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BLOW-UP FOR DEGENERATE PARABOLIC EQUATIONS WITH NONLOCAL SOURCE

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ABSTRACT. This paper deals with the blow-up properties of the solution to the degenerate nonlinear reaction diffusion equation with nonlocal source $x^q u_t - (x^\gamma u_x)_x = \int_0^a u^p dx$ in $(0,a)\times(0,T)$ subject to the homogeneous Dirichlet boundary conditions. The existence of a unique classical nonnegative solution is established and the sufficient conditions for the solution exists globally or blows up in finite time are obtained. Furthermore, it is proved that under certain conditions the blow-up set of the solution is the whole domain.

1. INTRODUCTION

Let $T \le +\infty, a, q, \gamma$ and p be constants with $T > 0, a > 0, \gamma \in [0,1)$, $|q| + \gamma \ne 0$ and $p > 1$. Let $D = (0,a)$ and $\Omega_t = D \times (0,t]$, and let $\bar{D}$ and $\overline{\Omega}_t$ be their respective closures. We consider the following initial boundary value problem of the degenerate nonlinear reaction diffusion equation with nonlocal source:

$$\begin{aligned} &x^q u_t - (x^\gamma u_x)_x = \textstyle\int_0^a u^p dx, && (x,t) \in (0,a)\times(0,T), \\ &u(0,t) = u(a,t) = 0, && t \in (0,T), \\ &u(x,0) = u_0(x), && x \in [0,a], \end{aligned} \tag{1.1}$$

where $u_0(x) \in C^{2+\alpha}(\overline{D})$ for some constant $\alpha \in (0,1)$, $u_0(0) = u_0(a) = 0, u_0(x) \ge 0$. Since $|q| + \gamma \ne 0$, the coefficients of u_t, u_x and u_{xx} may tend to 0 or ∞ as x tends to 0, we can regard the equation as degenerate and singular.

Floater [7] and Chan & Liu [4] investigated the blow-up properties of the following degenerate parabolic problem:

$$\begin{aligned} &x^q u_t - u_{xx} = u^p, && (x,t) \in (0,a)\times(0,T), \\ &u(0,t) = u(a,t) = 0, && t \in (0,T), \\ &u(x,0) = u_0(x), && x \in [0,a], \end{aligned} \tag{1.2}$$

where $q > 0$ and $p > 1$. Under certain conditions on the initial datum $u_0(x)$, Floater [7] proved that for the case $1 < p \le q+1$, if the solution $u(x,t)$ of (1.2) blows up in finite time, then it blows up at the boundary $x = 0$. This contrasts with a result of Friedman and Mcleod [9], who showed that for the case $q = 0$, the

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blow-up set of the solution $u(x,t)$ of (1.2) is a proper compact subset of D. The motivation for studying problem (1.2) comes from Ockendon's model (see [13]) for the flow in a channel of a fluid whose viscosity is temperature dependent

$$xu_t = u_{xx} + e^u, \tag{1.3}$$

where u represents the temperature of the fluid. Floater in [7] approximated e^u by u^p and considered equation (1.2). Budd, Galaktionov and Chen in [2] generalized the results of [7] to the following degenerate quasilinear parabolic equation:

$$x^q u_t = (u^m)_{xx} + u^p \tag{1.4}$$

subject to homogeneous Dirichlet conditions in the critical exponent case $q = (p-1)/m$, where $q > 0, m \geq 1$ and $p > 1$. They pointed out that the general classification of blow-up solutions for the degenerate equation (1.4) stays the same for the quasilinear equation

$$u_t = (u^m)_{xx} + u^p; \tag{1.5}$$

see [2] and [15, Chapter 4]. Chan & Liu in [4] continued to study problem (1.2) for the case $p > q+1$. They proved that under certain conditions $x = 0$ is not a blow-up point and the blow-up set is a proper compact subset of D.

In this paper, we continue to consider (1.2) with the reaction term u^p replaced by $\int_0^a u^p dx$ and investigate the effect of the singularity, degeneracy and nonlocal reaction on the behavior of the solution of (1.1). The difficulties are the establishment of the corresponding comparison principle and the construction of an upper solution of (1.1). It is different from [4] and [7]; we prove that under certain conditions the blow-up set of the solution of (1.1) is the whole domain. This is also consistent with the conclusion that in a nonlocal problem blow-up can be global (see [1, 16, 17]).

This paper is organized as follows: in section 2, we show the existence of a unique classical solution; in section 3, we give some criteria for the solution $u(x,t)$ to exist globally or to blow up in finite time.

2. Local existence

In order to prove the existence of a unique positive solution to (1.1), we first show the following comparison result.

Lemma 2.1. *Let $b(x,t)$ be a continuous nonnegative function defined on $[0,a] \times [0,r]$ for any $r \in (0,T)$, and $u(x,t) \in C(\overline{\Omega}_r) \cap C^{2,1}(\Omega_r)$ satisfies*

$$\begin{array}{ll} x^q u_t - (x^\gamma u_x)_x \geq \int_0^a b(x,t)u(x,t)dx, & (x,t) \in (0,a) \times (0,r], \\ u(0,t) \geq 0, u(a,t) \geq 0, & t \in (0,r], \\ u(x,0) \geq 0, & x \in [0,a]. \end{array} \tag{2.1}$$

Then $u(x,t) \geq 0$ on $[0,a] \times [0,T)$.

