THE FORM SUM AND THE FRIEDRICHS EXTENSION
OF SCHÖDINGER-TYPE OPERATORS
ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We consider $H_V = \Delta_M + V$, where $(M, g)$ is a Riemannian manifold (not necessarily complete), and $\Delta_M$ is the scalar Laplacian on $M$. We assume that $V = V_0 + V_1$, where $V_0 \in L^2_{\text{loc}}(M)$ and $-C \leq V_1 \in L^1_{\text{loc}}(M)$ ($C$ is a constant) are real-valued, and $\Delta_M + V_0$ is semibounded below on $C^\infty_c(M)$. Let $T_0$ be the Friedrichs extension of $(\Delta_M + V_0)|_{C^\infty_c(M)}$. We prove that the form sum $T_0 + V_1$ coincides with the self-adjoint operator $T_F$ associated to the closure of the restriction to $C^\infty_c(M) \times C^\infty_c(M)$ of the sum of two closed quadratic forms of $T_0$ and $V_1$. This is an extension of a result of Cycon. The proof adopts the scheme of Cycon, but requires the use of a more general version of Kato's inequality for operators on Riemannian manifolds.

1. Introduction and the main result

Let $(M, g)$ be a Riemannian manifold (i.e. $M$ is a $C^\infty$-manifold, $(g_{jk})$ is a Riemannian metric on $M$), $\dim M = n$. We will assume that $M$ is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \ldots, x^n$ there exists a strictly positive $C^1$-density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2\ldots dx^n$. We do not assume that $(M, g)$ is complete.

We will consider a Schrödinger-type operator of the form

$$H_V = \Delta_M + V.$$  

Here $\Delta_M := d^*d$, where $d: C^\infty(M) \to \Omega^1(M)$, and $V \in L^1_{\text{loc}}(M)$ is real-valued.

1.1. Maximal operator. We define the maximal operator $H_{V,\text{max}}$ associated to $H_V$ as an operator in $L^2(M)$ given by $H_{V,\text{max}}u = H_Vu$ with domain

$$\text{Dom}(H_{V,\text{max}}) = \{u \in L^2(M) : Vu \in L^1_{\text{loc}}(M), H_Vu \in L^2(M)\}.$$  

Here $\Delta_M u$ in $H_V u = \Delta_M u + Vu$ is understood in the distributional sense.

We make the following assumptions on $V$.

Assumption A. Assume $V = V_0 + V_1$, where

(i) $V_0 \in L^2_{\text{loc}}(M)$ and $\Delta_M + V_0$ is semibounded below on $C^\infty_c(M)$.

(ii) $V_1 \in L^1_{\text{loc}}(M)$ and $V_1 \geq -C$, where $C > 0$ is a constant.

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1.2. **Quadratic forms.** For any self-adjoint operator $T$: $\text{Dom}(T) \subset L^2(M) \to L^2(M)$ such that $T \geq -\alpha$, we will denote by $Q(T)$ the domain of the quadratic form $t$ associated to $T$. By Theorem 2.1 in [3], $t$ is a closed semibounded below form, i.e. $Q(T)$ is a Hilbert space with the inner product

\[(u, v)_t = t(u, v) + (1 + \alpha)(u, v)_{L^2(M)},\]

where $(\cdot, \cdot)$ is the sesquilinear form obtained by polarization of $t$.

1.3. **Form sum.** By (i) of Assumption A, $\Delta_M + V_0$ is symmetric and semibounded below on $C_0^\infty(M)$, so we can associate to it a semibounded below self-adjoint operator $T_0$ (Friedrichs extension, cf. Theorem 14.1 in [3]).

We will denote by $T_0 + V_1$ the form sum of $T_0$ and $V_1$. By Theorem 4.1 in [3], this is the self-adjoint operator associated to the semibounded below closed quadratic form $t_q$ given by the sum of two semibounded below closed quadratic forms corresponding to $T_0$ and $V_1$. By the same theorem, the following is true: $Q(T_0 + V_1) = Q(T_0) \cap Q(V_1)$. Clearly, $T_0 + V_1$ is a self-adjoint restriction of $H_{V_{\text{max}}}$.

