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A BERNSTEIN THEOREM FOR COMPLETE SPACELIKE HYPERSURFACES OF CONSTANT MEAN CURVATURE IN MINKOWSKI SPACE

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ABSTRACT. In this paper we prove a general Bernstein theorem on the complete spacelike constant mean curvature hypersurfaces in Minkowski space. The result generalizes the previous result of Cao-Shen-Zhu (1998) and Xin (1991). The proof again uses the fact that the Gauss map of a constant mean curvature hypersurface is harmonic, which was proved by K. T. Milnor (1983), and the maximum principle of S. T. Yau (1975).

1. Introduction

The classical Bernstein theorem says that a minimal graph in $\mathbf{R}^3$ is a plane. The high-dimensional generalization was extensively studied. In 1968, E. Calabi proposed to study a similar problem in the Minkowski space. He considered the maximum (the mean curvature $H = 0$) spacelike hypersurface M in the Lorentz-Minkowski space $\mathbf{R}_1^{n+1}$ with coordinates $(x_0, x_1, \cdots, x_n)$ and metric

$$ds^2 = -(dx_0)^2 + \sum_{i=1}^{n} (dx_i)^2.$$

If M is given by the graph of a function u, the equation has the following form:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial u}{\partial x_i}}{\sqrt{1 - |\nabla u|^2}} \right) = 0. \tag{1.1}$$

If $|\nabla u| < 1$, the graph u is spacelike. In [1], Calabi showed that for $2 \leq n \leq 4$, the graph of any solution u to (1.1) is a hyperplane. In [3], a much more general theorem was proved by Cheng and Yau. Namely they showed that any maximum spacelike hypersurface in R_1^{n+1} is a hyperplane. As a next step, the spacelike hypersurfaces with constant mean curvature were studied. The effective tool in this study is using the Harmonic maps theory. The hinge between the constant mean curvature hypersurfaces and the harmonic maps is the following results of K. T. Milnor:

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Theorem 1.1. *The spacelike hypersurface M in the Lorentz-Minkowski space $\mathbf{R}^{n+1}_1$ has constant mean curvature if and only if the Gauss map (see the following section for the definition) is a harmonic map from M to the hyperbolic space $\mathbf{H}^n$.*

Using this observation, several Bernstein type theorems have been proved. For example, it was proved by Xin in [8] that if the image of the Gauss map is inside a bounded subdomain of $\mathbf{H}^n$, M must be a hyperplane. A similar result was also proved earlier in [7] with extra assumptions. In [2], Cao, Shen and Zhu further extended the result by showing that if the image of the Gauss map lies inside a horoball, M is necessarily a hyperplane. There are also very important works of Treiberg and Choi-Treiberg [5]. In this paper, we will prove the following generalization of the above mentioned results:

Theorem 1.2. *Let M be a spacelike constant mean curvature hypersurface in the Lorentz-Minkowski space $\mathbf{R}^{n+1}_1$. Assume that the image of the Gauss map is a bounded distance away from a geodesic in $\mathbf{H}^n$. Then either M is a hyperplane or a hyperbolic cylinder.*

Our proof uses both methods of Cao-Shen-Zhu [2] and Xin [8], and a result of Choi-Treiberg [5]. The key here is the maximum principle of Cheng-Yau. The argument also simplifies the proofs in [2] and [8] quite a bit.

2. Preliminaries

In this section we shall collect some basic facts on Lorentz geometry and recall some formulas. We also derive Theorem 1.1 out of the well-known Gauss-Codazzi equation. We include these for the reader's convenience.

We use the moving frame notation. Let L be an $n+1$-Lorentzian manifold with metric $\widehat{ds}^2$ of signature $(-,+,\cdots,+)$. Let $(e_0,e_1,\cdots,e_n)$ be a local orthonormal moving frame and $(\omega_0,\omega_1,\cdots,\omega_n)$ is the dual frame so that $\widehat{ds}^2=-\omega_0^2+\sum_{i=1}^n\omega_i^2$. We have the structure equation

$$d\omega_0=\omega_{0i}\wedge\omega_i, \tag{2.1}$$

$$d\omega_i=-\omega_{i0}\wedge\omega_0+\omega_{ij}\wedge\omega_j, \tag{2.2}$$

$$\omega_{\alpha\beta}=-\omega_{\beta\alpha}. \tag{2.3}$$

We then use $\widehat{D}$ and $\widehat{\Omega}$ to denote the connection and the curvature form of L. They can be derived from the structure equation:

$$\begin{aligned}
\widehat{D}e_0 &= \omega_{0i}e_i,\\
\widehat{D}e_i &= \omega_{ij}e_j-\omega_{i0}e_0,\\
\widehat{\Omega}_{0i} &= d\omega_{0i}-\omega_{0k}\wedge\omega_{ki},\\
\widehat{\Omega}_{ij} &= d\omega_{ij}+\omega_{i0}\wedge\omega_{0j}-\omega_{ik}\wedge\omega_{kj},\\
\widehat{\Omega}_{\alpha\beta} &= -\frac{1}{2}\widehat{R}_{\alpha\beta\gamma\delta}\omega_\gamma\wedge\omega_\delta.
\end{aligned}$$

Let M be a spacelike hypersurface in L. We choose the local frame such that $(e_1,\cdots,e_n)$, restricted to M, is a local orthonormal frame for M and e_0 is normal to M. Then the induced Riemannian metric ds^2 on M is given by $ds^2=\sum_{i=1}^n\omega_i^2$.