Proof. First, similar to the proof of Lemma 2.1 in [18], by using the positive Lemma 2.2.1 for uniformly parabolic equations in [14], we can easily obtain the following conclusion:

If $w(x,t) \in C(\overline{\Omega}_r) \cap C^{2,1}(\Omega_r)$ satisfies

$$
(2.2) \qquad \begin{array}{ll}
x^q w_t - (x^\gamma w_x)_x \geq \int_0^a b(x,t)w(x,t)dx, & (x,t) \in (0,a) \times (0,r], \\
w(0,t) > 0, w(a,t) \geq 0, & t \in [0,r], \\
w(x,0) \geq 0, & x \in [0,a],
\end{array}
$$

then $w(x,t) > 0, (x,t) \in (0,a) \times (0,r]$.

Next, let $\gamma' \in (\gamma, 1)$ be a positive constant and

$$
w(x,t) = u(x,t) + \eta(1 + x^{\gamma'-\gamma})e^{ct},
$$

where $\eta > 0$ is sufficiently small and c is a positive constant to be determined. Then $w(x,t) > 0$ on the parabolic boundary of Ω_r and

$$
\begin{aligned}
& x^q w_t - (x^\gamma w_x)_x - \textstyle\int_0^a b(x,t)w(x,t)dx \\
& \geq x^q \eta(1 + x^{\gamma'-\gamma})ce^{ct} + (\gamma' - \gamma)(1 - \gamma')\eta e^{ct}/x^{2-\gamma'} - \textstyle\int_0^a b(x,t)\eta(1 + x^{\gamma'-\gamma})e^{ct}dx \\
& \geq \eta e^{ct}[cx^q + (\gamma' - \gamma)(1 - \gamma')/x^{2-\gamma'} - a(1 + a^{\gamma'-\gamma}) \max_{(x,t)\in[0,a]\times[0,r]} b(x,t)].
\end{aligned}
$$

If $\max_{(x,t)\in[0,a]\times[0,r]} b(x,t) \leq (\gamma' - \gamma)(1 - \gamma)/(a^{3-\gamma'}(1 + a^{\gamma'-\gamma}))$, then

$$
\begin{aligned}
& x^q w_t - (x^\gamma w_x)_x - \textstyle\int_0^a b(x,t)w(x,t)dx \\
& \geq \eta e^{ct}[(\gamma' - \gamma)(1 - \gamma')\frac{1}{x^{2-\gamma'}} - a(1 + a^{\gamma'-\gamma}) \max_{(x,t)\in[0,a]\times[0,r]} b(x,t)] \\
& \geq 0.
\end{aligned}
$$

On the other hand, if $\max_{(x,t)\in[0,a]\times[0,r]} b(x,t) > (\gamma'-\gamma)(1-\gamma')/(a^{3-\gamma'}(1+a^{\gamma'-\gamma}))$, let x_0 be the root of the algebraic equation

$$
a(1 + a^{\gamma'-\gamma}) \max_{(x,t)\in[0,a]\times[0,r]} b(x,t) = (\gamma' - \gamma)(1 - \gamma')/x^{2-\gamma'},
$$

and let $c > 0$ be sufficiently large such that

$$
(2.3) \qquad c > \begin{cases} (\max_{(x,t)\in[0,a]\times[0,r]} b(x,t))a(1 + a^{\gamma'-\gamma})/x_0^q & \text{for} \quad q \geq 0, \\ (\max_{(x,t)\in[0,a]\times[0,r]} b(x,t))a(1 + a^{\gamma'-\gamma})/a^q & \text{for} \quad q < 0. \end{cases}
$$

Then we have

$$
\begin{aligned}
& x^q w_t - (x^\gamma w_x)_x - \textstyle\int_0^a b(x,t)w(x,t)dx \\
& \geq \begin{cases} \eta e^{ct}[(\gamma' - \gamma)(1 - \gamma')/x^{2-\gamma'} - a(1 + a^{\gamma'-\gamma}) \max\limits_{(x,t)\in[0,a]\times[0,r]} b(x,t)] & \text{for } x \leq x_0, \\ \eta e^{ct}[cx^q - a(1 + a^{\gamma'-\gamma}) \max_{(x,t)\in[0,a]\times[0,r]} b(x,t)] & \text{for } x > x_0 \end{cases} \\
& \geq 0.
\end{aligned}
$$

From the above conclusion, we know that $w(x,t) > 0$ on $[0,a] \times [0,r]$. Letting $\eta \to 0^+$, we then have $u(x,t) \geq 0$ on $[0,a] \times [0,r]$. From the arbitrariness of $r \in (0,T)$, we complete the proof. □

Obviously, $\hat{u} = 0$ is a lower solution of (1.1); we need to construct an upper solution.

