$

Taking the infimum at both sides, we have $\lim_{A_1 \rightarrow \infty} \lambda_2(A_1) = \infty$. □

ACKNOWLEDGMENTS

This research was partially supported by the National Science Council, Republic of China. The author thanks the referee of this paper for his helpful suggestions and comments.

REFERENCES

1. C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, Landon, 1980. MR **81e**:35095
2. P. R. Beesack, *Integral Inequalities of the Wirtinger Type*, Duke. Math. J. vol. 25(1958), pp. 477-498. MR **20**:3947
3. R. Courant and D. Hilbert, *Methods of Mathematical Physics, vol. I*, Wiley-Interscience, New York-London, 1953. MR **16**:426a
4. G. H. Hardy, J. E. Littlewood and G. Polya. *Inequalities*, Cambridge University Press, 1952. MR **13**:727e
5. E. L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.
6. J. R. Kuttler and V. G. Sigillito, *Eigenvalues of the Laplacian in Two Dimensions*, SIAM Review, vol 26(1984), no. 2 pp. 163-193. MR **85k**:65086
7. C.-L. Shen and C.-T. Shieh, *Some properties of the first eigenvalue of the Laplace operator on the spherical bands in S^2* , SIAM J. Math. Anal., vol. 23(1992), 1305–1308. MR **93f**:35162
8. G. P. Tolstov, *Fourier Series*, Dover, New York, 1962.

DEPARTMENT OF MATHEMATICS, FU-JEN CATHOLIC UNIVERSITY, TAIPEI, TAIWAN, REPUBLIC OF CHINA

E-mail address: ctshieh@math.fju.edu.tw