PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 1, Pages 147–156
S 0002-9939(03)07075-8
Article electronically published on April 24, 2003

THE FORM SUM AND THE FRIEDRICHS EXTENSION OF SCHRÖDINGER-TYPE OPERATORS ON RIEMANNIAN MANIFOLDS

OGNJEN MILATOVIC

(Communicated by David S. Tartakoff)

ABSTRACT. We consider $H_V = \Delta_M + V$, where (M, g) is a Riemannian manifold (not necessarily complete), and Δ_M is the scalar Laplacian on M. We assume that $V = V_0 + V_1$, where $V_0 \in L^2_{\text{loc}}(M)$ and $-C \leq V_1 \in L^1_{\text{loc}}(M)$ (C is a constant) are real-valued, and $\Delta_M + V_0$ is semibounded below on $C_c^\infty(M)$. Let T_0 be the Friedrichs extension of $(\Delta_M + V_0)|_{C_c^\infty(M)}$. We prove that the form sum $T_0 \tilde{+} V_1$ coincides with the self-adjoint operator T_F associated to the closure of the restriction to $C_c^\infty(M) \times C_c^\infty(M)$ of the sum of two closed quadratic forms of T_0 and V_1. This is an extension of a result of Cycon. The proof adopts the scheme of Cycon, but requires the use of a more general version of Kato's inequality for operators on Riemannian manifolds.

1. Introduction and the main result

Let (M, g) be a Riemannian manifold (i.e. M is a C^∞-manifold, (g_{jk}) is a Riemannian metric on M), $\dim M = n$. We will assume that M is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \dots, x^n$ there exists a strictly positive C^∞-density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2 \dots dx^n$. We do not assume that (M, g) is complete.

We will consider a Schrödinger-type operator of the form

$$H_V = \Delta_M + V.$$

Here $\Delta_M := d^*d$, where $d\colon C^\infty(M) \to \Omega^1(M)$, and $V \in L^1_{\text{loc}}(M)$ is real-valued.

1.1. **Maximal operator.** We define the maximal operator $H_{V,\max}$ associated to H_V as an operator in $L^2(M)$ given by $H_{V,\max}u = H_V u$ with domain

$$\operatorname{Dom}(H_{V,\max}) = \{u \in L^2(M) : Vu \in L^1_{\text{loc}}(M),\ H_V u \in L^2(M)\}. \tag{1.1}$$

Here $\Delta_M u$ in $H_V u = \Delta_M u + Vu$ is understood in the distributional sense.

We make the following assumptions on V.

Assumption A. Assume $V = V_0 + V_1$, where

(i) $V_0 \in L^2_{\text{loc}}(M)$ and $\Delta_M + V_0$ is semibounded below on $C_c^\infty(M)$.
(ii) $V_1 \in L^1_{\text{loc}}(M)$ and $V_1 \geq -C$, where $C > 0$ is a constant.

Received by the editors August 20, 2002.
2000 *Mathematics Subject Classification.* Primary 35P05, 58G25; Secondary 47B25, 81Q10.

1.2. **Quadratic forms.** For any self-adjoint operator $T\colon \mathrm{Dom}(T) \subset L^2(M) \to L^2(M)$ such that $T \geq -\alpha$, we will denote by $Q(T)$ the domain of the quadratic form t associated to T. By Theorem 2.1 in [3], t is a closed semibounded below form, i.e. $Q(T)$ is a Hilbert space with the inner product

$$(u,v)_t = t(u,v) + (1+\alpha)(u,v)_{L^2(M)}, \tag{1.2}$$

where $t(\cdot,\cdot)$ is the sesquilinear form obtained by polarization of t.

1.3. **Form sum.** By (i) of Assumption A, $\Delta_M + V_0$ is symmetric and semibounded below on $C_c^\infty(M)$, so we can associate to it a semibounded below self-adjoint operator T_0 (Friedrichs extension, cf. Theorem 14.1 in [3]).

We will denote by $T_0 \tilde{+} V_1$ the form sum of T_0 and V_1. By Theorem 4.1 in [3], this is the self-adjoint operator associated to the semibounded below closed quadratic form t_q given by the sum of two semibounded below closed quadratic forms corresponding to T_0 and V_1. By the same theorem, the following is true: $Q(T_0 \tilde{+} V_1) = Q(T_0) \cap Q(V_1)$. Clearly, $T_0 \tilde{+} V_1$ is a self-adjoint restriction of $H_{V,\max}$.