Lemma 2.2. *There exists a positive constant $t_0 (t_0 < T)$ such that the problem (1.1) has an upper solution $h(x,t) \in C(\overline{\Omega_{t_0}}) \cap C^{2,1}(\Omega_{t_0})$.*

Proof. Let

$$\psi(x) = (\frac{x}{a})^{1-\gamma}(1-\frac{x}{a}) + (\frac{x}{a})^{\frac{1-\gamma}{2}}(1-\frac{x}{a})^{\frac{1}{2}},$$

and let k_0 be a positive constant such that $k_0\psi(x) \geq u_0(x)$. Denote the positive constant $\int_0^1 [s^{1-\gamma}(1-s) + s^{\frac{1-\gamma}{2}}(1-s)^{\frac{1}{2}}]^p ds$ by b_0. Let $k_2 \in (0, \frac{1-\gamma}{2-\gamma})$ be a positive constant such that

$$k_2 < (2^p a^{3-\gamma} b_0 k_0^{p-1})^{-\frac{2}{1-\gamma}}, \tag{2.4}$$

and let $k(t)$ be the positive solution of the following initial value problem:

$$\begin{cases} k'(t) = \begin{cases} \dfrac{b_0 k^p(t)}{a^{q-1} k_2^{q+\frac{1}{2}}(1-k_2)^{\frac{1-\gamma}{2}}[1 + k_2^{\frac{1}{2}}(1-k_2)^{\frac{1-\gamma}{2}}]}, & q \geq 0, \\ \dfrac{b_0 k^p(t)}{a^{q-1} k_2^{\frac{1}{2}}(1-k_2)^{q+\frac{1-\gamma}{2}}[1 + k_2^{\frac{1}{2}}(1-k_2)^{\frac{1-\gamma}{2}}]}, & q < 0, \end{cases} & t > 0, \\ k(0) = k_0. \end{cases} \tag{2.5}$$

Since $k(t)$ is increasing, we can choose $t_0 > 0$ such that $k(t) \leq 2k_0$ for all $t \in [0, t_0]$. Let $h(x,t) = k(t)\psi(x)$; then $h(x,t) \geq 0$ on $\overline{\Omega}_{t_0}$. We would like to show that $h(x,t)$ is an upper solution of (1.1) in Ω_{t_0}. To do this, let us construct a function J by

$$J = x^q h_t - (x^\gamma h_x)_x - \int_0^a h^p dx, \quad (x,t) \in \Omega_{t_0}.$$

Then,

$$\begin{aligned} J &= x^q k'(t)\psi(x) - (x^\gamma \psi'(x))' k(t) - \int_0^a k^p(t)\psi^p(x) dx \\ &= x^q k'(t)\psi(x) + [(2-\gamma)/a^{2-\gamma} + (\tfrac{(1-\gamma)^2}{4} x^{\frac{\gamma-3}{2}}(a-x)^{\frac{1}{2}} + \tfrac{1}{2} x^{\frac{\gamma-1}{2}}(a-x)^{-\frac{1}{2}} \\ &\quad + \tfrac{1}{4} x^{\frac{1+\gamma}{2}}(a-x)^{-\frac{3}{2}})/a^{1-\frac{\gamma}{2}}] k(t) - a b_0 k^p(t) \\ &\geq x^q k'(t)\psi(x) + x^{\frac{\gamma-1}{2}}(a-x)^{-\frac{1}{2}} k(t)/(2a^{1-\frac{\gamma}{2}}) - a b_0 k^p(t), \quad (x,t) \in \Omega_{t_0}. \end{aligned}$$

For $(x,t) \in (0, ak_2) \times (0, t_0] \cup (a(1-k_2), a) \times (0, t_0]$, by (2.4), we have

$$\begin{aligned} J &\geq x^{\frac{\gamma-1}{2}}(a-x)^{-\frac{1}{2}} k(t)/(2a^{1-\frac{\gamma}{2}}) - a b_0 k^p(t) \\ &\geq [k_2^{\frac{\gamma-1}{2}}/(2a^{2-\gamma}) - a b_0 k^{p-1}(t_0)] k(t) \\ &\geq [k_2^{\frac{\gamma-1}{2}}/(2a^{2-\gamma}) - a b_0 (2k_0)^{p-1}] k(t) \\ &\geq 0. \end{aligned}$$

For $(x,t) \in [ak_2, a(1-k_2)] \times (0, t_0]$, by (2.5), we have

$$\begin{aligned} J &\geq x^q k'(t)\psi(x) - a b_0 k^p(t) \\ &\geq \begin{cases} a^q k_2^q k'(t)[k_2(1-k_2)^{1-\gamma} + k_2^{\frac{1}{2}}(1-k_2)^{\frac{1-\gamma}{2}}] - a b_0 k^p(t), & q \geq 0, \\ a^q (1-k_2)^q k'(t)[k_2(1-k_2)^{1-\gamma} + k_2^{\frac{1}{2}}(1-k_2)^{\frac{1-\gamma}{2}}] - a b_0 k^p(t), & q < 0 \end{cases} \\ &\geq 0. \end{aligned}$$

Thus, $J(x,t) \geq 0$ in Ω_{t_0}. It follows from $h(0,t) = h(a,t) = 0$ and $h(x,0) = k_0\psi(x) \geq u_0(x)$ that $h(x,t)$ is an upper solution of (1.1) in Ω_{t_0}. ∎