1.4. **Operator $T_F$.** Denote by $t_{\text{min}}$ the restriction of $t_q$ to $C_0^\infty(M) \times C_0^\infty(M)$. Denote by $T_F$ the self-adjoint operator associated to the closure of $t_{\text{min}}$ in the sense of the norm in $Q(T_0 + V_1)$. Clearly, $T_F$ is a self-adjoint restriction of $H_{V_{\text{max}}}$.

We will give a sufficient condition for $T_F = T_0 + V_1$.

**Theorem 1.5.** Suppose that Assumption A holds. Then $T_F = T_0 + V_1$.

**Remark 1.6.** Theorem 1.5 was proven by Cycon [2] in the case of the operator $-\Delta + V$ in an open set $M \subset \mathbb{R}^n$, where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric. In the case $V_0 = 0$ and $M = \mathbb{R}^n$ with standard metric, Theorem 1.5 was proven in Simon [13].

2. **Operators with a positive form core**

**Definition 2.1.** Let $T: C_0^\infty(M) \subset L^2(M) \to L^2(M)$ be a symmetric semibounded below operator. Let $T_F$ denote its Friedrichs extension and $Q(T_F)^+$ the set of a.e. positive elements of $Q(T_F)$. We say that $T_F$ has a positive form core if for every $u \in Q(T_F)^+$ there exists a sequence $u_k \in C_0^\infty(M)^+$ such that

$$\|u_k - u\|_t \to 0 \quad \text{as} \quad k \to \infty,$$

where $\|\cdot\|_t$ is the norm associated to the closure of quadratic form $t(v, w) := (Tv, w)$ $(v, w \in C_0^\infty(M))$ via (1.2).

The main result of this section is

**Theorem 2.2.** Suppose that $\Delta_M + V_0$ is as in (i) of Assumption A. Let $T_0$ be the Friedrichs extension of $(\Delta_M + V_0)|_{C_0^\infty(M)}$. Then $T_0$ has a positive form core.

**Remark 2.3.** In the case of the operator $-\Delta + V_0$ in an open set $M \subset \mathbb{R}^n$, Theorem 2.2 was proven in [2] Th. 1.

We will first prove the following special case of Theorem 2.2

**Proposition 2.4.** Suppose that $-C \leq V_0 \in L^2_{\text{loc}}(M)$, where $C > 0$ is a constant. Let $T_0$ be the Friedrichs extension of $(\Delta_M + V_0)|_{C_0^\infty(M)}$. Then $T_0$ has a positive form core.
We begin with a few preliminary lemmas.

In what follows \( T_b \) is as in the hypothesis of Proposition 2.4, and \( T_b \) is the closed quadratic form associated with \( T_b \). Without loss of generality, we may and we will assume that \( V_0 \geq 0 \) so that \( T_b \) is a positive self-adjoint operator.

We will denote \( W^{1,2}(M) := \{ u \in L^2(M) : du \in L^2(T^*M) \} \). By \( W^{1,2}_0(M) \) we will denote the closure of \( \mathcal{C}_c^\infty(M) \) in the norm \( \| u \|_{W^{1,2}}^2 := \| du \|^2 + \| u \|^2 \), where \( \| \cdot \| \) is the \( L^2 \) norm. By \( Q(V_0) \) we will denote the set \( \{ u \in L^2(M) : V_0^{1/2}u \in L^2(M) \} \). Clearly, \( Q(V_0) \) is the closure of \( \mathcal{C}_c^\infty(M) \) in the norm
\[
\| u \|^2_{V_0} := \| V_0^{1/2}u \|^2 + \| u \|^2,
\]
where \( \| \cdot \| \) is the norm in \( L^2(M) \).

In the proofs of the following three lemmas, we will use the arguments from the proof of Lemma 1 in [5].

**Lemma 2.5.** \( Q(T_b) = W^{1,2}_0(M) \cap Q(V_0) \).

**Proof.** Denote by \( \mathcal{H}_1 := W^{1,2}_0(M) \cap Q(V_0) \). Consider a sesquilinear form \( S : \mathcal{H}_1 \times \mathcal{H}_1 \rightarrow \mathbb{C} \) given by
\[
S(u,v) := (du, dv) + (V_0^{1/2}u, V_0^{1/2}v),
\]
where \((\cdot, \cdot)\) is the inner product in \( L^2 \).