1.4. **Operator** T_F**.** Denote by $t_{\min}$ the restriction of t_q to $C_c^\infty(M) \times C_c^\infty(M)$. Denote by T_F the self-adjoint operator associated to the closure of $t_{\min}$ in the sense of the norm in $Q(T_0 \tilde{+} V_1)$. Clearly, T_F is a self-adjoint restriction of $H_{V,\max}$.

We will give a sufficient condition for $T_F = T_0 \tilde{+} V_1$.

Theorem 1.5. *Suppose that Assumption A holds. Then* $T_F = T_0 \tilde{+} V_1$*.*

Remark 1.6. Theorem 1.5 was proven by Cycon [2] in the case of the operator $-\Delta + V$ in an open set $M \subset \mathbb{R}^n$, where Δ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric. In the case $V_0 = 0$ and $M = \mathbb{R}^n$ with standard metric, Theorem 1.5 was proven in Simon [13].

2. Operators with a positive form core

Definition 2.1. Let $T\colon C_c^\infty(M) \subset L^2(M) \to L^2(M)$ be a symmetric semibounded below operator. Let T_F denote its Friedrichs extension and $Q(T_F)^+$ the set of a.e. positive elements of $Q(T_F)$. We say that T_F has a positive form core if for every $u \in Q(T_F)^+$ there exists a sequence $u_k \in C_c^\infty(M)^+$ such that

$$\|u_k - u\|_t \to 0 \qquad \text{as } k \to \infty,$$

where $\|\cdot\|_t$ is the norm associated to the closure of quadratic form $t(v,w) := (Tv, w)$ $(v, w \in C_c^\infty(M))$ via (1.2).

The main result of this section is

Theorem 2.2. *Suppose that* $\Delta_M + V_0$ *is as in* (i) *of Assumption* A*. Let* T_0 *be the Friedrichs extension of* $(\Delta_M + V_0)|_{C_c^\infty(M)}$*. Then* T_0 *has a positive form core.*

Remark 2.3. In the case of the operator $-\Delta + V_0$ in an open set $M \subset \mathbb{R}^n$, Theorem 2.2 was proven in [2, Th. 1].

We will first prove the following special case of Theorem 2.2

Proposition 2.4. *Suppose that* $-C \leq V_0 \in L^2_{\mathrm{loc}}(M)$*, where* $C > 0$ *is a constant. Let* T_b *be the Friedrichs extension of* $(\Delta_M + V_0)|_{C_c^\infty(M)}$*. Then* T_b *has a positive form core.*

We begin with a few preliminary lemmas.

In what follows T_b is as in the hypothesis of Proposition 2.4, and t_b is the closed quadratic form associated with T_b. Without loss of generality, we may and we will assume that $V_0 \geq 0$ so that T_b is a positive self-adjoint operator.

We will denote $W^{1,2}(M) := \{u \in L^2(M) \colon du \in L^2(T^*M)\}$. By $W_0^{1,2}(M)$ we will denote the closure of $C_c^\infty(M)$ in the norm $\|u\|^2_{W^{1,2}} := \|du\|^2 + \|u\|^2$, where $\|\cdot\|$ is the L^2 norm. By $Q(V_0)$ we will denote the set $\{u \in L^2(M) \colon V_0^{1/2}u \in L^2(M)\}$. Clearly, $Q(V_0)$ is the closure of $C_c^\infty(M)$ in the norm

$$\|u\|^2_{V_0} := \|V_0^{1/2}u\|^2 + \|u\|^2, \tag{2.1}$$

where $\|\cdot\|$ is the norm in $L^2(M)$.

In the proofs of the following three lemmas, we will use the arguments from the proof of Lemma 1 in [5].

Lemma 2.5. *$Q(T_b) = W_0^{1,2}(M) \cap Q(V_0)$.*

Proof. Denote by $\mathcal{H}_1 := W_0^{1,2}(M) \cap Q(V_0)$. Consider a sesquilinear form $S \colon \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$ given by

$$S(u,v) := (du, dv) + (V_0^{1/2}u, V_0^{1/2}v),$$

where $(\cdot,\cdot)$ is the inner product in L^2.

This sesquilinear form is closed, so the pre-Hilbert space $\mathcal{H}_1$ is complete in the norm

$$(u,v)_{t_b} := (du, dv) + (V_0^{1/2}u, V_0^{1/2}v) + (u,v). \tag{2.2}$$

By definition of $W_0^{1,2}(M)$ and $Q(V_0)$, it follows that $\mathcal{H}_1$ is the closure of $C_c^\infty(M)$ in the norm $\|\cdot\|_{t_b}$ corresponding to (2.2).

