THE FORM SUM AND THE FRIEDRICHS EXTENSION
OF SCHröDINGER-TYPE OPERATORS
ON RIEMANNIAN MANIFOLDS

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Abstract. We consider \(H_V = \Delta_M + V\), where \((M, g)\) is a Riemannian manifold (not necessarily complete), and \(\Delta_M\) is the scalar Laplacian on \(M\). We assume that \(V = V_0 + V_1\), where \(V_0 \in L^2_{\text{loc}}(M)\) and \(-C \leq V_1 \in L^1_{\text{loc}}(M)\) \((C\) is a constant) are real-valued, and \(\Delta_M + V_0\) is semibounded below on \(C^\infty_c(M)\). Let \(\mathcal{T}_0\) be the Friedrichs extension of \((\Delta_M + V_0)\). We prove that the form sum \(\mathcal{T}_0 + V_1\) coincides with the self-adjoint operator \(\mathcal{T}_F\) associated to the closure of the restriction to \(C^\infty_c(M)\) of the sum of two closed quadratic forms of \(\mathcal{T}_0\) and \(V_1\). This is an extension of a result of Cycon. The proof adopts the scheme of Cycon, but requires the use of a more general version of Kato’s inequality for operators on Riemannian manifolds.

1. Introduction and the main result

Let \((M, g)\) be a Riemannian manifold (i.e. \(M\) is a \(C^\infty\)-manifold, \((g_{jk})\) is a Riemannian metric on \(M\), \(\dim M = n\)). We will assume that \(M\) is connected. We will also assume that we are given a positive smooth measure \(d\mu\), i.e. in any local coordinates \(x^1, x^2, \ldots, x^n\) there exists a strictly positive \(C^\infty\)-density \(\rho(x)\) such that \(d\mu = \rho(x)dx^1dx^2\ldots dx^n\). We do not assume that \((M, g)\) is complete.

We will consider a Schrödinger-type operator of the form

\[H_V = \Delta_M + V.\]

Here \(\Delta_M := d^*d\), where \(d: C^\infty(M) \to \Omega^1(M)\), and \(V \in L^1_{\text{loc}}(M)\) is real-valued.

1.1. Maximal operator. We define the maximal operator \(H_{V,\text{max}}\) associated to \(H_V\) as an operator in \(L^2(M)\) given by \(H_{V,\text{max}}u = H_Vu\) with domain

\[\text{Dom}(H_{V,\text{max}}) = \{u \in L^2(M) : Vu \in L^1_{\text{loc}}(M), \ H_Vu \in L^2(M)\}.\]

Here \(\Delta_M u\) in \(H_Vu = \Delta_M u + Vu\) is understood in the distributional sense.

We make the following assumptions on \(V\).

Assumption A. Assume \(V = V_0 + V_1\), where

(i) \(V_0 \in L^2_{\text{loc}}(M)\) and \(\Delta_M + V_0\) is semibounded below on \(C^\infty_c(M)\).

(ii) \(V_1 \in L^1_{\text{loc}}(M)\) and \(V_1 \geq -C\), where \(C > 0\) is a constant.

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1.2. Quadratic forms. For any self-adjoint operator \( T : \text{Dom}(T) \subset L^2(M) \to L^2(M) \) such that \( T \geq -\alpha \), we will denote by \( Q(T) \) the domain of the quadratic form \( t \) associated to \( T \). By Theorem 2.1 in [3], \( t \) is a closed semibounded below form, i.e. \( Q(T) \) is a Hilbert space with the inner product
\[
(u,v)_t = t(u,v) + (1+\alpha)(u,v)_{L^2(M)},
\]
where \( t(\cdot,\cdot) \) is the sesquilinear form obtained by polarization of \( t \).

1.3. Form sum. By (i) of Assumption A, \( \Delta_M + V_0 \) is symmetric and semibounded below on \( C_c^\infty(M) \), so we can associate to it a semibounded below self-adjoint operator \( T_0 \) (Friedrichs extension, cf. Theorem 14.1 in [3]).