We also have the induced structure equations

$$\begin{aligned} d\omega_i &= \omega_{ik} \wedge \omega_k, \\ d\omega_{ij} &= \omega_{ik} \wedge \omega_{kj} - \omega_{i0} \wedge \omega_{0j} + \widehat{\Omega}_{ij}, \\ \omega_{ij} &= -\omega_{ji}, \\ \Omega_{ij} &= d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l. \end{aligned}$$

Here Ω is the curvature form of M. The second fundamental form $h = h_{ij}\omega_i \otimes \omega_j$ is given by Cartan's Lemma:

$$\omega_{0i} = h_{ij}\omega_j.$$

In case L is the Minkowski space we have the Gauss equation

$$R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}) \tag{2.4}$$

and the Codazzi equation

$$h_{ijk} = h_{ikj}, \tag{2.5}$$

where h_{ijk} is the covariant derivative of h_{ij} defined as

$$\sum h_{ijk}\omega_k = dh_{ij} + \sum h_{ik}\omega_{kj} + \sum h_{kj}\omega_{ki}.$$

The Gauss map is defined by e_0. Namely,

$$F = e_0 : M \to \mathbf{H}^n.$$

Clearly,

$$dF = h_{ik}\omega_i \otimes e_k. \tag{2.6}$$

Here we also use $\{e_k\}$ as an orthonormal frame of $\mathbf{H}^n$ near the image of F. Therefore we have that

$$\tau(F) = \sum h_{kii} e_k$$

where $\tau(F)$ is the tension field of F. By the Codazzi equation,

$$\tau(F) = \sum (H)_k e_k,$$

where $H = \sum h_{ii}$ is the mean curvature of M. From here on Theorem 1.1 follows easily. From (2.6) we also know that

$$2e(F) = \|h\|^2, \tag{2.7}$$

where $e(F)$ is the energy density of F.

3. Proof of the theorem

Now let M be a spacelike hypersurface of constant mean curvature in $\mathbf{R}_1^{n+1}$. By Theorem 1.1, we know that F is a harmonic map from M to $\mathbf{H}^n$. Before we prove the theorem we first derive some basic differential inequalities which can also be used to simplify the proof of the theorems in [2] and [8]. Let g be a function defined over the image of F, the Gauss map. Denote $f = g \circ F$. We calculate ∇f and Δf:

$$\begin{aligned} \nabla_i f &= \langle dF(e_i), \nabla g \rangle \\ &= \langle h_{ik} e_k, \nabla g \rangle, \\ \Delta f &= \nabla_i \langle dF(e_i), \nabla g \rangle \\ &= \langle \tau(F), \nabla g \rangle + \langle dF(e_i), \nabla_{dF(e_i)} \nabla g \rangle \\ &= \text{Hessian}(g)(dF(e_i), dF(e_i)). \end{aligned}$$

If the image of F lies inside a ball of finite radius, which is assumed in Xin's theorem, we can choose g to be the square of the distance function. We then have that

$$\Delta f \geq 2|dF(e_i)|^2$$

by Hessian comparison. Thus

$$\Delta f \geq 2|h|^2 \geq 2nH^2.$$

By the maximum principle of Cheng-Yau [4] it would imply $H \equiv 0$. We then know that M is a hyperplane by Cheng-Yau's theorem in [3]. Hence we proved Xin's theorem in [8]. Here since the Ricci curvature has a lower bound, from the Gauss equation we can apply Cheng-Yau's maximum principle. If the image of the Gauss map lies inside a horoball we can choose g to be the minus of the Buseman function. We then, by (31) in [2], have that

$$|\nabla f|^2 + \Delta f \geq \frac{n}{2}H^2.$$

Applying the maximum principle of Cheng-Yau again we proved the theorem of Cao-Shen-Zhu in [2]. The proof of Theorem 1.2 follows along the same line. By the assumption, we have a geodesic γ such that $F(M)$ lies inside a tubular neighborhood of γ. We choose g to be the square of the distance function to γ. We need to calculate the Hessian of g first. By the Jacobi field calculation we know that Hessian(g) has eigenvalues $2\rho\frac{\sinh\rho}{\cosh\rho}, 2\rho\frac{\cosh\rho}{\sinh\rho}, \cdots, 2\rho\frac{\cosh\rho}{\sinh\rho}, 2$. Here $\rho = \sqrt{g}$. Then for $f = g \circ F$ we have the following inequality:

$$\Delta f \geq \min\left(2, \rho\frac{\sinh\rho}{\cosh\rho}\right)nH^2. \tag{3.1}$$

By the maximum principle of Cheng-Yau we know that either $f \equiv 0$ or $H \equiv 0$. In the latter case we conclude that M is a hyperplane rightway. In the former case, we know that the Gauss map lies totally inside γ. If it is not a constant, by the splitting result of Choi-Treiberg [5] we know that M is a hyperbolic cylinder $\mathbf{H}^1 \times R^{n-1}$. If the Gauss map is a consant we again have $H \equiv 0$, therefore M is a hyperplane.

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