For all $u, v \in C_c^\infty(M)$, $(u,v)_{t_b} = (u,v) + (T_b u, v)$. By Theorem 14.1 in [3], $Q(T_b)$ is the closure of $C_c^\infty(M)$ in the norm $\|\cdot\|_{t_b}$ corresponding to (2.2), so $Q(T_b) = W_0^{1,2}(M) \cap Q(V_0)$. □

Lemma 2.6. *Assume that $u \in C_c^\infty(M)$. Then there exists a sequence $\phi_k \in C_c^\infty(M)^+$ such that $\|\phi_k - |u|\|_{t_b} \to 0$ as $k \to \infty$, where $\|\cdot\|_{t_b}$ is the norm corresponding to* (2.2)*.*

Proof. Let $u \in C_c^\infty(M)$. Then $|u| \in W^{1,2}_{\mathrm{comp}}(M)$. Using a partition of unity we may assume that u is supported in a coordinate neighborhood. Let $|u|^\rho = J^\rho|u|$, where J^ρ is the Friedrichs mollifying operator; cf. Sect. 5.11 in [1]. Then $|u|^\rho \in C_c^\infty(M)$. It is well-known that $|u|^\rho \to |u|$ as $\rho \to 0+$ both in the space $W^{1,2}_{\mathrm{comp}}(M)$ and in the space $L^2_{\mathrm{comp}}(M)$. Also, since $|u|$ is continuous compactly supported on M and $V_0 \in L^2_{\mathrm{loc}}(M)$, we have

$$\int V_0(|u|^\rho)^2\,d\mu \to \int V_0|u|^2\,d\mu \qquad \text{as } \rho \to 0+. \tag{2.3}$$

Therefore,

$$\||u|^\rho - |u|\|_{t_b} \to 0 \qquad \text{as } \rho \to 0+, \tag{2.4}$$

where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2). □

Lemma 2.7. *Suppose that $u \in Q(T_b)$. Then $|u| \in Q(T_b)$.*

Proof. Let $u \in Q(T_b)$. By Lemma 2.5, we get $u \in W_0^{1,2}(M) \cap Q(V_0)$. Since $u \in W_0^{1,2}(M)$, Lemma 7.6 from [4] gives $|u| \in W_0^{1,2}(M)$. From $u \in Q(V_0)$, we immediately get $|u| \in Q(V_0)$. Therefore, $|u| \in W_0^{1,2}(M) \cap Q(V_0)$, so by Lemma 2.5, we obtain $|u| \in Q(T_b)$. □

2.8. **Proof of Proposition 2.4.** We will follow the proof of Lemma 2 in [2].

Suppose that $u \in Q(T_b)^+$. By Lemma 2.5, there exists a sequence $\phi_j \in C_c^\infty(M)$ such that

$$\|\phi_j - u\|_{t_b} \to 0 \qquad \text{as } j \to \infty, \tag{2.5}$$

where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2).

In what follows, we will denote $(\operatorname{sign} w)(x) := \frac{w(x)}{|w(x)|}$ when $w(x) \neq 0$, and 0 otherwise.

We have

$$\begin{aligned} \||\phi_j| - u\|_{t_b}^2 &= \||\phi_j| - u\|^2 + \|d|\phi_j| - du\|^2 + \|V_0^{1/2}(|\phi_j| - u)\|^2 \\ &\leq \|\phi_j - u\|^2 + \|d|\phi_j| - du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2 \\ &= \|\phi_j - u\|^2 + \|\operatorname{Re}((\operatorname{sign}\bar{\phi}_j)d\phi_j) - du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2, \end{aligned} \tag{2.6}$$

where $\|\cdot\|$ denotes the norm L^2.

From (2.6) we obtain

$$\begin{aligned} \||\phi_j| - u\|_{t_b}^2 \leq{}& \|\phi_j - u\|^2 + [\|(\operatorname{sign}\bar{\phi}_j)(d\phi_j - du)\| + \|(\operatorname{sign}\bar{\phi}_j - 1)du\|]^2 \\ &+ \|V_0^{1/2}(\phi_j - u)\|^2 \\ \leq{}& \|\phi_j - u\|^2 + [\|d\phi_j - du\| + \|(\operatorname{sign}\bar{\phi}_j - 1)du\|]^2 + \|V_0^{1/2}(\phi_j - u)\|^2, \end{aligned} \tag{2.7}$$

where $\|\cdot\|$ denotes the norm in L^2.

By (2.5), the first, second and fourth term on the right-hand side of (2.7) go to 0 as $j \to \infty$.