We will denote by \( T_0 + V_1 \) the form sum of \( T_0 \) and \( V_1 \). By Theorem 4.1 in [3], this is the self-adjoint operator associated to the semibounded below closed quadratic form \( t_q \) given by the sum of two semibounded below closed quadratic forms corresponding to \( T_0 \) and \( V_1 \). By the same theorem, the following is true: \( Q(T_0 + V_1) = Q(T_0) \cap Q(V_1) \). Clearly, \( T_0 + V_1 \) is a self-adjoint restriction of \( H_{V_{\text{max}}} \).

1.4. Operator \( T_F \). Denote by \( t_{\text{min}} \) the restriction of \( t_q \) to \( C_c^\infty(M) \times C_c^\infty(M) \). Denote by \( T_F \) the self-adjoint operator associated to the closure of \( t_{\text{min}} \) in the sense of the norm in \( Q(T_0 + V_1) \). Clearly, \( T_F \) is a self-adjoint restriction of \( H_{V_{\text{max}}} \).

We will give a sufficient condition for \( T_F = T_0 + V_1 \).

**Theorem 1.5.** Suppose that Assumption A holds. Then \( T_F = T_0 + V_1 \).

**Remark 1.6.** Theorem 1.5 was proven by Cycon [2] in the case of the operator \( -\Delta + V \) in an open set \( M \subset \mathbb{R}^n \), where \( \Delta \) is the standard Laplacian on \( \mathbb{R}^n \) with the standard metric. In the case \( V_0 = 0 \) and \( M = \mathbb{R}^n \) with standard metric, Theorem 1.5 was proven in Simon [13].

2. Operators with a positive form core

**Definition 2.1.** Let \( T : C_c^\infty(M) \subset L^2(M) \to L^2(M) \) be a symmetric semibounded below operator. Let \( T_F \) denote its Friedrichs extension and \( Q(T_F)^+ \) the set of a.e. positive elements of \( Q(T_F) \). We say that \( T_F \) has a positive form core if for every \( u \in Q(T_F)^+ \) there exists a sequence \( u_k \in C_c^\infty(M)^+ \) such that
\[
\| u_k - u \|_t \to 0 \quad \text{as } k \to \infty,
\]
where \( \| \cdot \|_t \) is the norm associated to the closure of quadratic form \( t(v,w) := (Tv,w) \) \((v,w) \in C_c^\infty(M) \) via (1.2).

The main result of this section is

**Theorem 2.2.** Suppose that \( \Delta_M + V_0 \) is as in (i) of Assumption A. Let \( T_0 \) be the Friedrichs extension of \( (\Delta_M + V_0)|_{C_c^\infty(M)} \). Then \( T_0 \) has a positive form core.

**Remark 2.3.** In the case of the operator \( -\Delta + V_0 \) in an open set \( M \subset \mathbb{R}^n \), Theorem 2.2 was proven in [2, Th. 1].

We will first prove the following special case of Theorem 2.2

**Proposition 2.4.** Suppose that \( -\alpha \leq V_0 \in L^2_{\text{loc}}(M) \), where \( \alpha > 0 \) is a constant. Let \( T_0 \) be the Friedrichs extension of \( (\Delta_M + V_0)|_{C_c^\infty(M)} \). Then \( T_0 \) has a positive form core.
We begin with a few preliminary lemmas.

In what follows $T_b$ is as in the hypothesis of Proposition 2.4, and $t_b$ is the closed quadratic form associated with $T_b$. Without loss of generality, we may and we will assume that $V_0 \geq 0$ so that $T_b$ is a positive self-adjoint operator.

We will denote $W^{1,2}(M) := \{ u \in L^2(M) : du \in L^2(T^*M) \}$. By $W^{1,2}_0(M)$ we will denote the closure of $C_c^\infty(M)$ in the norm $\| u \|_{W^{1,2}}^2 := \| du \|^2 + \| u \|^2$, where $\| \cdot \|$ is the $L^2$ norm. By $Q(V_0)$ we will denote the set $\{ u \in L^2(M) : V_0^{1/2} u \in L^2(M) \}$.

Clearly, $Q(V_0)$ is the closure of $C_c^\infty(M)$ in the norm

\[
\| u \|_{V_0}^2 := \| V_0^{1/2} u \|^2 + \| u \|^2,
\]

where $\| \cdot \|$ is the norm in $L^2(M)$.

In the proofs of the following three lemmas, we will use the arguments from the proof of Lemma 1 in [3].