To show the existence of the classical solution $u(x,t)$ of (1.1), let us introduce a "cut-off function" $\rho(x)$. By Dunford and Schwartz [6, p. 1640], there exists a nondecreasing function $\rho(x) \in C^3(R)$ such that $\rho(x) = 0$ if $x \le 0$, and $\rho(x) = 1$ if $x \ge 1$. Let $0 < \delta < \frac{1-\gamma}{2-\gamma}a$,

$$\rho_\delta(x) = \begin{cases} 0, & x \le \delta, \\ \rho(\frac{x}{\delta} - 1), & \delta < x < 2\delta, \\ 1, & x \ge 2\delta, \end{cases}$$

and let $u_{0\delta}(x) = \rho_\delta(x)u_0(x)$. We note that

$$\frac{\partial}{\partial\delta}u_{0\delta}(x) = \begin{cases} 0, & x \le \delta, \\ -\frac{x}{\delta^2}\rho'(\frac{x}{\delta} - 1)u_0(x), & \delta < x < 2\delta, \\ 0, & x \ge 2\delta. \end{cases}$$

Since ρ is nondecreasing, we have $\frac{\partial}{\partial\delta}u_{0\delta}(x) \le 0$. From $0 \le \rho(x) \le 1$, we have $u_0(x) \ge u_{0\delta}(x)$ and $\lim_{\delta\to 0} u_{0\delta}(x) = u_0(x)$.

Let $D_\delta = (\delta, a), \omega_\delta = D_\delta \times (0, t_0]$, $\overline{D}_\delta$ and $\overline{\omega}_\delta$ be their respective closures, and let $S_\delta = \{\delta, a\} \times (0, t_0]$. We consider the following regularized problem:

$$\begin{aligned} &x^q u_{\delta t} - (x^\gamma u_{\delta x})_x = \textstyle\int_\delta^a u_\delta^p dx, && (x,t) \in \omega_\delta, \\ &u_\delta(\delta, t) = u_\delta(a,t) = 0, && t \in (0, t_0], \\ &u_\delta(x, 0) = u_{0\delta}(x), && x \in \overline{D}_\delta. \end{aligned} \tag{2.6}$$

By using Schauder's fixed point theorem, we have

Theorem 2.3. *Problem* (2.6) *admits a unique nonnegative solution* $u_\delta \in C^{2+\alpha,1+\frac{\alpha}{2}}$ $(\overline{\omega}_\delta)$. *Moreover,* $0 \le u_\delta(x,t) \le h(x,t)$, $(x,t) \in \overline{\omega}_\delta$, *where* $h(x,t)$ *is given by Lemma* 2.2.

Proof. From the proof of Lemma 2.1, we know that there exists at most one nonnegative solution u_δ. To prove existence, we use Schauder's fixed point theorem. Let

$$X = \{v \in C^{\alpha,\frac{\alpha}{2}}(\overline{\omega}_\delta) : 0 \le v(x,t) \le h(x,t), (x,t) \in \overline{\omega}_\delta\}.$$

Obviously, X is a closed convex subset of the Banach space $C^{\alpha,\frac{\alpha}{2}}(\overline{\omega}_\delta)$. For any $v \in X$, let us consider the following linearized uniformly parabolic problem:

$$\begin{aligned} &x^q w_{\delta t} - (x^\gamma w_{\delta x})_x = \textstyle\int_\delta^a v^p dx, && (x,t) \in \omega_\delta, \\ &w_\delta(\delta, t) = w_\delta(a,t) = 0, && t \in (0, t_0), \\ &w_\delta(x, 0) = u_{0\delta}(x), && x \in [\delta, a]. \end{aligned} \tag{2.7}$$

It is easy to see that $\hat{w}(x,t) = 0$ and $\tilde{w}(x,t) = h(x,t)$ are lower and upper solutions of the problem (2.7). We also note that $x^{-q+\gamma}, x^{-q-1+\gamma}, x^{-q} \in C^{\alpha,\frac{\alpha}{2}}(\overline{\omega}_\delta)$, $x^{-q}\int_\delta^a v^p dx \in C^{\alpha,\frac{\alpha}{2}}(\overline{\omega}_\delta)$, $u_{0\delta} \in C^{2+\alpha}(\overline{D}_\delta)$. It follows from Theorem 4.2.2 of Ladde et al. [10, p. 143] that the problem (2.7) has a unique solution $w_\delta(x,t;v) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\omega}_\delta)$, which satisfies $0 \le w_\delta(x,t;v) \le h(x,t)$. Thus, we can define a mapping Z from X into $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\omega_\delta})$ such that $Z\,v(x,t) = w_\delta(x,t;v)$, where $w_\delta(x,t;v)$ denotes the unique solution of the problem (2.7) corresponding to $v(x,t) \in X$. To use Schauder's fixed point theorem, we need to verify that Z maps X into itself and Z is continuous and compact.