It remains to show that

$$\|(\operatorname{sign}\bar{\phi}_j - 1)du\| \to 0 \qquad \text{as } j \to \infty. \tag{2.8}$$

Since $\phi_j \to u$ in $L^2(M)$, a lemma of Riesz shows that there exists a subsequence ϕ_{j_k} such that $\phi_{j_k} \to u$ a.e. $d\mu$, as $k \to \infty$. By Lemma 7.7 from [4], it follows that $du = 0$ almost everywhere on $\{x \in M : u(x) = 0\}$. Hence, as $k \to \infty$, $\operatorname{sign}\bar{\phi}_{j_k} \to 1$ a.e. on M. Since $du \in L^2(T^*M)$, dominated convergence theorem immediately implies (2.8) (after passing to the chosen subsequence ϕ_{j_k}).

This shows that

$$\||\phi_{j_k}| - u\|_{t_b} \to 0 \qquad \text{as } k \to \infty. \tag{2.9}$$

By (2.9) and Lemma 2.6, there exists a sequence ψ_l in $C_c^\infty(M)^+$ such that $\|\psi_l - u\|_{t_b} \to 0$ as $l \to \infty$. By Definition 2.1 it follows that T_b has a positive form core. □

In what follows, we will use a version of Kato's inequality. For the proof of this inequality in general setting, cf. Theorem 5.6 in [1].

Theorem 2.9. *Let E be a Hermitian vector bundle on M, and let $\nabla\colon C^\infty(E) \to C^\infty(T^*M \otimes E)$ be a Hermitian connection on E. Let $\nabla^*\colon C^\infty(T^*M \otimes E) \to C^\infty(E)$*

be formal adjoint of ∇ with respect to the usual inner product on $L^2(E)$. Assume that $u \in L^1_{\mathrm{loc}}(E)$ and $\nabla^*\nabla u \in L^1_{\mathrm{loc}}(E)$. Then

$$\Delta_M|u| \;\leq\; \mathrm{Re}\langle \nabla^*\nabla u, \operatorname{sign} u\rangle, \tag{2.10}$$

where

$$\operatorname{sign} u(x) \;=\; \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.10. Let (X, μ) be a measure space. A bounded linear operator $A\colon L^2(X,\mu) \to L^2(X,\mu)$ is said to be *positivity preserving* if for every $0 \leq u \in L^2(X,\mu)$ we have $Au \geq 0$.

We will also use the following abstract theorem due to Simon; cf. Theorem 2.1 in [11].

Theorem 2.11 (Simon [11])**.** *Suppose that (X,μ) is a measure space. Suppose that H is a positive self-adjoint operator in $L^2(X,\mu)$. Then $(H+1)^{-1}$ is positivity preserving if and only if the following two conditions are satisfied*:

(i) *For every $u \in Q(H)$, we have $|u| \in Q(H)$.*
(ii) *For every $u \in \mathrm{Dom}(H)$ and $0 \leq v \in Q(H)$, the following is true*:

$$\mathrm{Re}[h(|u|, v)] \leq \mathrm{Re}((\operatorname{sign} u)v, Hu),$$

where h is the quadratic form associated to H, and $(\operatorname{sign} u)(x) = \frac{u(x)}{|u|}$ whenever $u(x) \neq 0$, and 0 otherwise.

The following lemma extends Lemma 1 from [5] to the case of Riemannian manifolds.

Lemma 2.12. *The operator $(T_b+1)^{-1}$ is positivity preserving.*

Proof. Let t_b be the quadratic form associated to T_b. By Theorem 2.11, it suffices to check the following conditions:

(i) For every $u \in Q(T_b)$, we have $|u| \in Q(T_b)$ and
(ii) For every $u \in \mathrm{Dom}(T_b)$ and $0 \leq v \in Q(T_b)$, the following is true:

$$\mathrm{Re}[t_b(|u|, v)] \leq \mathrm{Re}((\operatorname{sign} u)v, T_b u).$$

Condition (i) follows immediately by Lemma 2.7.

We now prove that condition (ii) holds. Let $u \in \mathrm{Dom}(T_b)$. Then $(\Delta_M + V_0)u \in L^2(M)$ and hence $\Delta_M u \in L^1_{\mathrm{loc}}(M)$.