**Lemma 2.5.** $Q(T_b) = W^{1,2}_0(M) \cap Q(V_0)$.

**Proof.** Denote by $\mathcal{H}_1 := W^{1,2}_0(M) \cap Q(V_0)$. Consider a sesquilinear form $S : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$ given by

\[
S(u, v) := (du, dv) + (V_0^{1/2} u, V_0^{1/2} v),
\]

where $(\cdot, \cdot)$ is the inner product in $L^2$.

This sesquilinear form is closed, so the pre-Hilbert space $\mathcal{H}_1$ is complete in the norm

\[
(u, v)_{t_b} := (du, dv) + (V_0^{1/2} u, V_0^{1/2} v) + (u, v).
\]

By definition of $W^{1,2}_0(M)$ and $Q(V_0)$, it follows that $\mathcal{H}_1$ is the closure of $C_c^\infty(M)$ in the norm $\| \cdot \|_{t_b}$ corresponding to (2.2).

For all $u, v \in C_c^\infty(M)$, $(u, v)_{t_b} = (u, v) + (T_b u, v)$. By Theorem 14.1 in [3], $Q(T_b)$ is the closure of $C_c^\infty(M)$ in the norm $\| \cdot \|_{t_b}$ corresponding to (2.2), so $Q(T_b) = W^{1,2}_0(M) \cap Q(V_0)$. \hfill $\square$

**Lemma 2.6.** Assume that $u \in C_c^\infty(M)$. Then there exists a sequence $\phi_k \in C_c^\infty(M)^+$ such that $\| \phi_k - |u| \|_{t_b} \to 0$ as $k \to \infty$, where $\| \cdot \|_{t_b}$ is the norm corresponding to (2.2).

**Proof.** Let $u \in C_c^\infty(M)$. Then $|u| \in W^{1,2}_{\text{comp}}(M)$. Using a partition of unity we may assume that $u$ is supported in a coordinate neighborhood. Let $|u|^\rho = J^\rho |u|$, where $J^\rho$ is the Friedrichs mollifying operator; cf. Sect. 5.11 in [1]. Then $|u|^\rho \in C_c^\infty(M)$. It is well-known that $|u|^\rho \to |u|$ as $\rho \to 0+$ both in the space $W^{1,2}_{\text{comp}}(M)$ and in the space $L^2_{\text{comp}}(M)$. Also, since $|u|$ is continuous compactly supported on $M$ and $V_0 \in L^2_{\text{loc}}(M)$, we have

\[
\int V_0(|u|^\rho)^2 d\mu \to \int V_0 |u|^2 d\mu \quad \text{as} \quad \rho \to 0 +.
\]

Therefore,

\[
\| |u|^\rho - |u| \|_{t_b} \to 0 \quad \text{as} \quad \rho \to 0+,
\]

where $\| \cdot \|_{t_b}$ is the norm corresponding to (2.2). \hfill $\square$

**Lemma 2.7.** Suppose that $u \in Q(T_b)$. Then $|u| \in Q(T_b)$.
Proof. Let $u \in Q(T_b)$. By Lemma 2.9, we get $u \in W^{1,2}_0(M) \cap Q(V_0)$. Since $u \in W^{1,2}_0(M)$, Lemma 7.6 from [4] gives $|u| \in W^{1,2}_0(M)$. From $u \in Q(V_0)$, we immediately get $|u| \in Q(V_0)$. Therefore, $|u| \in W^{1,2}_0(M) \cap Q(V_0)$, so by Lemma 2.5 we obtain $|u| \in Q(T_b)$.

2.8. Proof of Proposition 2.4. We will follow the proof of Lemma 2 in [2].

Suppose that $u \in Q(T_b)^+$. By Lemma 2.5 there exists a sequence $\phi_j \in C_c^\infty(M)$ such that

$$\|\phi_j - u\|_{t_b} \to 0 \quad \text{as} \quad j \to \infty, \quad \text{(2.5)}$$

where $\|\cdot\|_{t_b}$ is the norm corresponding to (2.2).

In what follows, we will denote $(\text{sign } w)(x) := \frac{w(x)}{|w(x)|}$ when $w(x) \neq 0$, and 0 otherwise.