In fact, $Z\,X \subset X$ and the embedding operator from the Banach space $C^{2+\alpha,1+\frac{\alpha}{2}}$ $(\overline{\omega}_\delta)$ to the Banach space $C^{\alpha,\frac{\alpha}{2}}(\overline{\omega}_\delta)$ is compact. Therefore Z is compact. To show Z is continuous, let us consider in X a sequence $\{v_n(x,t)\}$ which converges to $v(x,t)$ uniformly in the norm $\|\cdot\|_{\alpha,\frac{\alpha}{2}}$. We know that $v(x,t) \in X$. Let $w_{\delta n}(x,t)$ and $w_\delta(x,t)$ be the solutions of the problem (2.7) corresponding to $v_n(x,t)$ and $v(x,t)$, respectively. Without loss of generality, let us assume that

$$\|v_n(x,t)\|_{\alpha,\frac{\alpha}{2}} \leq \|v(x,t)\|_{\alpha,\frac{\alpha}{2}} + 1 \text{ for any } n \geq 1.$$

Let $w(x,t) = w_{\delta n}(x,t) - w_\delta(x,t)$. Then we have

$$\begin{array}{ll} x^q w_t - (x^\gamma w_x)_x = \int_\delta^a (v_n^p - v^p)dx, & (x,t) \in \omega_\delta, \\ w(\delta,t) = 0, w(a,t) = 0, & t \in (0,t_0], \\ w(x,0) = 0, & x \in \overline{D}_\delta. \end{array}$$

From theorem 4.5.2 of Ladyzenskaya et al. [11, p. 320], there exists a positive constant C (independent of v_n and v) such that

$$\begin{aligned} \|w\|_{2+\alpha,1+\frac{\alpha}{2}} &\leq C\| \textstyle\int_\delta^a (v_n^p - v^p)dx\|_{\alpha,\frac{\alpha}{2}} \\ &\leq C\ a\ p\|(v+\tau(v_n - v))^{p-1}\|_{\alpha,\frac{\alpha}{2}}\|v_n - v\|_{\alpha,\frac{\alpha}{2}} \\ &\leq C\ a\ p[3(\|v\|_{\alpha,\frac{\alpha}{2}} + 1)]^{p-1}\|v_n - v\|_{\alpha,\frac{\alpha}{2}}, \end{aligned}$$

where $\tau \in (0,1)$. This shows that the mapping Z is continuous. By Schauder's fixed point theorem, we complete the proof. □

Now we can prove the following local existence result.

Theorem 2.4. *There exists some $t_0 (< T)$ such that problem* (1.1) *has a unique nonnegative solution $u(x,t) \in C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$.*

Proof. From Theorem 2.3, the problem (2.6) has a unique nonnegative solution $u_\delta \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\omega}_\delta)$. It follows from Lemma 2.1 that $u_{\delta_1}(x,t) \leq u_{\delta_2}(x,t)$ in ω_{δ_1} if $\delta_1 > \delta_2$. Therefore $\lim_{\delta\to 0} u_\delta(x,t)$ exists for all $(x,t) \in (0,a] \times [0,t_0]$. Let $u(x,t) = \lim_{\delta\to 0} u_\delta(x,t), (x,t) \in (0,a]\times[0,t_0]$ and define $u(0,t) = 0, t \in [0,t_0]$. We would like to show that $u(x,t)$ is a classical solution of (1.1) in Ω_{t_0}. For any $(x_1,t_1) \in \Omega_{t_0}$, there exist three domains $Q' = (a_1', a_2') \times (t_2', t_3']$, $Q'' = (a_1'', a_2'') \times (t_2'', t_3'']$, and $Q''' = (a_1''', a_2''') \times (t_2''', t_3''']$ such that $(x_1,t_1) \in Q' \subset Q'' \subset Q''' \subset (0,a) \times (0,t_0]$ with $0 < a_1''' < a_1'' < a_1' < x_1 < a_2' < a_2'' < a_2''' < a$, $0 \leq t_2''' \leq t_2'' \leq t_2' < t_1 < t_3' \leq t_3'' \leq t_3''' \leq t_0$. Since $u_\delta(x,t) \leq h(x,t)$ in Q''' and $h(x,t)$ is finite on $\overline{Q}'''$, we have for any constant $\tilde{q} > 1$,

(i) $\|u_\delta\|_{L^{\tilde{q}}(Q''')} \leq \|h\|_{L^{\tilde{q}}(Q''')} \leq k_3$ for some positive constant k_3.

(ii) $\|x^{-q}\int_\delta^a u_\delta^p dx\|_{L^{\tilde{q}}(Q''')} \leq (a^*)^{-q}\|\int_0^a h^p(x,t)dx\|_{L^{\tilde{q}}(Q''')} \leq k_4$ for some positive constant k_4, where $a^* = a_1'''$ if $q \geq 0$ and $a^* = a_2'''$ if $q < 0$.