This sesquilinear form is closed, so the pre-Hilbert space \( \mathcal{H}_1 \) is complete in the norm
\[
(u,v)_{V_0} := (du, dv) + (V_0^{1/2}u, V_0^{1/2}v) + (u,v).
\]
By definition of \( W^{1,2}_0(M) \) and \( Q(V_0) \), it follows that \( \mathcal{H}_1 \) is the closure of \( \mathcal{C}_c^\infty(M) \) in the norm \( \| \cdot \|_{V_0} \) corresponding to (2.2).

For all \( u,v \in \mathcal{C}_c^\infty(M) \), \((u,v)_{V_0} = (u,v) + (T_b u,v) \). By Theorem 14.1 in [3], \( Q(T_b) \) is the closure of \( \mathcal{C}_c^\infty(M) \) in the norm \( \| \cdot \|_{V_0} \) corresponding to (2.2), so \( Q(T_b) = W^{1,2}_0(M) \cap Q(V_0) \). \( \square \)

**Lemma 2.6.** Assume that \( u \in \mathcal{C}_c^\infty(M) \). Then there exists a sequence \( \phi_k \in \mathcal{C}_c^\infty(M)^+ \) such that \( \| \phi_k - |u| \|_{V_0} \rightarrow 0 \) as \( k \rightarrow \infty \), where \( \| \cdot \|_{V_0} \) is the norm corresponding to (2.2).

**Proof.** Let \( u \in \mathcal{C}_c^\infty(M) \). Then \( |u| \in W^{1,2}_{\text{comp}}(M) \). Using a partition of unity we may assume that \( u \) is supported in a coordinate neighborhood. Let \( |u|^{\rho} = J^{\rho}|u| \), where \( J^{\rho} \) is the Friedrichs mollifying operator; cf. Sect. 5.11 in [1]. Then \( |u|^{\rho} \in \mathcal{C}_c^\infty(M) \).

It is well-known that \( |u|^{\rho} \rightarrow |u| \) as \( \rho \rightarrow 0+ \) both in the space \( W^{1,2}_{\text{comp}}(M) \) and in the space \( L^2_{\text{comp}}(M) \). Also, since \( |u| \) is continuous compactly supported on \( M \) and \( V_0 \in L^2_{\text{loc}}(M) \), we have
\[
\int V_0(|u|^{\rho})^2 d\mu \rightarrow \int V_0|u|^2 d\mu \quad \text{as} \; \rho \rightarrow 0+.
\]
Therefore,
\[
|||u|^{\rho} - |u|||_{V_0} \rightarrow 0 \quad \text{as} \; \rho \rightarrow 0+,
\]
where \( ||| \cdot |||_{V_0} \) is the norm corresponding to (2.2). \( \square \)

**Lemma 2.7.** Suppose that \( u \in Q(T_b) \). Then \( |u| \in Q(T_b) \).
Proof. Let \( u \in Q(T_b) \). By Lemma 2.9, we get \( u \in W_{0}^{1,2}(M) \cap Q(V_0) \). Since \( u \in W_{0}^{1,2}(M) \), Lemma 7.6 from [4] gives \( |u| \in W_{0}^{1,2}(M) \). From \( u \in Q(V_0) \), we immediately get \( |u| \in Q(V_0) \). Therefore, \( |u| \in W_{0}^{1,2}(M) \cap Q(V_0) \), so by Lemma 2.5 we obtain \( |u| \in Q(T_b) \).

2.8. Proof of Proposition 2.4. We will follow the proof of Lemma 2 in [2]. Suppose that \( u \in Q(T_b)^{+} \). By Lemma 5.6 there exists a sequence \( \phi_j \in C_{0}^{c}(M) \) such that

\[
\|\phi_j - u\|_{tb} \to 0 \quad \text{as } j \to \infty, \tag{2.5}
\]

where \( \|\cdot\|_{tb} \) is the norm corresponding to \( (2.2) \).

In what follows, we will denote \( (\text{sign } w)(x) := \frac{w(x)}{|w(x)|} \) when \( w(x) \neq 0 \), and 0 otherwise.

We have

\[
(\|\phi_j - u\|_{tb})^2 = \|\phi_j - u\|^2 + |d\phi_j| - du|^2 + \|V_{0}^{1/2}(|\phi_j)| - u||^2 \tag{2.6}
\]

\[
\leq \|\phi_j - u\|^2 + |d\phi_j| - du|^2 + \|V_{0}^{1/2}(\phi_j - u)||^2 \tag{2.6}
\]

\[
= \|\phi_j - u\|^2 + \|\text{Re}(\text{sign } \phi_j) d\phi_j - du||^2 + \|V_{0}^{1/2}(\phi_j - u)||^2, \tag{2.6}
\]

where \( \|\cdot\| \) denotes the norm \( L^2 \).