We have

$$\|\phi_j - u\|_{t_b}^2 = \|\phi_j - u\|^2 + \|d\phi_j - du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2 \leq \|\phi_j - u\|^2 + \|d\phi_j - du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2 \quad \text{(2.6)}$$

$$= \|\phi_j - u\|^2 + \|\text{Re}(\text{sign } \phi_j) d\phi_j - du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2,$$

where $\|\cdot\|$ denotes the norm $L^2$.

From (2.6) we obtain

$$\|\phi_j - u\|_{t_b}^2 \leq \|\phi_j - u\|^2 + \|\text{Re}(\text{sign } \phi_j) d\phi_j - du\|^2 + \|\text{Re}(\text{sign } \phi_j - 1) du\|^2$$

$$+ \|V_0^{1/2}(\phi_j - u)\|^2 \leq |\phi_j - u|^2 + \|d\phi_j - du\|^2 + \|\text{Re}(\text{sign } \phi_j - 1) du\|^2 + \|V_0^{1/2}(\phi_j - u)\|^2,$$

where $\|\cdot\|$ denotes the norm in $L^2$.

By (2.5), the first, second and fourth term on the right-hand side of (2.7) go to 0 as $j \to \infty$.

It remains to show that

$$\|\text{Re}(\text{sign } \phi_j - 1) du\| \to 0 \quad \text{as} \quad j \to \infty. \quad \text{(2.8)}$$

Since $\phi_j \to u$ in $L^2(M)$, a lemma of Riesz shows that there exists a subsequence $\phi_{j_k}$ such that $\phi_{j_k} \to u$ a.e. $d\mu$, as $k \to \infty$. By Lemma 7.7 from [4], it follows that $du = 0$ almost everywhere on $\{x \in M : u(x) = 0\}$. Hence, as $k \to \infty$, sign $\phi_{j_k} \to 1$ a.e. on $M$. Since $du \in L^2(T^*M)$, dominated convergence theorem immediately implies (2.8) (after passing to the chosen subsequence $\phi_{j_k}$).

This shows that

$$\|\phi_{j_k} - u\|_{t_b} \to 0 \quad \text{as} \quad k \to \infty. \quad \text{(2.9)}$$

By (2.9) and Lemma 2.6 there exists a sequence $\psi_l$ in $C_c^\infty(M)^+$ such that $\|\psi_l - u\|_{t_b} \to 0$ as $l \to \infty$. By Definition 2.1 it follows that $T_b$ has a positive form core.

In what follows, we will use a version of Kato’s inequality. For the proof of this inequality in general setting, cf. Theorem 5.6 in [1].

Theorem 2.9. Let $E$ be a Hermitian vector bundle on $M$, and let $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ be a Hermitian connection on $E$. Let $\nabla^* : C^\infty(T^*M \otimes E) \to C^\infty(E)$
be formal adjoint of $\nabla$ with respect to the usual inner product on $L^2(E)$. Assume
that $u \in L^1_{\text{loc}}(E)$ and $\nabla^* \nabla u \in L^1_{\text{loc}}(E)$. Then
\begin{equation}
\Delta_M |u| \leq \text{Re} \langle \nabla^* \nabla u, \text{sign } u \rangle,
\end{equation}
where
\[
\text{sign } u(x) = \begin{cases}
\frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

**Definition 2.10.** Let $(X, \mu)$ be a measure space. A bounded linear operator $A : L^2(X, \mu) \to L^2(X, \mu)$ is said to be *positivity preserving* if for every $0 \leq u \in L^2(X, \mu)$ we have $Au \geq 0$.

We will also use the following abstract theorem due to Simon; cf. Theorem 2.1 in [11].

**Theorem 2.11 (Simon [11]).** Suppose that $(X, \mu)$ is a measure space. Suppose that $H$ is a positive self-adjoint operator in $L^2(X, \mu)$. Then $(H + 1)^{-1}$ is positivity preserving if and only if the following two conditions are satisfied:

(i) For every $u \in Q(H)$, we have $|u| \in Q(H)$.

(ii) For every $u \in \text{Dom}(H)$ and $0 \leq v \in Q(H)$, the following is true:
\[
\text{Re}[h(|u|, v)] \leq \text{Re}((\text{sign } u)v, Hu),
\]
where $h$ is the quadratic form associated to $H$, and $(\text{sign } u)(x) = \frac{u(x)}{|u(x)|}$ whenever $u(x) \neq 0$, and $0$ otherwise.