By the local L^p estimate of Ladyzenskaya et al. [11, pp. 341-342, 352], $u_\delta \in W_{\tilde{q}}^{2,1}(Q'')$. By embedding theorems [11, pp. 61 and 80], $W_{\tilde{q}}^{2,1}(Q'') \hookrightarrow H^{\alpha,\frac{\alpha}{2}}(Q'')$ if we choose $\tilde{q} > \frac{2}{1-\alpha}$. Then, $\|u_\delta\|_{H^{\alpha,\frac{\alpha}{2}}(Q'')} \le k_5$ for some positive constant k_5. Now

$$\begin{aligned}
&\|x^{-q}\textstyle\int_\delta^a u_\delta^p dx\|_{H^{\alpha,\frac{\alpha}{2}}(Q'')}\\
&\le (a^*)^{-q}\|\textstyle\int_\delta^a h^p(x,t)dx\|_\infty + \sup_{\substack{(x,t)\in Q''\\(\tilde{x},t)\in Q''}} \frac{|\int_\delta^a u_\delta^p dx|\cdot|x^{-q}-(\tilde{x})^{-q}|}{|x-\tilde{x}|^\alpha}\\
&+ \sup_{\substack{(\tilde{x},t)\in Q''\\(\tilde{x},\tilde{t})\in Q''}} \frac{|(\tilde{x})^{-q}|\cdot|\int_\delta^a p(u_\delta(x,\tilde{t})+\tau(u_\delta(x,t)-u_\delta(x,\tilde{t})))^{p-1}(u_\delta(x,t)-u_\delta(x,\tilde{t}))dx|}{|t-\tilde{t}|^{\frac{\alpha}{2}}}\\
&\le (a^*)^{-q}\|\textstyle\int_0^a h^p(x,t)dx\|_\infty + \|\int_0^a h^p(x,t)dx\|_\infty \|x^{-q}\|_{H^\alpha(a_1'',a_2'')}\\
&\quad +(a^*)^{-q}\|\textstyle\int_0^a ph^{p-1}(x,t)dx\|_\infty \|u_\delta\|_{H^{\alpha,\frac{\alpha}{2}}(Q'')}\\
&\le k_6
\end{aligned}$$

for some positive constant k_6 which is independent of δ, where $\tau \in (0,1)$. By Theorem 4.10.1 of Ladyzenskaya et al. [11, pp. 351-352], we have

$$\|u_\delta\|_{H^{2+\alpha,1+\frac{\alpha}{2}}(Q')} \le k_7$$

for some positive constant k_7 independent of δ. This implies that $u_\delta, u_{\delta t}, u_{\delta x}$ and $u_{\delta xx}$ are equicontinuous in Q'. By the Ascoli-Arzela theorem, we know that

$$\|u\|_{H^{2+\alpha',1+\frac{\alpha'}{2}}(Q')} \le k_8$$

for some $\alpha' \in (0,\alpha)$ and some positive constant k_8 independent of δ, and that the derivatives of u are the uniform limits of the corresponding partial derivatives of u_δ. Hence u satisfies (1.1) and $\lim_{t\to 0} u(x,t) = \lim_{t\to 0}\lim_{\delta\to 0} u_\delta(x,t) = \lim_{\delta\to 0} u_{0\delta}(x) = u_0(x)$. Also from $0 \le u(x,t) \le h(x,t)$ and $h(x,t) \to 0$ as $x \to 0$ or $x \to a$, we have $\lim_{x\to 0} u(x,t) = \lim_{x\to a} u(x,t) = 0$, thus $u \in C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$ is the solution of (1.1) in Ω_{t_0}. We complete the proof. □

By using Lemma 2.1, there exists at most one nonnegative solution u of (1.1). A proof similar to that of Theorem 2.5 of Floater [7] gives the following continuational result.

Theorem 2.5. *Let T be the supremum over t_0 for which there is a unique nonnegative solution $u(x,t) \in C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$ of (1.1). Then (1.1) has a unique nonnegative solution $u(x,t) \in C([0,a]\times[0,T)) \cap C^{2,1}((0,a)\times(0,T))$. If $T < +\infty$, then $\limsup_{t\to T}\max_{x\in[0,a]} u(x,t) = +\infty$.*

3. Blow-up of solution

In this section, we will give some global existence and blow-up results of the solution of (1.1).

3.1. **Existence and nonexistence of the global solution.** In this subsection, we would assume that $q > \gamma - 1$.

First, the solution of the following elliptic boundary value problem

$$-(x^\gamma\psi'(x))' = 1,\ x \in (0,a);\quad \psi(0) = \psi(a) = 0$$

is given by $\psi(x) = \frac{a^{2-\gamma}}{2-\gamma}(\frac{x}{a})^{1-\gamma}(1-\frac{x}{a})$. By direct computation,

$$\int_0^a \psi^p(x)dx = a^{(2-\gamma)p+1}B(p(1-\gamma)+1, p+1)/(2-\gamma)^p,$$

where $B(l,m)$ is a Beta function which is defined by $B(l,m)=\int_0^1 x^{l-1}(1-x)^{m-1}dx$. Let

$$a_1 = [\ a^{(2-\gamma)p+1}B(p(1-\gamma)+1,p+1)/(2-\gamma)^p\ ]^{-\frac{1}{p-1}}.$$

Then we have the following global existence result.