From (2.6) we obtain

\[
(\|\phi_j - u\|_{tb})^2 \leq \|\phi_j - u\|^2 + \|d\phi_j - du\|^2 + \|\text{sign } \phi_j - 1||^2 du||^2 \tag{2.7}
\]

\[
+ \|V_{0}^{1/2}(\phi_j - u)||^2 \leq \|\phi_j - u\|^2 + |d\phi_j - du|^2 + \|\text{sign } \phi_j - 1||^2 du||^2 + \|V_{0}^{1/2}(\phi_j - u)||^2, \tag{2.7}
\]

where \( \|\cdot\| \) denotes the norm in \( L^2 \).

By (2.5), the first, second and fourth term on the right-hand side of (2.7) go to 0 as \( j \to \infty \).

It remains to show that

\[
((\text{sign } \phi_j - 1)du) \to 0 \quad \text{as } j \to \infty. \tag{2.8}
\]

By \( \phi_j \to u \) in \( L^2(M) \), a lemma of Riesz shows that there exists a subsequence \( \phi_{jk} \) such that \( \phi_{jk} \to u \) a.e. \( du \), as \( k \to \infty \). By Lemma 7.7 from [4], it follows that \( du = 0 \) almost everywhere on \( \{x \in M : u(x) = 0\} \). Hence, as \( k \to \infty \), sign \( \phi_{jk} \to 1 \) a.e. on \( M \). Since \( du \in L^2(T^*M) \), dominated convergence theorem immediately implies (2.8) (after passing to the chosen subsequence \( \phi_{jk} \)).

This shows that

\[
\|\phi_{jk} - u\|_{tb} \to 0 \quad \text{as } k \to \infty. \tag{2.9}
\]

By (2.5) and Lemma 2.6 there exists a sequence \( \psi_l \in C_{0}^{c}(M)^{+} \) such that \( \|\psi_l - u\|_{tb} \to 0 \) as \( l \to \infty \). By Definition 2.1 it follows that \( T_b \) has a positive form core.

In what follows, we will use a version of Kato’s inequality. For the proof of this inequality in general setting, cf. Theorem 5.6 in [1].

Theorem 2.9. Let \( E \) be a Hermitian vector bundle on \( M \), and let \( \nabla : C_{0}^{c}(E) \to C_{0}^{c}(T^*M \otimes E) \) be a Hermitian connection on \( E \). Let \( \nabla^* : C_{0}^{c}(T^*M \otimes E) \to C_{0}^{c}(E) \)
be formal adjoint of $\nabla$ with respect to the usual inner product on $L^2(E)$. Assume that $u \in L^1_{loc}(E)$ and $\nabla^* \nabla u \in L^1_{loc}(E)$. Then
\begin{equation}
\Delta_M |u| \leq \text{Re}\langle \nabla^* \nabla u, \text{sign } u \rangle,
\end{equation}
where
\[
\text{sign } u(x) = \begin{cases} 
\frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

**Definition 2.10.** Let $(X, \mu)$ be a measure space. A bounded linear operator $A: L^2(X, \mu) \to L^2(X, \mu)$ is said to be positivity preserving if for every $0 \leq u \in L^2(X, \mu)$ we have $Au \geq 0$.

We will also use the following abstract theorem due to Simon; cf. Theorem 2.1 in [11].

**Theorem 2.11 (Simon [11]).** Suppose that $(X, \mu)$ is a measure space. Suppose that $H$ is a positive self-adjoint operator in $L^2(X, \mu)$. Then $(H + 1)^{-1}$ is positivity preserving if and only if the following two conditions are satisfied:

$(i)$ For every $u \in Q(H)$, we have $|u| \in Q(H)$.

$(ii)$ For every $u \in \text{Dom}(H)$ and $0 \leq v \in Q(H)$, the following is true:

\[
\text{Re}[h(|u|, v)] \leq \text{Re}((\text{sign } u)v, Hu),
\]
where $h$ is the quadratic form associated to $H$, and $(\text{sign } u)(x) = \frac{u(x)}{|u|}$ whenever $u(x) \neq 0$, and 0 otherwise.