The following lemma extends Lemma 1 from [5] to the case of Riemannian manifolds.

**Lemma 2.12.** The operator $(T_b + 1)^{-1}$ is positivity preserving.

*Proof.* Let $t_b$ be the quadratic form associated to $T_b$. By Theorem 2.11 it suffices to check the following conditions:

(i) For every $u \in Q(T_b)$, we have $|u| \in Q(T_b)$ and

(ii) For every $u \in \text{Dom}(T_b)$ and $0 \leq v \in Q(T_b)$, the following is true:
\[
\text{Re}[t_b(|u|, v)] \leq \text{Re}((\text{sign } u)v, T_b u).
\]

Condition (i) follows immediately by Lemma 2.7.

We now prove that condition (ii) holds. Let $u \in \text{Dom}(T_b)$. Then $(\Delta_M + V_0)u \in L^2(M)$ and hence $\Delta_M u \in L^1_{\text{loc}}(M)$.

For $u \in \text{Dom}(T_b)$ and $0 \leq \phi \in C^\infty_c(M)$ we have
\[
\text{Re}[t_b(|u|, \phi)] = \text{Re}([|u|, (\Delta_M + V_0)\phi]) = ([|u|, \Delta_M \phi] + ([|u|, V_0] \phi)
\]
\[
= (\Delta_M |u|, \phi) + (V_0 |u|, \phi)
\]
\[
\leq \text{Re}((\text{sign } u)\Delta_M u, \phi) + ((\text{sign } u)V_0 u, \phi)
\]
\[
= \text{Re}((\text{sign } u)T_b u, \phi) = \text{Re}((\text{sign } u)\phi, T_b u).
\]

Here we used integration by parts and the special case of Kato inequality (2.10) for $\Delta_M$.

Let $0 \leq v \in Q(T_b)$. By Proposition 2.7 there exists a sequence $\phi_j \in C^\infty_c(M)$ such that $\|\phi_j - v\|_{T_b} \to 0$ as $j \to \infty$, where $\| \cdot \|_{T_b}$ is the norm corresponding to (2.2).

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From (2.11), we obtain
\[ \text{Re}(t_b(u, v)) = \lim_{j \to \infty} \text{Re}(t_b(|u|, |v|)) \leq \lim_{j \to \infty} \text{Re}((\text{sign } u)\phi_j, T_b u) = \text{Re}((\text{sign } u)v, T_b u). \]
This proves condition (ii) and the lemma.

In what follows \( T_0 \) is as in the hypothesis of Theorem 2.2. Without loss of generality we may and we will assume that \( T_0 \) is a positive self-adjoint operator.

We will also use the notation \( Z_+ := \{1, 2, 3, \ldots \} \).

**Proposition 2.13.** \((T_0 + 1)^{-1}\) is positivity preserving.

**Proof.** We will adopt the arguments from the proof of Lemma 2 in [5] to our setting.

For every \( k \in \mathbb{Z}_+ \) and \( x \in M \), define
\[ Q_k(x) = \begin{cases} V_0(x) & \text{if } V_0(x) \geq -k, \\ -k & \text{if } V_0(x) < -k. \end{cases} \]

Let \( T_k \) be the Friedrichs extension of \((\Delta_M + Q_k)|C^\infty_c(M)\). Then for all \( k \in \mathbb{Z}_+ \) and \( u \in C^\infty_c(M) \), we have
\[(2.12) \quad (u, T_k u) \geq (u, T_0 u) \geq 0, \]
where \((\cdot, \cdot)\) is the inner product in \( L^2(M) \).

From (2.12) it follows that
\[(2.13) \quad T_0 \leq T_k \quad \text{for all } k \in \mathbb{Z}_+, \]
i.e. \( Q(T_k) \subset Q(T_0) \), and for all \( u \in Q(T_k) \), \( t_0(u, u) \leq t_k(u, u) \), where \( t_0 \) and \( t_k \) are the quadratic forms associated to \( T_0 \) and \( T_k \), respectively.