Theorem 3.1. *Let $u(x,t)$ be the solution of* (1.1). *If $u_0(x)\le a_1\psi(x)$, then $u(x,t)$ exists globally.*

Proof. Since $a_1=\left[a^{(2-\gamma)p+1}B(p(1-\gamma)+1,p+1)/(2-\gamma)^p\right]^{-\frac{1}{p-1}}$, we have

$$a_1=[a^{(2-\gamma)p+1}B(p(1-\gamma)+1,p+1)/(2-\gamma)^p]a_1^p.$$

Let $\tilde{u}(x,t)=a_1\psi(x)$. Then we have

$$\begin{aligned}
&x^q\tilde{u}_t(x,t)-(x^\gamma\tilde{u}_x(x,t))_x=-(x^\gamma a_1\psi'(x))'=a_1\\
&=a_1^p[a^{(2-\gamma)p+1}B(p(1-\gamma)+1,p+1)/(2-\gamma)^p]\\
&=\int_0^a[\ a_1\psi(x)\ ]^p dx=\int_0^a\tilde{u}^p(x,t)dx, && (x,t)\in(0,a)\times(0,T),\\
&\tilde{u}(0,t)=\tilde{u}(a,t)=0, && t\in(0,T),\\
&\tilde{u}(x,0)=a_1\psi(x)\ge u_0(x), && x\in[0,a],
\end{aligned}$$

that is to say, $\tilde{u}(x,t)=a_1\psi(x)$ is an upper solution of problem (1.1). By Theorem 2.5, $T=+\infty$, i.e., $u(x,t)$ exists globally. □

Next, we consider the following eigenvalue problem:

$$\begin{aligned}
&-(x^\gamma\varphi'(x))'=\lambda x^q\varphi(x), \quad x\in(0,a),\\
&\varphi(0)=\varphi(a)=0.
\end{aligned}\tag{3.1}$$

By transformation $\varphi(x)=x^{\frac{1-\gamma}{2}}y(x)$, the above differential equation becomes

$$x^2y''(x)+xy'(x)-\tfrac{(1-\gamma)^2}{4}y(x)+\lambda x^{q+2-\gamma}y(x)=0,\quad x\in(0,a).$$

Again by transformation $x=z^{\frac{2}{q+2-\gamma}}$, problem (3.1) becomes

$$\begin{aligned}
&z^2y''(z)+zy'(z)+[\tfrac{4\lambda z^2}{(q+2-\gamma)^2}-\tfrac{(1-\gamma)^2}{(q+2-\gamma)^2}]y(z)=0,\quad z\in(0,b),\\
&y(0)=y(b)=0,
\end{aligned}\tag{3.2}$$

where $b=a^{\frac{q+2-\gamma}{2}}$. Equation (3.2) is a Bessel equation. Its general solution is given by

$$y(z)=AJ_{\frac{1-\gamma}{q+2-\gamma}}\left(\ \tfrac{2\sqrt{\lambda}}{q+2-\gamma}z\ \right)+BJ_{-\frac{1-\gamma}{q+2-\gamma}}\left(\ \tfrac{2\sqrt{\lambda}}{q+2-\gamma}z\ \right),$$

where A and B are arbitrary constants, and $J_{\frac{1-\gamma}{q+2-\gamma}}$ and $J_{-\frac{1-\gamma}{q+2-\gamma}}$ denote Bessel functions of the first kind of orders $\frac{1-\gamma}{q+2-\gamma}$ and $-\frac{1-\gamma}{q+2-\gamma}$, respectively. Let μ be the first root of $J_{\frac{1-\gamma}{q+2-\gamma}}(\ \frac{2\sqrt{\lambda}}{q+2-\gamma}b\ )$. By Mclachlan [12, pp. 29 and 75], it is positive. It is obvious that μ is the first eigenvalue of problem (3.1); also we can easily obtain the corresponding eigenfunction

$$\varphi(x)=kx^{\frac{1-\gamma}{2}}J_{\frac{1-\gamma}{q+2-\gamma}}\left(\ \tfrac{2\sqrt{\mu}}{q+2-\gamma}x^{\frac{q+2-\gamma}{2}}\ \right),\tag{3.3}$$

which is positive for $x\in(0,a)$. Since $q>\gamma-1$, we can choose $k>0$ such that $\max_{x\in[0,a]}x^q\varphi(x)=1$. Then, we have

Theorem 3.2. *Let $u(x,t)$ be the solution of the problem* (1.1)*. If*

$$\int_0^a x^q\varphi(x)u_0(x)dx > \left(\frac{\mu}{\int_0^a \varphi(x)dx}\right)^{\frac{1}{p-1}} \int_0^a x^q\varphi(x)dx,$$

then $u(x,t)$ blows up in finite time.

Proof. We set

$$U(t) = \int_0^a x^q\varphi(x)u(x,t)dx.$$

Multiplying (1.1) by $\varphi(x)$ and integrating it over x from 0 to a leads to

$$\int_0^a x^q u_t\varphi dx = \int_0^a (x^\gamma u_x)_x\varphi dx + \int_0^a \varphi dx \int_0^a u^p dx.$$

Integration by parts and Jensen's inequality imply that

$$\begin{aligned} U'(t) &= \int_0^a x^q\varphi(x)u_t(x,t)dx \\ &\geq -\mu\int_0^a x^q\varphi(x)u(x,t)dx + \int_0^a \varphi(x)dx \int_0^a x^q\varphi(x)u^p(x,t)dx \\ &\geq -\mu U(t) + \int_0^a \varphi(x)dx(\int_0^a x^q\varphi(x)dx)^{1-p}(\int_0^a x^q\varphi(x)u(x,t)dx)^p \\ &= -\mu U(t) + \int_0^a \varphi(x)dx(\int_0^a x^q\varphi(x)dx)^{1-p}U^p(t). \end{aligned}$$