The following lemma extends Lemma 1 from [5] to the case of Riemannian manifolds.

**Lemma 2.12.** The operator $(T_b + 1)^{-1}$ is positivity preserving.

**Proof.** Let $t_b$ be the quadratic form associated to $T_b$. By Theorem 2.11, it suffices to check the following conditions:

$(i)$ For every $u \in Q(T_b)$, we have $|u| \in Q(T_b)$ and

$(ii)$ For every $u \in \text{Dom}(T_b)$ and $0 \leq v \in Q(T_b)$, the following is true:

\[
\text{Re}[t_b(|u|, v)] \leq \text{Re}((\text{sign } u)v, T_b u).
\]

Condition $(i)$ follows immediately by Lemma 2.7.

We now prove that condition $(ii)$ holds. Let $u \in \text{Dom}(T_b)$. Then $(\Delta_M + V_0)u \in L^2(M)$ and hence $\Delta_M u \in L^1_{loc}(M)$.

For $u \in \text{Dom}(T_b)$ and $0 \leq \phi \in C^\infty_c(M)$ we have

\[
\text{Re}[t_b(|u|, \phi)] = \text{Re}(|u|, (\Delta_M + V_0)\phi) = (|u|, \Delta_M \phi) + (|u|, V_0 \phi)
\]
\[
= (\Delta_M |u|, \phi) + (V_0 |u|, \phi)
\]
\[
\leq \text{Re}((\text{sign } u)\Delta_M u, \phi) + ((\text{sign } \bar{u})V_0 u, \phi)
\]
\[
= \text{Re}((\text{sign } u)T_b u, \phi) = \text{Re}((\text{sign } u)\phi, T_b^2 u).
\]

Here we used integration by parts and the special case of Kato inequality (2.10) for $\Delta_M$.

Let $0 \leq v \in Q(T_b)$. By Proposition 2.13, there exists a sequence $\phi_j \in C^\infty_c(M)^+$ such that $\|\phi_j - v\|_{t_b} \to 0$ as $j \to \infty$, where $\| \cdot \|_{t_b}$ is the norm corresponding to (2.2).
From (2.11), we obtain
\[
\Re(t_b(|u|, v)) = \lim_{j \to \infty} \Re(t_b(|u|, \phi_j)) \leq \lim_{j \to \infty} \Re((\text{sign } u)\phi_j, T_b u) = \Re((\text{sign } u)v, T_b u).
\]
This proves condition (ii) and the lemma.

In what follows, \( T_0 \) is as in the hypothesis of Theorem 2.2. Without loss of generality, we may and we will assume that \( T_0 \) is a positive self-adjoint operator.

We will also use the notation \( Z_+: = \{1, 2, 3, \ldots\} \).

**Proposition 2.13.** \((T_0 + 1)^{-1}\) is positivity preserving.

**Proof.** We will adopt the arguments from the proof of Lemma 2 in [5] to our setting.

For every \( k \in Z_+ \) and \( x \in M \), define
\[
Q_k(x) = \begin{cases} V_0(x) & \text{if } V_0(x) \geq -k, \\ -k & \text{if } V_0(x) < -k. \end{cases}
\]

Let \( T_k \) be the Friedrichs extension of \((\Delta_M + Q_k)C^\infty_c(M)\). Then for all \( k \in Z_+ \) and \( u \in C^\infty_c(M) \), we have
\[
(u, T_k u) \geq (u, T_0 u) \geq 0,
\]
where \((\cdot, \cdot)\) is the inner product in \( L^2(M) \).

From (2.12) it follows that
\[
T_0 \leq T_k \quad \text{for all } k \in Z_+,
\]
and for all \( u \in Q(T_k) \), \( t_0(u, u) \leq t_k(u, u) \), where \( t_0 \) and \( t_k \) are the quadratic forms associated to \( T_0 \) and \( T_k \), respectively.

Furthermore, for all \( u \in C^\infty_c(M) \), the following is true:
\[
(u, T_k u) \to (u, T_0 u) \quad \text{as } k \to \infty.
\]

Clearly, \( C^\infty_c(M) \subset Q(T_k) \) for all \( k \in Z_+ \). By definition of Friedrichs extension, it follows that \( C^\infty_c(M) \) is dense in \( Q(T_0) \) (in the norm of \( Q(T_0) \)).