Furthermore, for all \( u \in C^\infty_c(M) \), the following is true:
\[(2.14) \quad (u, T_k u) \to (u, T_0 u) \quad \text{as } k \to \infty. \]

Clearly, \( C^\infty_c(M) \subset Q(T_k) \) for all \( k \in \mathbb{Z}_+ \). By definition of Friedrichs extension, it follows that \( C^\infty_c(M) \) is dense in \( Q(T_0) \) (in the norm of \( Q(T_0) \)).

This, (2.13) and (2.14) show that the hypotheses of abstract Theorem 7.9 from [3] are satisfied.

Therefore, as \( k \to \infty \), \( T_k \to T_0 \) in the strong resolvent sense.

By Lemma 2.12, \((T_k + 1)^{-1}\) is positivity preserving for all \( k \in \mathbb{Z}_+ \). Therefore, \((T_0 + 1)^{-1}\) is also positivity preserving.

**Corollary 2.14.** Assume that \( u \in Q(T_0) \). Then \(|u| \in Q(T_0)\).

**Proof.** \( T_0 \) is a positive self-adjoint operator in \( L^2(M) \). By Proposition 2.13, the operator \((T_0 + 1)^{-1}\) is positivity preserving. Now the corollary follows immediately from Theorem 2.12. \( \square \)

2.15. **Truncation operators corresponding to \( T_0 \).** Let \( T_0 \) be as in the hypothesis of Theorem 2.2.

Define \( V_0^+ := \max\{V_0, 0\} \), \( V_0^- := \max\{-V_0, 0\} \), and for each \( k \in \mathbb{Z}_+ \), let \( V_k^0 := \min\{k, V_0^-\} \).

Denote by \( T_+ \) and \( T_k \) the Friedrichs extension of \((\Delta_M + V_0^+)|C^\infty_c(M)\) and \((\Delta_M + V_0^+ - V_k^0)|C^\infty_c(M)\), respectively.

Let \( t_0, t_+ \) and \( t_k \) \((k \in \mathbb{Z}_+)\) be the quadratic forms associated to \( T_0, T_+ \) and \( T_k \), respectively.
The following lemma is analogous to Lemma 3 in [2].

**Lemma 2.16.** With the notations of Section 2.15,

(i) $T_k \to T_0$ in the strong resolvent sense as $k \to \infty$.

(ii) $Q(T_+) \subset Q(T_0)$.

**Proof.** For all $k \in \mathbb{Z}_+$, we clearly have $T_0 \leq T_k$. Also, $C_c^\infty(M) \subset Q(T_k)$ for all $k \in \mathbb{Z}_+$. By definition of $T_0$ it follows that $C_c^\infty(M)$ is dense in $Q(T_0)$ (in the norm of $Q(T_0)$).

Clearly, for all $w \in C_c^\infty(M),$ 

$$(w, T_k w) \to (w, T_0 w) \quad \text{as} \quad k \to \infty.$$ 

We now apply Theorem 7.9 in [3] to conclude the proof of (i).

Property (ii) follows immediately since $T_+ \geq T_0$.

2.17. **Proof of Theorem 2.2.** We will adopt the arguments from the proof of Theorem 1 in [2].

By the proof of Lemma 2.16 it follows that

$$(2.15) \quad Q(T_+) \subset Q(T_k) \subset Q(T_0)$$

and

$$(2.16) \quad \| \cdot \|_{t_0} \leq \| \cdot \|_{t_k} \leq \| \cdot \|_{t_+},$$

where $\| \cdot \|_{t_0}$, $\| \cdot \|_{t_k}$ and $\| \cdot \|_{t_+}$ are the norms associated to $t_0$, $t_k$ and $t_+$, respectively; cf. (1.2).

In fact, $Q(T_k) = Q(T_+)$ since the norms $\| \cdot \|_{t_k}$ and $\| \cdot \|_{t_+}$ are equivalent, because $V_0^+ - V_0^+$ and $V_0^+$ differ by a bounded function.