For $u \in \mathrm{Dom}(T_b)$ and $0 \leq \phi \in C_c^\infty(M)$ we have

$$\begin{aligned} \mathrm{Re}[t_b(|u|, \phi)] &= \mathrm{Re}(|u|, (\Delta_M + V_0)\phi) = (|u|, \Delta_M\phi) + (|u|, V_0\phi) \\ &= (\Delta_M|u|, \phi) + (V_0|u|, \phi) \\ &\leq \mathrm{Re}((\operatorname{sign}\bar{u})\Delta_M u, \phi) + ((\operatorname{sign}\bar{u})V_0 u, \phi) \\ &= \mathrm{Re}((\operatorname{sign}\bar{u})T_b u, \phi) = \mathrm{Re}((\operatorname{sign} u)\phi, T_b u). \end{aligned} \tag{2.11}$$

Here we used integration by parts and the special case of Kato inequality (2.10) for Δ_M.

Let $0 \leq v \in Q(T_b)$. By Proposition 2.4, there exists a sequence $\phi_j \in C_c^\infty(M)^+$ such that $\|\phi_j - v\|_{t_b} \to 0$ as $j \to \infty$, where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2).

From (2.11), we obtain

$$\operatorname{Re}[t_b(|u|, v)] = \lim_{j\to\infty} \operatorname{Re}[t_b(|u|, \phi_j)] \le \lim_{j\to\infty} \operatorname{Re}((\operatorname{sign} u)\phi_j, T_b u) = \operatorname{Re}((\operatorname{sign} u)v, T_b u).$$

This proves condition (ii) and the lemma. □

In what follows T_0 is as in the hypothesis of Theorem 2.2. Without loss of generality we may and we will assume that T_0 is a positive self-adjoint operator.

We will also use the notation $\mathbb{Z}_+ := \{1, 2, 3, \dots\}$.

Proposition 2.13. $(T_0 + 1)^{-1}$ *is positivity preserving.*

Proof. We will adopt the arguments from the proof of Lemma 2 in [5] to our setting.

For every $k \in \mathbb{Z}_+$ and $x \in M$, define

$$Q_k(x) \;=\; \begin{cases} V_0(x) & \text{if } \; V_0(x) \ge -k \ , \\ -k & \text{if } \; V_0(x) < -k. \end{cases}$$

Let T_k be the Friedrichs extension of $(\Delta_M + Q_k)|C_c^\infty(M)$. Then for all $k \in \mathbb{Z}_+$ and $u \in C_c^\infty(M)$, we have

$$(u, T_k u) \ge (u, T_0 u) \ge 0, \tag{2.12}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(M)$.

From (2.12) it follows that

$$T_0 \le T_k \qquad \text{for all } k \in \mathbb{Z}_+, \tag{2.13}$$

i.e. $Q(T_k) \subset Q(T_0)$, and for all $u \in Q(T_k)$, $t_0(u,u) \le t_k(u,u)$, where t_0 and t_k are the quadratic forms associated to T_0 and T_k, respectively.

Furthermore, for all $u \in C_c^\infty(M)$, the following is true:

$$(u, T_k u) \to (u, T_0 u) \qquad \text{as } k \to \infty. \tag{2.14}$$

Clearly, $C_c^\infty(M) \subset Q(T_k)$ for all $k \in \mathbb{Z}_+$. By definition of Friedrichs extension, it follows that $C_c^\infty(M)$ is dense in $Q(T_0)$ (in the norm of $Q(T_0)$).

This, (2.13) and (2.14) show that the hypotheses of abstract Theorem 7.9 from [3] are satisfied.

Therefore, as $k \to \infty$, $T_k \to T_0$ in the strong resolvent sense.

By Lemma 2.12, $(T_k + 1)^{-1}$ is positivity preserving for all $k \in \mathbb{Z}_+$. Therefore, $(T_0 + 1)^{-1}$ is also positivity preserving. □

Corollary 2.14. *Assume that* $u \in Q(T_0)$*. Then* $|u| \in Q(T_0)$*.*

Proof. T_0 is a positive self-adjoint operator in $L^2(M)$. By Proposition 2.13, the operator $(T_0 + 1)^{-1}$ is positivity preserving. Now the corollary follows immediately from Theorem 2.11. □

2.15. **Truncation operators corresponding to T_0.** Let T_0 be as in the hypothesis of Theorem 2.2.

Define $V_0^+ := \max\{V_0, 0\}$, $V_0^- := \max\{-V_0, 0\}$, and for each $k \in \mathbb{Z}_+$, let $V_0^k := \min\{k, V_0^-\}$.

Denote by T_+ and T_k the Friedrichs extension of $(\Delta_M + V_0^+)|_{C_c^\infty(M)}$ and $(\Delta_M + V_0^+ - V_0^k)|_{C_c^\infty(M)}$, respectively.