This, (2.13) and (2.14) show that the hypotheses of abstract Theorem 7.9 from [3] are satisfied.

Therefore, as \( k \to \infty \), \( T_k \to T_0 \) in the strong resolvent sense.

By Lemma 2.12, \((T_k + 1)^{-1}\) is positivity preserving for all \( k \in Z_+ \). Therefore, \((T_0 + 1)^{-1}\) is also positivity preserving.

**Corollary 2.14.** Assume that \( u \in Q(T_0) \). Then \(|u| \in Q(T_0)\).

**Proof.** \( T_0 \) is a positive self-adjoint operator in \( L^2(M) \). By Proposition 2.13, the operator \((T_0 + 1)^{-1}\) is positivity preserving. Now the corollary follows immediately from Theorem 2.12. \( \square \)

2.15. **Truncation operators corresponding to** \( T_0 \). Let \( T_0 \) be as in the hypothesis of Theorem 2.2.

Define \( V_0^+ := \max\{V_0, 0\} \), \( V_0^- := \max\{-V_0, 0\} \), and for each \( k \in Z_+ \), let \( V_k^0 := \min\{k, V_0^-\} \).

Denote by \( T_+ \) and \( T_k \) the Friedrichs extension of \((\Delta_M + V_0^+)|_{C^\infty_c(M)}\) and \((\Delta_M + V_k^+ - V_0^-)|_{C^\infty_c(M)}\), respectively.

Let \( t_0, t_+ \) and \( t_k \) \( (k \in Z_+) \) be the quadratic forms associated to \( T_0, T_+ \) and \( T_k \), respectively.
The following lemma is analogous to Lemma 3 in [2].

**Lemma 2.16.** With the notations of Section 2.15,

(i) $T_k \to T_0$ in the strong resolvent sense as $k \to \infty$.

(ii) $Q(T_+) \subset Q(T_0)$.

**Proof.** For all $k \in \mathbb{Z}_+$, we clearly have $T_0 \leq T_k$. Also, $C_c^\infty(M) \subset Q(T_k)$ for all $k \in \mathbb{Z}_+$. By definition of $T_0$ it follows that $C_c^\infty(M)$ is dense in $Q(T_0)$ (in the norm of $Q(T_0)$).

Clearly, for all $w \in C_c^\infty(M)$,

$$(w, T_k w) \to (w, T_0 w) \quad \text{as } k \to \infty.$$  

We now apply Theorem 7.9 in [3] to conclude the proof of (i).

Property (ii) follows immediately since $T_+ \geq T_0$. ∎

2.17. **Proof of Theorem 2.2.** We will adopt the arguments from the proof of Theorem 1 in [2].

By the proof of Lemma 2.16 it follows that

$$Q(T_+) \subset Q(T_k) \subset Q(T_0) \quad (2.15)$$

and

$$\| \cdot \|_{t_0} \leq \| \cdot \|_{t_k} \leq \| \cdot \|_{t_+}, \quad (2.16)$$

where $\| \cdot \|_{t_0}$, $\| \cdot \|_{t_k}$ and $\| \cdot \|_{t_+}$ are the norms associated to $t_0$, $t_k$ and $t_+$, respectively; cf. (1.2).

In fact, $Q(T_k) = Q(T_+)$ since the norms $\| \cdot \|_{t_k}$ and $\| \cdot \|_{t_+}$ are equivalent, because $V_0^+ - V_k^+$ and $V_0^+$ differ by a bounded function.

By Proposition 2.4, $T_+$ has a positive form core, i.e. for every $u \in Q(T_+)^+$ there exists a sequence $\phi_j \in C_c^\infty(M)^+$ such that $\| \phi_j - u \|_{t_+} \to 0$ as $j \to \infty$. By (2.16) it follows that

$$\| \phi_j - u \|_{t_0} \to 0 \quad \text{as } j \to \infty.$$  

To prove the theorem, it remains to show that for every $w \in Q(T_0)^+$, there exists a sequence $w_j \in Q(T_+)^+$ such that

$$\| w_j - w \|_{t_0} \to 0 \quad \text{as } j \to \infty. \quad (2.17)$$

Let $w \in Q(T_0)^+$. For every $k, l \in \mathbb{Z}_+$ define

$$w_l := \left( \frac{1}{l} T_0 + 1 \right)^{-1} w$$

and

$$w_k^l := \left( \frac{1}{l} T_k + 1 \right)^{-1} w.$$  

This makes sense since $0 \leq T_0 \leq T_k$ are self-adjoint operators.