By Proposition 2.4, $T_+$ has a positive form core, i.e. for every $u \in Q(T_+)^+$ there exists a sequence $\phi_j \in C_c^\infty(M)^+$ such that $\| \phi_j - u \|_{t_+} \to 0$ as $j \to \infty$. By (2.16) it follows that

$$\| \phi_j - u \|_{t_0} \to 0 \quad \text{as} \quad j \to \infty.$$

To prove the theorem, it remains to show that for every $w \in Q(T_0)^+$, there exists a sequence $w_j \in Q(T_+)^+$ such that

$$(2.17) \quad \| w_j - w \|_{t_0} \to 0 \quad \text{as} \quad j \to \infty.$$

Let $w \in Q(T_0)^+$. For every $k, l \in \mathbb{Z}_+$ define

$$w_l := \left( \frac{1}{l} T_0 + 1 \right)^{-1} w$$

and

$$w_k^l := \left( \frac{1}{l} T_k + 1 \right)^{-1} w.$$ 

This makes sense since $0 \leq T_0 \leq T_k$ are self-adjoint operators.

By Lemma 2.12, the operator $(T_k + 1)^{-1} \subset Q(T_0)^+$ is positivity preserving. Hence $w_k^l \in Dom(T_+)^{-1}$ is positivity preserving. Hence $w_k^l \in Dom(T_+)^{-1}$.

Since the operators $(T_0 + 1)^{-1/2}$ and $(T_0/l + 1)^{-1}$ commute, we have

$$(2.18) \quad \| w_l - w \|_{t_0} = \left\| \left( \frac{1}{l} T_0 + 1 \right)^{-1} - 1 \right\| (T_0 + 1)^{1/2} w ||<1.$$
Clearly,
\[
\left( \frac{1}{l} T_0 + 1 \right)^{-1} \to 1 \quad \text{strongly as } l \to \infty.
\]

This and (2.19) show that
\[(2.19) \quad \|w_l - w\|_{t_0 + l} \to 0 \quad \text{as } l \to \infty.
\]

Fix \( l \in \mathbb{Z}_+ \). For each \( k \in \mathbb{Z}_+ \), let \( t_0 + l \) and \( t_k + l \) denote the quadratic forms corresponding to (positive self-adjoint) operators \( T_0 + l \) and \( T_k + l \), respectively. Let \( \| \cdot \|_{t_0 + l} \) and \( \| \cdot \|_{t_k + l} \) denote the norms in \( Q(T_0 + l) \) and \( Q(T_k + l) \), respectively; cf. (1.2). The corresponding inner products will be denoted by \( (\cdot, \cdot)_{t_0 + l} \) and \( (\cdot, \cdot)_{t_k + l} \).

Using (2.15), (2.16) and the Cauchy-Schwarz inequality we have for all \( w \in Q(T_0)^+ \)
\[(2.20) \quad \|(T_k + l)^{-1} w - (T_0 + l)^{-1} w\|^2_{t_0 + l}
\leq (1 - l)\|\|T_0^{-1} w\|^2 + \|T_0^{-1} w\|\|^2 - 2((T_k + l)^{-1} w, (T_0 + l)^{-1} w)\|T_0^{-1} w\|,
\quad \|T_0^{-1} w\| - ((T_k + l)^{-1} w, (T_0 + l)^{-1} w) \leq \|T_0^{-1} w\| - (T_k + l)^{-1} w\|\|w\|,
\]
where \( (\cdot, \cdot) \) is the inner product in \( L^2(M) \) and \( \| \cdot \| \) is the norm in \( L^2(M) \).

By Lemma 2.16 it follows that for fixed \( l \in \mathbb{Z}_+ \), \( T_k + l \to T_0 + l \) in the strong resolvent sense as \( k \to \infty \).

Clearly, for any positive self-adjoint operator \( A \), \( (A/l+1)^{-1} = l(A+l)^{-1} \). Therefore by (2.20), for a fixed \( l \in \mathbb{Z}_+ \),
\[\|w_k^l - w_l\|_{t_0 + l} \to 0 \quad \text{as } k \to \infty.\]

This is equivalent to
\[(2.21) \quad \|w_k^l - w_l\|_{t_0} \to 0 \quad \text{as } k \to \infty.\]

Since \( w_k^l \in Q(T_0)^+ \), we can use (2.19) and (2.21) to choose a subsequence \( \{w_j\} \) from \( \{w_k^l\} \) so that (2.17) holds.

This concludes the proof of the theorem. \( \square \)

3. PROOF OF THEOREM 1.5

We essentially follow the proof of Theorem 2 in [2], however, we need to use Kato inequality (2.10) for operators on manifolds.

Without loss of generality, we may and we will assume that \( \Delta_M + V_0 \geq 0 \) and \( V_1 \geq 0 \).