Let t_0, t_+ and t_k ($k \in \mathbb{Z}_+$) be the quadratic forms associated to T_0, T_+ and T_k, respectively.

The following lemma is analogous to Lemma 3 in [2].

Lemma 2.16. *With the notations of Section 2.15,*

(i) $T_k \to T_0$ *in the strong resolvent sense as* $k \to \infty$.

(ii) $Q(T_+) \subset Q(T_0)$.

Proof. For all $k \in \mathbb{Z}_+$, we clearly have $T_0 \le T_k$. Also, $C_c^\infty(M) \subset Q(T_k)$ for all $k \in \mathbb{Z}_+$. By definition of T_0 it follows that $C_c^\infty(M)$ is dense in $Q(T_0)$ (in the norm of $Q(T_0)$).

Clearly, for all $w \in C_c^\infty(M)$,

$$(w, T_k w) \to (w, T_0 w) \qquad \text{as } k \to \infty.$$

We now apply Theorem 7.9 in [3] to conclude the proof of (i).

Property (ii) follows immediately since $T_+ \ge T_0$. □

2.17. **Proof of Theorem 2.2.** We will adopt the arguments from the proof of Theorem 1 in [2].

By the proof of Lemma 2.16 it follows that

$$Q(T_+) \subset Q(T_k) \subset Q(T_0) \tag{2.15}$$

and

$$\|\cdot\|_{t_0} \le \|\cdot\|_{t_k} \le \|\cdot\|_{t_+}, \tag{2.16}$$

where $\|\cdot\|_{t_0}$, $\|\cdot\|_{t_k}$ and $\|\cdot\|_{t_+}$ are the norms associated to t_0, t_k and t_+, respectively; cf. (1.2).

In fact, $Q(T_k) = Q(T_+)$ since the norms $\|\cdot\|_{t_k}$ and $\|\cdot\|_{t_+}$ are equivalent, because $V_0^+ - V_0^k$ and V_0^+ differ by a bounded function.

By Proposition 2.4, T_+ has a positive form core, i.e. for every $u \in Q(T_+)^+$ there exists a sequence $\phi_j \in C_c^\infty(M)^+$ such that $\|\phi_j - u\|_{t_+} \to 0$ as $j \to \infty$. By (2.16) it follows that

$$\|\phi_j - u\|_{t_0} \to 0 \qquad \text{as } j \to \infty.$$

To prove the theorem, it remains to show that for every $w \in Q(T_0)^+$, there exists a sequence $w_j \in Q(T_+)^+$ such that

$$\|w_j - w\|_{t_0} \to 0 \qquad \text{as } j \to \infty. \tag{2.17}$$

Let $w \in Q(T_0)^+$. For every $k, l \in \mathbb{Z}_+$ define

$$w_l := \left(\frac{1}{l}T_0 + 1\right)^{-1} w$$

and

$$w_l^k := \left(\frac{1}{l}T_k + 1\right)^{-1} w.$$

This makes sense since $0 \le T_0 \le T_k$ are self-adjoint operators.

By Lemma 2.12, the operator $(T_k + 1)^{-1}$ is positivity preserving. Hence $w_l^k \in \mathrm{Dom}(T_k)^+ \subset Q(T_k)^+ = Q(T_+)^+$.

Since the operators $(T_0 + 1)^{1/2}$ and $(T_0/l + 1)^{-1}$ commute, we have

$$\|w_l - w\|_{t_0} = \left\| \left(\left(\frac{1}{l}T_0 + 1\right)^{-1} - 1 \right) (T_0 + 1)^{1/2} w \right\|, \tag{2.18}$$

where $\|\cdot\|$ denotes $L^2(M)$ norm.

Clearly,

$$\left(\frac{1}{l}T_0+1\right)^{-1} \to 1 \qquad \text{strongly as } l\to\infty.$$

This and (2.18) show that

$$\|w_l - w\|_{t_0} \to 0 \qquad \text{as } l\to\infty. \tag{2.19}$$

Fix $l\in\mathbb{Z}_+$. For each $k\in\mathbb{Z}_+$, let t_0+l and t_k+l denote the quadratic forms corresponding to (positive self-adjoint) operators T_0+l and T_k+l, respectively. Let $\|\cdot\|_{t_0+l}$ and $\|\cdot\|_{t_k+l}$ denote the norms in $Q(T_0+l)$ and $Q(T_k+l)$, respectively; cf. (1.2). The corresponding inner products will be denoted by $(\cdot,\cdot)_{t_0+l}$ and $(\cdot,\cdot)_{t_k+l}$.