By Lemma 2.12 the operator $(T_k + 1)^{-1}$ is positivity preserving. Hence $w_k^l \in \text{Dom}(T_k)^+ \subset Q(T_k)^+ = Q(T_+)^+$.

Since the operators $(T_0 + 1)^{1/2}$ and $(T_0/l + 1)^{-1}$ commute, we have

$$\| w_l - w \|_{t_0} = \left\| \left( \frac{1}{l} T_0 + 1 \right)^{-1} - 1 \right\| (T_0 + 1)^{1/2} w,$$  

where $\| \cdot \|$ denotes $L^2(M)$ norm.
Clearly, \[ \left( \frac{1}{l}T_0 + 1 \right)^{-1} \rightarrow 1 \quad \text{strongly as} \quad l \rightarrow \infty. \]

This and (2.18) show that \[ (2.19) \quad \|w_l - w\|_{t_0 + l} \rightarrow 0 \quad \text{as} \quad l \rightarrow \infty. \]

Fix \( l \in \mathbb{Z}_+ \). For each \( k \in \mathbb{Z}_+ \), let \( t_0 + l \) and \( t_k + l \) denote the quadratic forms corresponding to (positive self-adjoint) operators \( T_0 + l \) and \( T_k + l \), respectively. Let \( \| \cdot \|_{t_0 + l} \) and \( \| \cdot \|_{t_k + l} \) denote the norms in \( Q(T_0 + l) \) and \( Q(T_k + l) \), respectively; cf. (1.2). The corresponding inner products will be denoted by \( \langle \cdot, \cdot \rangle_{t_0 + l} \) and \( \langle \cdot, \cdot \rangle_{t_k + l} \).

Using (2.15), (2.16) and the Cauchy-Schwarz inequality we have for all \( w \in Q(T_0^+) \)
\[ (2.20) \quad \|(T_k + l)^{-1}w - (T_0 + l)^{-1}w\|_{t_0 + l}^2 \\
= \|((T_k + l)^{-1}w)_0 + ((T_0 + l)^{-1}w)_0 - 2((T_k + l)^{-1}w, (T_0 + l)^{-1}w)_0 + ((T_0 + l)^{-1}w, (T_k + l)^{-1}w)\|_{t_0 + l}^2 \\
\leq \|(T_k + l)^{-1}w\|_{t_0 + l}^2 + \|((T_0 + l)^{-1}w\|_{t_0 + l}^2 - 2((T_k + l)^{-1}w, (T_0 + l)^{-1}w)\|_{t_0 + l}
\]
\[ + (1 - l)\|((T_k + l)^{-1}w\|_{t_0 + l}^2 + \|((T_0 + l)^{-1}w\|_{t_0 + l}^2 - 2((T_k + l)^{-1}w, (T_0 + l)^{-1}w)\|_{t_0 + l}
\]
\[ \leq \|((T_0 + l)^{-1}w, (T_0 + l)^{-1}w - ((T_k + l)^{-1}w, (T_k + l)^{-1}w)\|_{t_0 + l}^2 - \|((T_0 + l)^{-1}w - (T_k + l)^{-1}w\|_{t_0 + l}^2, \]
where \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^2(M) \) and \( \| \cdot \| \) is the norm in \( L^2(M) \).

By Lemma 2.16 it follows that for fixed \( l \in \mathbb{Z}_+ \), \( T_k + l \rightarrow T_0 + l \) in the strong resolvent sense as \( k \rightarrow \infty \).

Clearly, for any positive self-adjoint operator \( A, (A/l+1)^{-1} = l(A+l)^{-1} \). Therefore by (2.22), for a fixed \( l \in \mathbb{Z}_+ \),
\[ \|w_l^k - w_l\|_{t_0 + l} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]

This is equivalent to
\[ (2.21) \quad \|w_l^k - w_l\|_{t_0} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]

Since \( w_l^k \in Q(T_0^+) \), we can use (2.19) and (2.21) to choose a subsequence \( \{w_j\} \) from \( \{w_l^k\} \) so that (2.17) holds.

This concludes the proof of the theorem. \( \square \)

3. Proof of Theorem 1.5

We essentially follow the proof of Theorem 2 in [2]; however, we need to use Kato inequality (2.10) for operators on manifolds.