Therefore $U(t)$ satisfies the following relation:

$$\begin{aligned} &U'(t) \geq U(t)(-\mu + \int_0^a \varphi(x)dxU^{p-1}(t)/(\int_0^a x^q\varphi(x)dx)^{p-1}), \\ &U(0) = \int_0^a x^q\varphi(x)u_0(x)dx. \end{aligned}$$

By the hypothesis,

$$U(0) = \int_0^a x^q\varphi(x)u_0(x) > \left(\frac{\mu}{\int_0^a \varphi(x)dx}\right)^{\frac{1}{p-1}} \int_0^a x^q\varphi(x)dx,$$

hence $U(t)$ tends to infinity in finite time. Therefore, $u(x,t)$ ceases to exist at some finite time; that is to say, $u(x,t)$ blows up in finite time. □

3.2. **Global blow-up.** In this subsection, we would assume that $q > 0$ and $\gamma = 0$. Chan & Chan [3] showed that the Green's function $G(x,\xi,t-\tau)$ associated with the operator $L = x^q\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, subject to the first boundary condition, exists. For ease of reference, we state their Lemmas 2 and 4 in the following lemma.

Lemma 3.3. (a) *For $t > \tau$, $G(x,\xi,t-\tau)$ is continuous for $(x,t,\xi,\tau) \in ([0,a] \times (0,T]) \times ((0,a] \times [0,T))$.*

(b) *For each fixed $(\xi,\tau) \in (0,a] \times [0,T), G_t(x,\xi,t-\tau) \in C([0,a] \times (\tau,T])$.*

(c) *In $\{(x,t,\xi,\tau) : x$ and ξ are in $(0,a), T \geq t > \tau \geq 0\}$, $G(x,\xi,t-\tau)$ is positive.*

In Lemma 7 of [5], Chan & Yang gave some additional properties of $G(x,\xi,t-\tau)$.

Lemma 3.4. *For fixed $x_0 \in (0,a]$, given any $x \in (0,a)$ and any finite time T, there exist positive constants C_1(depending on x and T) and C_2(depending on T) such that*

$$\begin{aligned} &\int_0^a G(x,\xi,t)d\xi > C_1 \quad \textit{for} \quad 0 \leq t \leq T, \\ &\int_0^a G(x_0,\xi,t)d\xi < C_2 \quad \textit{for} \quad 0 \leq t \leq T. \end{aligned}$$

Now we give the global blow-up result

Theorem 3.5. *If the solution $u(x,t)$ of* (1.1) *blows up at the point $x_0 \in (0,a)$ and in finite time T, then the blow-up set of $u(x,t)$ is $[0,a]$.*

Proof. By Green's second identity, we have

$$u(x,t) = \int_0^a \xi^q G(x,\xi,t)u_0(\xi)d\xi + \int_0^t \int_0^a G(x,\xi,t-\tau) \int_0^a u^p(y,\tau)dyd\xi d\tau \tag{3.4}$$

for any $(x,t) \in (0,a) \times (0,T)$. According to the conditions given in this theorem, the solution $u(x,t)$ of (1.1) blows up at $x = x_0$ and in finite time T. Then $\limsup_{t\to T} u(x_0,t) = +\infty$. By (3.4) and Lemma 3.4, we have

$$\begin{aligned} u(x_0,t) &= \int_0^a \xi^q G(x_0,\xi,t)u_0(\xi)d\xi + \int_0^t \int_0^a G(x_0,\xi,\tau) \int_0^a u^p(y,t-\tau)dyd\xi d\tau \\ &\le C_2 a^q \max_{x\in[0,a]} u_0(x) + C_2 \int_0^t \int_0^a u^p(y,t-\tau)dyd\tau. \end{aligned}$$

Since $\limsup_{t\to T} u(x_0,t) = +\infty$, we have

$$\lim_{t\to T} \int_0^t \int_0^a u^p(y,t-\tau)dyd\tau = +\infty. \tag{3.5}$$

On the other hand, for any given $x \in (0,a)$, we have

$$\begin{aligned} u(x,t) &\ge \int_0^a \xi^q G(x,\xi,t)u_0(\xi)d\xi + C_1 \int_0^t \int_0^a u^p(y,t-\tau)dyd\tau \\ &\ge C_1 \int_0^t \int_0^a u^p(y,t-\tau)dyd\tau, \ t \in (0,T). \end{aligned} \tag{3.6}$$

It follows from the above inequality and (3.5) that $\limsup_{t\to T} u(x,t) = +\infty$.

For any $\tilde{x} \in \{0,a\}$, we can choose a sequence $\{x_n,t_n\}$ such that $(x_n,t_n) \to (\tilde{x},T)(n \to +\infty)$ and $\lim_{n\to+\infty} u(x_n,t_n) = +\infty$. Thus the blow-up set is the whole domain $[0,a]$, and we complete the proof. □

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