Using (2.15), (2.16) and the Cauchy-Schwarz inequality we have for all $w\in Q(T_0)^+$

$$\begin{aligned}
&\|(T_k+l)^{-1}w-(T_0+l)^{-1}w\|^2_{t_0+l} \\
&= \|(T_k+l)^{-1}w\|^2_{t_0+l}+\|(T_0+l)^{-1}w\|^2_{t_0+l}-2((T_k+l)^{-1}w,(T_0+l)^{-1}w)_{t_0+l}\\
&\le \|(T_k+l)^{-1}w\|^2_{t_k+l}+\|(T_0+l)^{-1}w\|^2_{t_0+l}-2((T_k+l)^{-1}w,(T_0+l)^{-1}w)_{t_0+l}\\
&= ((T_0+l)^{-1}w,w)-((T_k+l)^{-1}w,w)\\
&+ (1-l)[\|(T_k+l)^{-1}w\|^2+\|(T_0+l)^{-1}w\|^2-2((T_k+l)^{-1}w,(T_0+l)^{-1}w)]\\
&\le ((T_0+l)^{-1}w,w)-((T_k+l)^{-1}w,w) \le \|(T_0+l)^{-1}w-(T_k+l)^{-1}w\|\|w\|,
\end{aligned} \tag{2.20}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(M)$ and $\|\cdot\|$ is the norm in $L^2(M)$.

By Lemma 2.16, it follows that for fixed $l\in\mathbb{Z}_+$, $T_k+l\to T_0+l$ in the strong resolvent sense as $k\to\infty$.

Clearly, for any positive self-adjoint operator A, $(A/l+1)^{-1}=l(A+l)^{-1}$. Therefore by (2.20), for a fixed $l\in\mathbb{Z}_+$,

$$\|w_l^k - w_l\|_{t_0+l}\to 0 \qquad \text{as } k\to\infty.$$

This is equivalent to

$$\|w_l^k - w_l\|_{t_0}\to 0 \qquad \text{as } k\to\infty. \tag{2.21}$$

Since $w_l^k\in Q(T_+)^+$, we can use (2.19) and (2.21) to choose a subsequence $\{w_j\}$ from $\{w_l^k\}$ so that (2.17) holds.

This concludes the proof of the theorem. □

3. Proof of Theorem 1.5

We essentially follow the proof of Theorem 2 in [2]; however, we need to use Kato inequality (2.10) for operators on manifolds.

Without loss of generality, we may and we will assume that $\Delta_M+V_0\ge 0$ and $V_1\ge 0$.

Let us denote $T_q:=T_0\tilde{+}V_1$ and let $t_{\min}$ and t_q be as in Sections 1.4 and 1.3. Since $t_{\min}$ and t_q coincide on $C_c^\infty(M)$, it is sufficient to show that $C_c^\infty(M)$ is dense in the Hilbert space $Q(T_q)=Q(T_0)\cap Q(V_1)$ with the inner product

$$(\cdot,\cdot)_{t_q}:=t_q(\cdot,\cdot)+(\cdot,\cdot)_{L^2(M)},$$

where $t_q(\cdot,\cdot)$ is the sesquilinear form obtained by polarization of t_q.

Let $v\in Q(T_q)$ be orthogonal to $C_c^\infty(M)$ in $(\cdot,\cdot)_{t_q}$. This means that for all $w\in C_c^\infty(M)$,

$$((\Delta_M+V_0+V_1)v,w)_{L^2(M)}+(v,w)_{L^2(M)}=0.$$

This leads to the following distributional equality:

$$\Delta_M v = -(V_0 + V_1 + 1)v. \tag{3.1}$$

Since $V_1 \in L^1_{\mathrm{loc}}(M)$ and $v \in Q(V_1)$, we have

$$2|V_1 v| = 2|V_1||v| \leq |V_1| + |V_1||v|^2$$

which immediately gives $V_1 v \in L^1_{\mathrm{loc}}(M)$.

Since $V_0 \in L^2_{\mathrm{loc}}(M)$, it follows that $V_0 v \in L^1_{\mathrm{loc}}(M)$. From (3.1) we obtain $\Delta_M v \in L^1_{\mathrm{loc}}(M)$.

Using Kato inequality (2.10) in case $\nabla = d$ and (3.1), we get

$$\Delta_M |v| \leq \mathrm{Re}(\operatorname{sign} \bar{v} \Delta_M v) = -V_0|v| - V_1|v| - |v| \leq -(V_0 + 1)|v|. \tag{3.2}$$

The last inequality in (3.2) holds since $V_1 \geq 0$.