Let us denote \( T_q := T_0 + V_1 \) and let \( t_{\min} \) and \( t_q \) be as in Sections 1.3 and 1.5.

Since \( t_{\min} \) and \( t_q \) coincide on \( C_c^\infty(M) \), it is sufficient to show that \( C_c^\infty(M) \) is dense in the Hilbert space \( Q(T_q) = Q(T_0) \cap Q(V_1) \) with the inner product
\[(\cdot, \cdot)_{t_q} := t_q(\cdot, \cdot) + (\cdot, \cdot)_{L^2(M)},
\]
where \( t_q(\cdot, \cdot) \) is the sesquilinear form obtained by polarization of \( t_q \).

Let \( v \in Q(T_q) \) be orthogonal to \( C_c^\infty(M) \) in \( (\cdot, \cdot)_{t_q} \). This means that for all \( w \in C_c^\infty(M) \),
\[(\Delta_M + V_0 + V_1) v, w)_{L^2(M)} + (v, w)_{L^2(M)} = 0.\]
This leads to the following distributional equality:
\begin{equation}
(3.1) \quad \Delta_M v = -(V_0 + V_1 + 1)v.
\end{equation}
Since \( V_1 \in L^1_{\text{loc}}(M) \) and \( v \in Q(V_1) \), we have
\begin{equation}
2|V_1 v| = 2|V_1||v| \leq |V_1| + |V_1||v|^2
\end{equation}
which immediately gives \( V_1 v \in L^1_{\text{loc}}(M) \).
Since \( V_0 \in L^2_{\text{loc}}(M) \), it follows that \( V_0 v \in L^1_{\text{loc}}(M) \). From (3.1) we obtain
\begin{equation}
\Delta_M v \in L^1_{\text{loc}}(M).
\end{equation}
Using Kato inequality (2.10) in case \( \nabla = d \) and (3.1), we get
\begin{equation}
(3.2) \quad \Delta_M |v| \leq \text{Re}(\text{sign} \tilde{v} \Delta_M v) = -V_0|v| - V_1|v| - |v| \leq -(V_0 + 1)|v|.
\end{equation}
The last inequality in (3.2) holds since \( V_1 \geq 0 \).
From (3.2), we obtain the following distributional inequality:
\begin{equation}
(3.3) \quad (\Delta_M + V_0 + 1)|v| \leq 0.
\end{equation}
Let \( T_0 \) be as in the hypothesis, and let \( t_0 \) denote the closed quadratic form associated to \( T_0 \).
Using (3.3), we get
\begin{equation}
(3.4) \quad ((T_0 + 1)w, |v|)_L^2(M) \leq 0 \quad \text{for all } w \in C_c^\infty(M)\,^+.
\end{equation}
Since \( v \in Q(T_0) \), Corollary 2.11 gives \( |v| \in Q(T_0) \). Therefore, we can write (3.4) as
\begin{equation}
(3.5) \quad (w, |v|)_{t_0} \leq 0 \quad \text{for all } w \in C_c^\infty(M)\,^+,
\end{equation}
where \((\cdot, \cdot)_{t_0} = t_0(\cdot, \cdot)_{L^2(M)}\) denotes the inner product in \( Q(T_0) \).
Let \( f := (T_0 + 1)^{-1}|v| \). By Proposition 2.13, \((T_0 + 1)^{-1}\) is positivity preserving, so \( f \in \text{Dom}(T_0)^+ \subset Q(T_0)^+ \).
By Theorem 2.2, \( T_0 \) has a positive form core. Therefore, there exists a sequence \( f_k \in C_c^\infty(M)\,^+ \) such that
\begin{equation}
(3.6) \quad \lim_{k \to \infty} (f_k, |v|)_{t_0} = (f, |v|)_{t_0} = ((T_0 + 1)^{-1}|v|, |v|)_{t_0} = \|v\|^2,
\end{equation}
where \( v \) and \((\cdot, \cdot)_{t_0}\) are as in (3.5), and \( \| \cdot \| \) is the norm in \( L^2(M) \).
From (3.5) and (3.6) we obtain \( \|v\|^2 \leq 0 \), i.e. \( v = 0 \).
This shows that \( C_c^\infty(M) \) is dense in \( Q(T_0) \), and the theorem is proven. \( \square \)

References


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