Without loss of generality, we may and we will assume that \( \Delta_M + V_0 \geq 0 \) and \( V_1 \geq 0 \).

Let us denote \( T_q := T_0 + V_1 \) and let \( t_{\min} \) and \( t_q \) be as in Sections 1.3 and 1.5. Since \( t_{\min} \) and \( t_q \) coincide on \( C_c^\infty(M) \), it is sufficient to show that \( C_c^\infty(M) \) is dense in the Hilbert space \( Q(T_q) = Q(T_0) \cap Q(V_1) \) with the inner product
\[ \langle \cdot, \cdot \rangle_{t_q} := t_q \langle \cdot, \cdot \rangle + \langle \cdot, \cdot \rangle_{L^2(M)}, \]
where \( t_q (\cdot, \cdot) \) is the sesquilinear form obtained by polarization of \( t_q \).

Let \( v \in Q(T_q) \) be orthogonal to \( C_c^\infty(M) \) in \( \langle \cdot, \cdot \rangle_{t_q} \). This means that for all \( w \in C_c^\infty(M) \),
\[ \langle (\Delta_M + V_0 + V_1)v, w \rangle_{L^2(M)} + \langle v, w \rangle_{L^2(M)} = 0. \]
This leads to the following distributional equality:

\[ (3.1) \quad \Delta_M v = -(V_0 + V_1 + 1)v. \]

Since \( V_1 \in L^1_{\text{loc}}(M) \) and \( v \in Q(V_1) \), we have

\[ 2|V_1v| = 2|V_1||v| \leq |V_1| + |V_1||v|^2 \]

which immediately gives \( V_1 v \in L^1_{\text{loc}}(M) \).

Since \( V_0 \in L^2_{\text{loc}}(M) \), it follows that \( V_0 v \in L^1_{\text{loc}}(M) \). From (3.1) we obtain \( \Delta_M v \in L^1_{\text{loc}}(M) \).

Using Kato inequality (2.10) in case \( \nabla = d \) and (3.1), we get

\[ (3.2) \quad \Delta_M |v| \leq \text{Re}(\text{sign} \nabla \Delta_M v) = -V_0 |v| - V_1 |v| - |v| \leq -(V_0 + 1)|v|. \]

The last inequality in (3.2) holds since \( V_1 \geq 0 \).

From (3.2), we obtain the following distributional inequality:

\[ (3.3) \quad (\Delta_M + V_0 + 1)|v| \leq 0. \]

Let \( T_0 \) be as in the hypothesis, and let \( t_0 \) denote the closed quadratic form associated to \( T_0 \).

Using (3.3), we get

\[ (3.4) \quad ( (T_0 + 1)w, |v| )_{L^2(M)} \leq 0 \quad \text{for all } w \in C_c^\infty(M)^+. \]

Since \( v \in Q(T_0) \), Corollary 2.11 gives \( |v| \in Q(T_0) \). Therefore, we can write (3.4) as

\[ (3.5) \quad (w, |v| )_{t_0} \leq 0 \quad \text{for all } w \in C_c^\infty(M)^+, \]

where \( (\cdot, \cdot)_{t_0} = t_0(\cdot, \cdot)_{L^2(M)} \) denotes the inner product in \( Q(T_0) \).

Let \( f := (T_0 + 1)^{-1}|v| \). By Proposition 2.13, \( (T_0 + 1)^{-1} \) is positivity preserving, so \( f \in \text{Dom}(T_0)^+ \subset Q(T_0)^+ \).

By Theorem 2.2, \( T_0 \) has a positive form core. Therefore, there exists a sequence \( f_k \in C_c^\infty(M)^+ \) such that

\[ (3.6) \quad \lim_{k \to \infty} (f_k, |v| )_{t_0} = (f, |v| )_{t_0} = ((T_0 + 1)^{-1}|v|, |v| )_{t_0} = \|v\|^2, \]

where \( v \) and \( (\cdot, \cdot)_{t_0} \) are as in (3.5), and \( \| \cdot \| \) is the norm in \( L^2(M) \).

From (3.5) and (3.6) we obtain \( \|v\|^2 \leq 0 \), i.e. \( v = 0 \).

This shows that \( C_c^\infty(M) \) is dense in \( Q(T_0) \), and the theorem is proven. \([\square]\)

References


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