From (3.2), we obtain the following distributional inequality:

$$(\Delta_M + V_0 + 1)|v| \leq 0. \tag{3.3}$$

Let T_0 be as in the hypothesis, and let t_0 denote the closed quadratic form associated to T_0.

Using (3.3), we get

$$((T_0 + 1)w, |v|)_{L^2(M)} \leq 0 \qquad \text{for all } w \in C_c^\infty(M)^+. \tag{3.4}$$

Since $v \in Q(T_0)$, Corollary 2.14 gives $|v| \in Q(T_0)$. Therefore, we can write (3.4) as

$$(w, |v|)_{t_0} \leq 0 \qquad \text{for all } w \in C_c^\infty(M)^+, \tag{3.5}$$

where $(\cdot,\cdot)_{t_0} = t_0(\cdot,\cdot) + (\cdot,\cdot)_{L^2(M)}$ denotes the inner product in $Q(T_0)$.

Let $f := (T_0 + 1)^{-1}|v|$. By Proposition 2.13, $(T_0 + 1)^{-1}$ is positivity preserving, so $f \in \mathrm{Dom}(T_0)^+ \subset Q(T_0)^+$.

By Theorem 2.2, T_0 has a positive form core. Therefore, there exists a sequence $f_k \in C_c^\infty(M)^+$ such that

$$\lim_{k\to\infty} (f_k, |v|)_{t_0} = (f, |v|)_{t_0} = ((T_0 + 1)^{-1}|v|, |v|)_{t_0} = \|v\|^2, \tag{3.6}$$

where v and $(\cdot,\cdot)_{t_0}$ are as in (3.5), and $\|\cdot\|$ is the norm in $L^2(M)$.

From (3.5) and (3.6) we obtain $\|v\|^2 \leq 0$, i.e. $v = 0$.

This shows that $C_c^\infty(M)$ is dense in $Q(T_q)$, and the theorem is proven. $\square$

References

[1] M. Braverman, O. Milatovic, M. A. Shubin, *Essential self-adjointness of Schrödinger type operators on manifolds*, Russian Math. Surveys, **57** (4) (2002), 641–692.

[2] H. L. Cycon, *On the form sum and the Friedrichs extension of Schrödinger operators with singular potentials*, J. Operator Theory, **6** (1981), 75–86. MR **82k:**47068

[3] W. G. Faris, *Self-adjoint Operators*, Lecture Notes in Mathematics No. 433, Springer-Verlag, Berlin e.a., 1975. MR **57:**7207

[4] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, New York, 1977. MR **57:**13109

[5] H.-W. Goelden, *On non-degeneracy of the ground state of Schrödinger operators*, Math. Z., **155** (1977), 239–247. MR **58:**29426

[6] H. Hess, R. Schrader, D.A. Uhlenbrock, *Domination of semigroups and generalization of Kato's inequality,* Duke Math. J., **44** (1977), 893–904. MR **56:**16446

[7] H. Hess, R. Schrader, D. A. Uhlenbrock, *Kato's inequality and the spectral distribution of Laplacians on compact Riemannian manifold*, J. Differential Geom., **15** (1980), 27-37. MR **82g:**58090

[8] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1980. reprint MR **96a:**47025

[9] T. Kato, *A second look at the essential selfadjointness of the Schrödinger operators*, Physical reality and mathematical description, Reidel, Dordrecht, 1974, 193–201. MR **57:**16958

[10] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I, II: Functional analysis. Fourier analysis, self-adjointness*, Academic Press, New York e.a., 1975. MR **58:**12429a; MR **58:**12429b

[11] B. Simon, *An abstract Kato's inequality for generators of positivity preserving semigroups*, Indiana Univ. Mat. J., **26** (1977), 1067–1073. MR **57:**1194

[12] B. Simon, *Kato's inequality and the comparison of semigroups*, J. Funct. Anal., **32** (1979), 97–101. MR **80e:**47036

[13] B. Simon, *Maximal and minimal Schrödinger forms*, J. Operator Theory, **1** (1979), 37–47. MR **81m:**35104

[14] M. Taylor, *Partial Differential Equations II: Qualitative Studies of Linear Equations*, Springer-Verlag, New York e.a., 1996. MR **98b:**35003

Department of Mathematics, Northeastern University, Boston, Massachusetts 02115

E-mail address: `ogmilato@lynx.neu.edu`

Current address: Department of Mathematics, Fitchburg State College, Fitchburg, Massachusetts 01420

E-mail address: `omilatovic@fsc.edu`